

Partial Derivatives

For $f = f(x, y)$ we can discuss derivatives with respect to x, y independently:

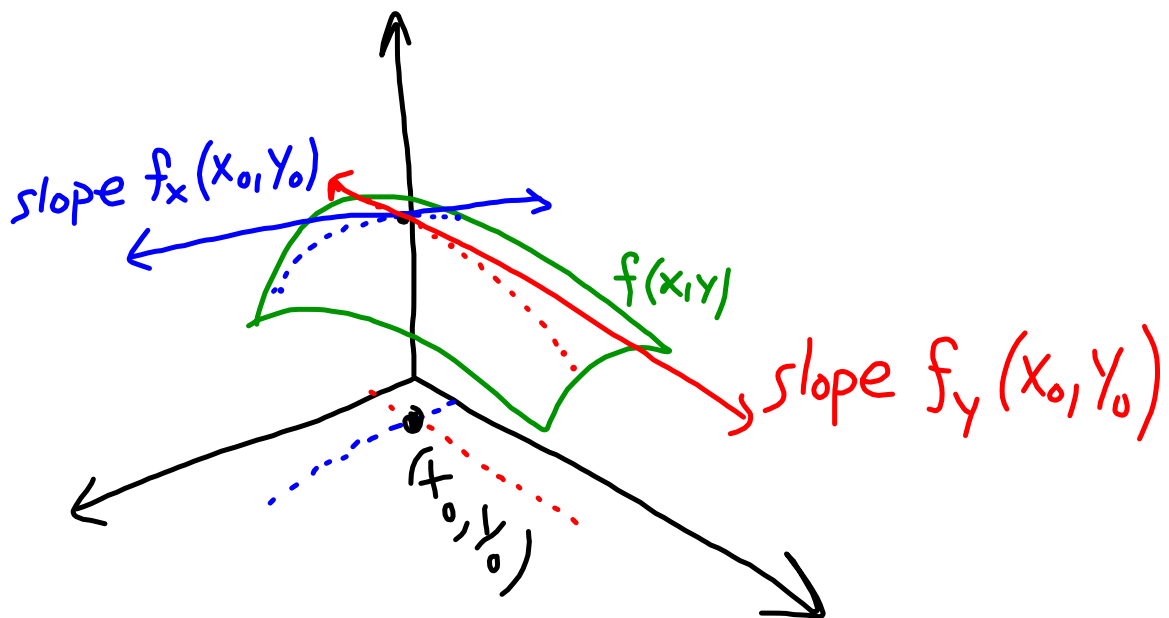
$$f_x(x, y) = \frac{\partial}{\partial x} f(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

y is constant

$$f_y(x, y) = \frac{\partial}{\partial y} f(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

x is constant

Partial derivatives are the slopes of the tangent lines in the coordinate directions.



EX: Find f_x, f_y if $f(x,y) = x + y + \sin(x)\cos(y)$.

Hold y fixed...

$$f_x = 1 + \cos(x)\cos(y).$$

Hold x fixed...

$$f_y = 1 - \sin(x)\sin(y).$$

EX: Calculate f_x, f_y for

$$(a) f(x, y) = \ln(x^2 + xy + y^2)$$

$$f_x = \frac{1}{x^2 + xy + y^2} \frac{\partial}{\partial x} (x^2 + xy + y^2)$$

$$\Rightarrow f_x = \frac{2x + y}{x^2 + xy + y^2}$$

$$f_y = \frac{2y + x}{x^2 + xy + y^2}$$

$$(b) f(x, y) = xy \sin(x+y)$$

$$f_x = y \sin(x+y) + xy \cos(x+y)$$

$$f_y = x \sin(x+y) + xy \cos(x+y)$$

Implicit differentiation

Let $z = z(x, y)$ satisfy

$$x^3 + y^2 + z^4 + xyz = 1.$$

Then find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ by differentiating through the whole equation...

$$\frac{\partial}{\partial x} (x^3 + y^2 + z^4 + xyz) = \frac{\partial}{\partial x} (1) = 0$$

$$\Rightarrow 3x^2 + 4z^3 \frac{\partial z}{\partial x} + yz + xy \frac{\partial z}{\partial x} = 0.$$

Now group $\frac{\partial z}{\partial x}$ terms and factor it out:

$$3x^2 + yz + \frac{\partial z}{\partial x} \{4z^3 + xy\} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-3x^2 - yz}{4z^3 + xy}$$

$$\frac{\partial}{\partial y} (x^3 + y^2 + z^4 + xyz) = \frac{\partial}{\partial y} (1) = 0$$

$$\Rightarrow 2y + 4z \frac{\partial z}{\partial y} + xz + xy \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial y}(4z^3 + xy) = -xz - 2y$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{-xz - 2y}{4z^3 + xy}}$$

* If $(x, y, z(x, y))$ solve the original equation then you can plug them in to get the partial derivatives.

Higher-order derivatives

2nd-order of $f(x, y)$: f_{xx} f_{xy}
 f_{yx} f_{yy}

THEOREM : If f_{xy} & f_{yx} are both continuous then $f_{xy} = f_{yx}$.

EX: Find all 2nd-order derivatives of
 $f(x,y) = x^3 + y^3 + x^2y^2 + xy + 1.$

$$f_x = 3x^2 + 2xy^2 + y \quad f_y = 3y^2 + 2x^2y + x$$

$$\begin{array}{ccc} \downarrow & \text{equal} & \downarrow \\ f_{xy} = 4xy + 1 & \longleftrightarrow & f_{yx} = 4xy + 1 \end{array}$$

$$f_{xx} = 6x + 2y^2 \quad f_{yy} = 6y + 2x^2$$

Some other notations

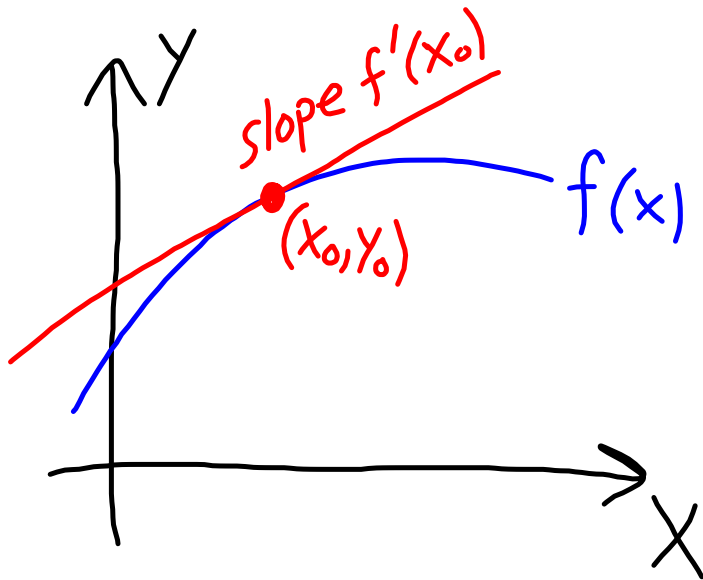
$$\frac{\partial}{\partial x} f = \frac{\partial f}{\partial x} = D_x f = f_x$$

$$\frac{\partial^2}{\partial x \partial y} f = \frac{\partial^2 f}{\partial x \partial y} = D_{xy} f = f_{xy}$$

These have various uses depending on the circumstances and are worth being familiar with for convenience in writing.

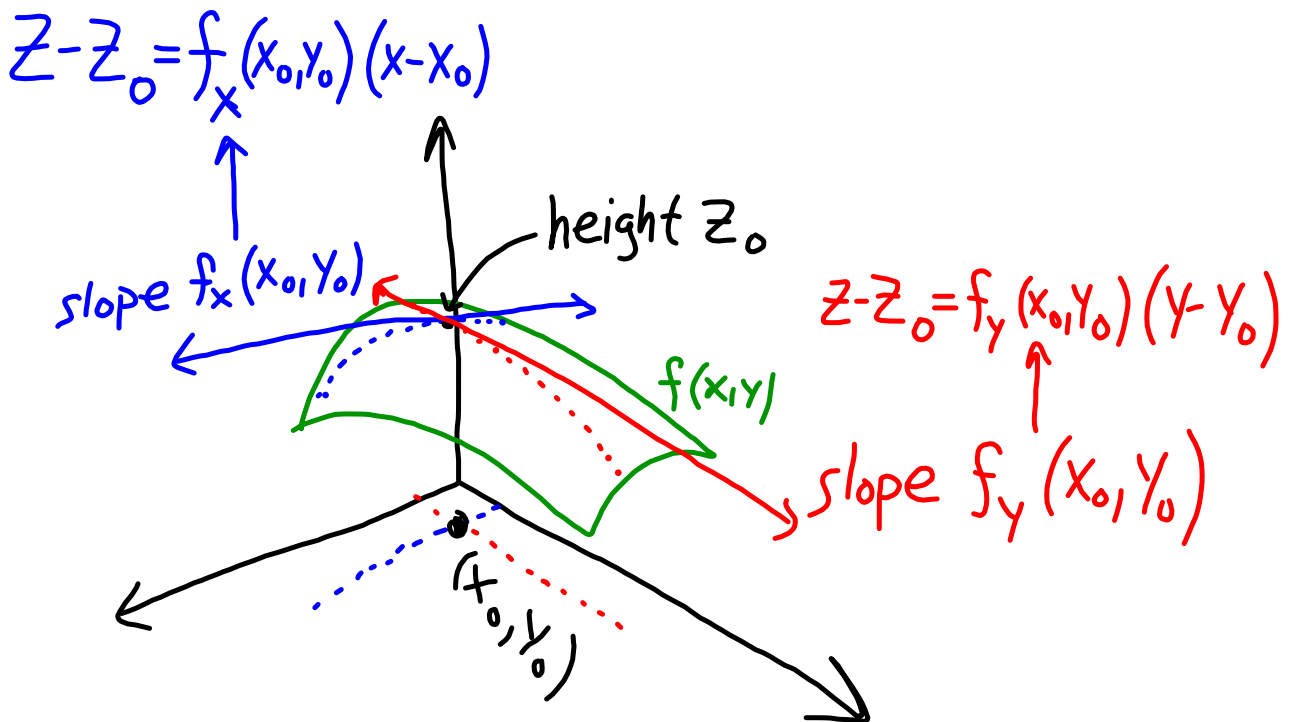
Tangent planes

Let's recall tangent lines in 1D:



$$\begin{aligned}y - y_0 &= m(x - x_0) \\ &= f'(x_0)(x - x_0).\end{aligned}$$

Now recall the tangent lines on a surface. The plane containing these is the tangent plane.



Equation for the tangent plane

Recall the plane equation of the form $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$$\Rightarrow c(z-z_0) = -a(x-x_0) - b(y-y_0)$$

$$\Rightarrow z-z_0 = -\frac{a}{c}(x-x_0) - \frac{b}{c}(y-y_0)$$

$$\Rightarrow z-z_0 = A(x-x_0) + B(y-y_0).$$

$$y=y_0 \text{ fixed} \Rightarrow z-z_0 = A(x-x_0) \\ = \frac{\partial f}{\partial x}(x_0, y_0)(x-x_0)$$

$$x=x_0 \text{ fixed} \Rightarrow z-z_0 = B(y-y_0) \\ = \frac{\partial f}{\partial y}(x_0, y_0)(y-y_0)$$

$$\Rightarrow z-z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x-x_0) \\ + \frac{\partial f}{\partial y}(x_0, y_0)(y-y_0).$$

TANGENT
PLANE

EX: Let $f(x,y) = \ln(x^2+y^2)$... find the tangent plane equation at $(1, 1, \ln(2))$.

$$\frac{\partial f}{\partial x} \Big|_{(1,1)} = \left(\frac{2x}{x^2+y^2} \right) \Big|_{(1,1)} = \frac{2}{2} = 1.$$

$$\frac{\partial f}{\partial y} \Big|_{(1,1)} = \left(\frac{2y}{x^2+y^2} \right) \Big|_{(1,1)} = 1$$

$$z - \ln(2) = (x-1) + (y-1) = x+y-2.$$

EX: Let $f(x,y) = \sqrt{x^2 - y^2 + 1}$. Find the equation of the tangent plane at the point $(\sqrt{8}, 0, 3)$.

$$f_x(\sqrt{8}, 0) = \frac{2x}{2\sqrt{x^2 - y^2 + 1}} \Big|_{(\sqrt{8}, 0)} = \frac{\sqrt{8}}{\sqrt{9}} = \frac{1}{3}\sqrt{8}.$$

$$f_y(\sqrt{8}, 0) = \frac{-2y}{2\sqrt{x^2 - y^2 + 1}} \Big|_{(\sqrt{8}, 0)} = 0.$$

$$\Rightarrow z - 3 = \frac{\sqrt{8}}{3}(x - \sqrt{8})$$

Differentials (particularly useful in engineering)

$$\underbrace{z - z_0}_{\Delta z} = (f_x) \underbrace{(x - x_0)}_{\Delta x} + (f_y) \underbrace{(y - y_0)}_{\Delta y}$$

$$\Delta z = \frac{\partial f}{\partial x} \cdot \Delta x + \frac{\partial f}{\partial y} \cdot \Delta y \approx \Delta f$$

For small perturbations of design variables we may estimate the effect on the output in this way.

EX: A cylinder has volume $V = \pi r^2 h$
 r : radius h : height

Consider machining a cylinder (solid metal)
and can control r to $\pm 0.1\%$ and h to $\pm 0.01\%$.
Put limits on how big/small the cylinder
could turn out.

$$\Delta V \approx V_r \Delta r + V_h \Delta h = 2\pi r h \Delta r + \pi r^2 \Delta h.$$

$$\Delta r = 0.001 \cdot r \quad \Delta h = 0.0001 \cdot h$$

MAX V:

$$V_{\text{MAX}} \approx \pi r^2 h + (0.001 \cdot 2\pi r^2 h + 0.0001 \cdot \pi r^2 h)$$

$\approx V + \Delta V$

$$V_{\text{MIN}} \approx \pi r^2 h - (0.001 \cdot 2\pi r^2 h + 0.0001 \cdot \pi r^2 h)$$

$\approx V - \Delta V$

Practice!

#1 Find all 2nd-order partial derivatives of $f(x,y) = \sin(x^2y^2)$.

$$f_x = \cos(x^2y^2) \frac{\partial}{\partial x}(x^2y^2) \\ = 2xy^2 \cos(x^2y^2)$$

$$f_y = 2yx^2 \cos(x^2y^2)$$

$$f_{xx} = 2y^2 \cos(x^2y^2) - 2xy^2 \cdot 2xy^2 \sin(x^2y^2)$$

Product rule Chain Rule

$$\text{So } f_{xx} = 2y^2 \cos(x^2 y^2) - 4x^2 y^4 \sin(x^2 y^2)$$

$$f_{xy} = f_{yx} = \frac{\partial}{\partial y} \left\{ 2xy^2 \cos(x^2 y^2) \right\}$$

$$= 4xy \cos(x^2 y^2) - 2xy^2 \cdot 2x^2 y \sin(x^2 y^2)$$

$$= 4xy \cos(x^2 y^2) - 4x^3 y^3 \sin(x^2 y^2).$$

$$f_{yy} = 2x^2 \cos(x^2 y^2) - 4y^2 x^4 \sin(x^2 y^2)$$

↑ just switch x & y in f_{xx} formula because for this example $f = \sin(x^2 y^2)$ is symmetric with respect to x & y .

#2 Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ at $(1, 1, -1)$ if

$z = z(x, y)$ satisfies

$$xy + xz^2 + y^2z = 1.$$

check: these should solve the equation

$$\frac{\partial}{\partial x} (xy + xz^2 + y^2z) = \frac{\partial}{\partial x} (1) = 0$$

$$y + z^2 + 2xz \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial x} = 0$$

Easiest to plug in $x=y=1$ & $z=-1$ now...

$$1 + 1 - 2 \frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} = 0 \quad \left. \vphantom{\frac{\partial z}{\partial x}} \right\} \begin{array}{l} \text{solve now} \\ \text{for } \frac{\partial z}{\partial x} \dots \end{array}$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = 1 + 1 = 2.}$$

Now for $\partial z / \partial y$,

$$\frac{\partial}{\partial y} (xy + xz^2 + y^2z) = \frac{\partial}{\partial y} (1) = 0$$

$$x + 2xz \frac{\partial z}{\partial y} + 2yz + y^2 \frac{\partial z}{\partial y} = 0$$

$$\left. \begin{array}{l} x=y=1 \\ z=-1 \end{array} \right\} \Rightarrow 1 - 2 \frac{\partial z}{\partial y} - 2 + \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow -1 - \frac{\partial z}{\partial y} = 0 \Rightarrow$$

$$\boxed{\frac{\partial z}{\partial y} = -1.}$$

#3 Find the tangent plane to
 $f(x, y) = x^2 + xy + y^3$ when $x=2, y=-1$.

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

\uparrow \uparrow \uparrow
 $z_0 = f(x_0, y_0) = 4 - 2 - 1 = 1$ $x_0 = 2$ $y_0 = -1$

$$f_x = 2x + y \Rightarrow f_x(2, -1) = 4 - 1 = 3$$

$$f_y = x + 3y^2 \Rightarrow f_y(2, -1) = 2 + 3 = 5$$

$$z - 1 = 3(x - 2) + 5(y + 1)$$

#4 Consider a box of length L , width W that is 4 ft. high. You want to construct the box with $L=2$ ft and $W=1$ ft, but are only able to measure these to within $1/8$ inch precision. Estimate the largest volume the box could potentially have by applying differentials.

$$\text{So } V = 4 \cdot L \cdot W \Rightarrow \Delta V \approx \frac{\partial V}{\partial L} \Delta L + \frac{\partial V}{\partial W} \Delta W$$

$$\Rightarrow \Delta V \approx 4W\Delta L + 4L\Delta W \left. \begin{array}{l} W = 1 \text{ ft.} \\ L = 2 \text{ ft.} \\ \text{use } \Delta L = \Delta W = \frac{1}{96} \text{ ft.} \end{array} \right\}$$

$$\begin{aligned} \text{Then } \Delta V_{\text{MAX}} &\approx 4 \cdot 1 \cdot \frac{1}{96} + 4 \cdot 2 \cdot \frac{1}{96} \\ &= \frac{12}{96} = \frac{1}{8} \text{ (ft.}^3\text{)} \end{aligned}$$

$$\begin{aligned} \Rightarrow V_{\text{MAX}} &= V + \Delta V_{\text{MAX}} = 4 \cdot 2 \cdot 1 + \frac{1}{8} \\ &= 8 + \frac{1}{8} = \frac{65}{8} \text{ ft.}^3 \end{aligned}$$