Math 2110Q Worksheet 15
November 7, 2016

1. Let $\mathscr{D} = \{(x,y) \,|\, 0 \le x \le 1,\ -x \le y \le 2x\}$. Use the transformation $u = x+y$, $v = 2x-y$ to calculate

$$\int\int_{\mathscr{D}} 2(x+y)(2x-y-3)\,dA.$$

Solution: We find the corresponding bounds in uv-space by transforming each boundary edge in turn. Along the edge $y=-x$, $x+y=0$ so that $u=0$. Along the edge $y=2x$, $2x-y=0$ so that $v=0$. Then there is the edge where $x=1$. Then we have both $u=1+y$ and $v=2-y$. Then $u-1=y$ and we substitute for y in the v-equation to get $v=2-(u-1)=3-u$, which is a line in uv-space. So the transformation corresponds to a triangle $0 \le u \le 3$, $0 \le v \le 3-u$ in uv-space. Now, we find the magnitude of the Jacobian; first we invert our transformation. We could substitute $x=u-y$ into the v-equation to get $v=2(u-y)-y=2u-3y$. Then $y=(2u-v)/3$. Substitute this back into $x=u-y=u-(2u-v)/3=(u+v)/3$. Therefore, the Jacobian is

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}.$$

Recall we take the absolute value of the Jacobian when we apply the change of variables, so that

$$\begin{aligned}\int\int_{\mathscr{D}} 2(x+y)(2x-y-3)\,dA &= \int_0^3\int_0^{3-u} 2u(v-3)\frac{1}{3}\,dv\,du = \frac{1}{3}\int_0^3 u(v-3)^2|_{v=0}^{v=3-u}\,du \\ &= \frac{1}{3}\int_0^3 u^3-9u\,du = \frac{1}{3}\left[\frac{1}{4}u^4-\frac{9}{2}u^2\right]|_0^3 \\ &= \frac{1}{3}\left(\frac{81}{4}-\frac{81}{2}\right) = -\frac{27}{4}.\end{aligned}$$

Some comments:

1. There are many ways to explain the domain transformation. Since the change of variables is linear, we could note that the new region in uv-space must also be triangular, and simply map the vertices of the triangle in xy-space to those in uv-space, then connect the dots.
2. One could also find the inverse transformation first, then insert the new equations for $x=x(u,v)$ and $y=y(u,v)$ into the equations for the boundary edges of the triangle in xy-space to convert it to uv-space. In this case, this is probably the most work. However, it is the most "robust" approach, meaning it will work in the most cases.

2. Describe the circle of intersection of the spheres $x^2+y^2+z^2=16$ and $x^2+y^2+(z-3)^2=4$ using spherical coordinates. (4 pts.)

Solution: convert these equations to spherical coordinates. The first sphere is just $\rho=4$. The second is

$$\begin{aligned}\rho^2\sin^2(\phi)+(\rho\cos(\phi)-3)^2=4 &\Rightarrow \rho^2\sin^2(\phi)+\rho^2\cos^2(\phi)-6\rho\cos(\phi)+9=4 \\ &\Rightarrow \rho^2-6\rho\cos(\phi)+5=0.\end{aligned}$$

The intersection occurs when $\rho=4$, so that

$$16-24\cos(\phi)+5=21-24\cos(\phi)=0 \Rightarrow \cos(\phi)=\frac{21}{24}=\frac{7}{8} \Rightarrow \phi=\cos^{-1}\left(\frac{7}{8}\right).$$

The circle of intersection wraps around the z-axis. In spherical coordinates, the curve is completely described by

$$\rho=4,\ \phi=\cos^{-1}\left(\frac{7}{8}\right),\ 0\le\theta\le 2\pi.$$