Math 2110Q Worksheet 10
October 12, 2016

1. Let $T \subset \mathbb{R}^2$ be the triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$ (including the boundary of the triangle). Find the global maximum and minimum values of $f(x,y) = x^2 + 2y^2 - y - x$ on the domain T. (5 points).
Solution: Find critical points inside T by solving

$$f_x = 2x - 1 = 0, \quad f_y = 4y - 1 = 0,$$

which yields one critical point $(1/2, 1/4)$. This point is within the interior of the domain, so consider $f(1/2, 1/4) = -3/8$ as a candidate value.
We need to check the behavior of f along the sides and at the corners of T. At the corners, we have $f(0,0) = 0$, $f(1,0) = 0$ and $f(1,1) = 1$. Along the edge $y = 0$, note $f_x = 0$ when $x = 1/2$, so we also consider $f(1/2, 0) = -1/4$. Along $x = 1$, $f_y = 0$ when $y = 1/4$, so calculate $f(1, 1/4) = -1/8$. Finally, along the edge $y = x$ we have $f(x,y) = f(x,x) = 3x^2 - 2x$. We set the derivative of this function to zero and find $x = 1/3$, so calculate $f(1/3, 1/3) = -1/3$.
Upon comparing all function values, the maximum is $f(1,1) = 1$ and the minimum is $f(1/2, 1/4) = -3/8$.

2. Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x,y) = x + y^2$ on the circle $x^2 + y^2 = 1$ (other methods are possible here, but you must use Lagrange multipliers) (5 points).
Solution: Then $g(x,y) = x^2 + y^2 - 1 = 0$ and there are three equations:

$$\begin{aligned} f_x = 1 &= 2x\lambda = \lambda g_x \\ f_y = 2y &= 2y\lambda = \lambda g_y \\ g = x^2 + y^2 - 1 &= 0. \end{aligned}$$

The first task is to identify **all** possible solutions. One way is to notice that from the second equation, which may be expressed as $2y(\lambda - 1) = 0$, there are only two possibilities: $\lambda = 1$ or $y = 0$. So consider both cases:
Case $\lambda = 1$. Then the first equation implies $x = 1/2$. The third equation then tells us $y^2 = 1 - x^2 = 1 - 1/4 = 3/4$, so that $f = x + y^2 = 1/2 + 3/4 = 5/4$. This is the only possible value of f for $\lambda = 1$.
Case $y = 0$. Then from the third (constraint) equation, $x^2 = 1$ and hence $x = \pm 1$. Then $f = x + y^2 = x = \pm 1$. There are no more possibilities.
In summary, the maximum of f is $5/4$ and the minimum is -1.