Math 2110Q Worksheet 13 Solutions
October 31, 2016

1. Consider a laminar structure that occupies the region $\mathscr{D} = \{(x,y)\,|\,x^2+y^2 \leq 1,\ y \geq 0 \text{ if } x \geq 0\}$. This is the unit disc in 2D with the part in the fourth quadrant removed. If the mass density is a constant value over the given region, find the center of mass.
Solution: The region is part of a circle with (polar coordinates) $0 \leq \theta \leq 3\pi/2$ for $0 \leq r \leq 1$. Let ρ denote the constant density. The total mass is then just ρ multiplied by the area of the region, which is $3/4$ of the area of a circle of radius 1; $m = 3\rho\pi/4$. The long way:

$$m = \int_0^{3\pi/2} \int_0^1 \rho\, r\, dr\, d\theta = \frac{\rho}{2}\frac{3\pi}{2} = \frac{3\pi\rho}{4}.$$

Next, we calculate

$$\bar{x} = \frac{1}{m}\int\int_{\mathscr{D}} x\rho\, dA = \frac{4}{3\pi\rho}\rho \int_0^{3\pi/2}\int_0^1 (r\cos(\theta))r\,dr\,d\theta = \frac{4}{3\pi}\int_0^{3\pi/2}\cos(\theta)\,d\theta \int_0^1 r^2\,dr$$
$$= \frac{4}{3\pi}(-1)\frac{1}{3} = -\frac{4}{9\pi}.$$

Since the density is constant, due to the shape of the region we must have $\bar{y} = -\bar{x} = 4/(9\pi)$.

2. Consider a laminar structure that occupies the region in $\mathbb{R}^2$ bounded by the curves $y^2 = x$ and $y = x^2$. If the mass density is denoted by $\rho(x,y)$, provide the general formulas for the moments of inertia about the x and y axes.
Solution: There are various ways to express these integrals:

$$\begin{aligned} I_x &= \int_0^1 \int_{x^2}^{\sqrt{x}} y^2 \rho(x,y)\,dy\,dx \\ \text{and } I_y &= \int_0^1 \int_{x^2}^{\sqrt{x}} x^2 \rho(x,y)\,dy\,dx, \end{aligned}$$

or if the order of integration is reversed,

$$\begin{aligned} I_x &= \int_0^1 \int_{y^2}^{\sqrt{y}} y^2 \rho(x,y)\,dx\,dy \\ \text{and } I_y &= \int_0^1 \int_{y^2}^{\sqrt{y}} y^2 \rho(x,y)\,dx\,dy. \end{aligned}$$

3. Find the surface area of the portion of the surface $f(x,y) = 1 + y + x^2$ that lies over the rectangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$ in the xy-plane.
Solution: In the formula for the surface area, we need

$$\sqrt{1+(f_x)^2+(f_y)^2} = \sqrt{1+(2x)^2+1^2} = \sqrt{2+4x^2}.$$

The integral is then

$$\int_0^1 \int_0^x \sqrt{2+4x^2}\,dy\,dx = \int_0^1 x\sqrt{2+4x^2}\,dx = \frac{1}{8}\int_2^6 \sqrt{u}\,du = \frac{1}{8}\frac{2}{3}(6^{3/2}-2^{3/2}) = \frac{1}{12}(6^{3/2}-2^{3/2}).$$