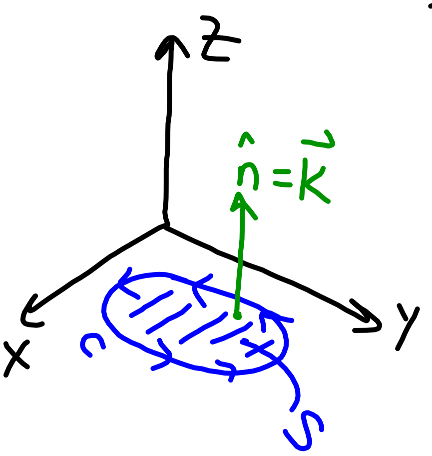
Green's Theorem is a special case of Stoke's



If S is a "surface" in the xy-plane, $\hat{n} = \vec{k}$

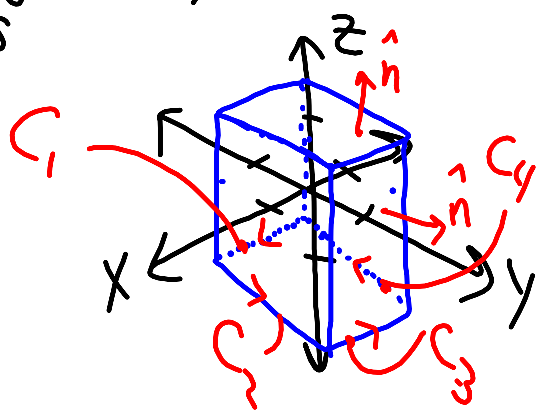
$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\vec{k}\, dA$$

(Stoke's)

But we previously showed that Green's Theorem may be written this way.

Some examples for Stoke's Theorem

Ex: let S be the 4 sides & top (not bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, outward orientation. Find $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$ if $\vec{F} = xyz\,\vec{i} + xy\,\vec{j} + x^2yz\,\vec{k}$.



Use Stoke's to write as a line integral over C, the union of C_1, C_2, C_3, C_4.

$$\iint_S (\nabla \times \vec{F})\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \int_C xyz\,dx + yx\,dy + x^2yz\,dz.$$

Note $dz = 0$ since $z=-1$, constant on C.

$$\Rightarrow \int_C xy(-1)\,dx + xy\,dy \quad \Big\}\ \text{calculate on each piece } C_i,\ i=1,2,3,4$$

$$\int_{C_1} -xy\,dx + xy\,dy = \int_{C_1} -x(-1)\,dx = \int_{-1}^{1} x\,dx = \frac{1}{2}x^2\Big|_{-1}^{1} = 0.$$

$y=-1$, $dy=0$

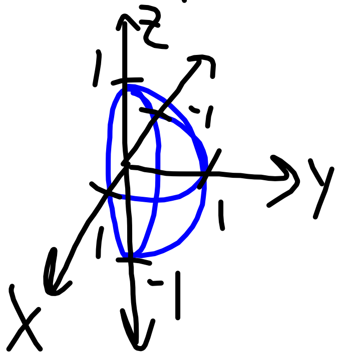
$$\int_{C_2} -xy\,dx + xy\,dy = \int_{C_2} (1)\,y\,dy = \int_{-1}^{1} y\,dy = 0.$$

$x=1$
$dx=0$

$$\int_{C_3} -xy\,dx + xy\,dy = 0 = \int_{C_4} -xy\,dx + xy\,dy$$

$$\Rightarrow \boxed{\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = 0.}$$

EX: Verify Stoke's Theorem for S the half of the surface of the unit sphere with $y \geq 0$, oriented in $+y$-direction, and $\vec{F} = \langle y, z, x \rangle$.



S-parameterization

$$\left.\begin{aligned} x &= \cos\theta \sin\phi \\ y &= \sin\theta \sin\phi \\ z &= \cos\phi \end{aligned}\right\} \quad \begin{aligned} &0 \leq \theta \leq \pi \\ &0 \leq \phi \leq \pi \end{aligned}$$

Previous results...

$$\vec{r}(\phi,\theta)=\langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$$

$$\Rightarrow \vec{r}_\phi \times \vec{r}_\theta = \langle \cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \sin\phi\cos\phi\rangle$$

(correct orientation)

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = \langle -1, -1, -1\rangle$$

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_0^{\pi}\int_0^{\pi} \langle -1,-1,-1\rangle \cdot \left(\vec{r}_\phi \times \vec{r}_\theta\right) d\phi\, d\theta$$

$$= -\int_0^{\pi}\int_0^{\pi} (\cos\theta + \sin\theta)\sin^2\phi + \sin\phi\cos\phi \; d\phi\, d\theta$$

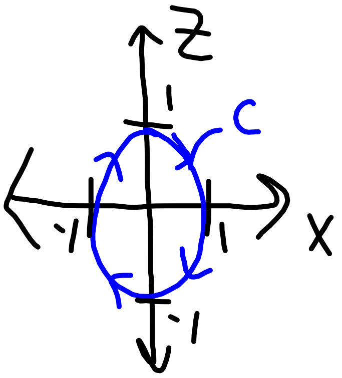
$$= -\left(\int_0^{\pi} \cos\theta + \sin\theta \, d\theta\right)\left(\int_0^{\pi} \sin^2\phi \, d\phi\right) - \pi \cancelto{0}{\int_0^{\pi} \frac{1}{2}\frac{d}{d\phi}\sin^2\phi \, d\phi}$$

$$= -\Big[\sin\theta - \cos\theta\Big]_0^{\pi} \int_0^{\pi} \frac{1}{2}(1 - \cos 2\phi)\, d\phi$$

$$\iint_S (\nabla\times\vec{F})\cdot d\vec{S} = -2\cdot\frac{1}{2}\cdot\left[\pi - \frac{1}{2}\underbrace{\sin 2\phi\Big|_0^{\pi}}_{0}\right]$$

$$= \boxed{-\pi}$$

Now we need to show $\int_C \vec{F}\cdot d\vec{r} = -\pi$.



$$\begin{cases} z = \cos(t) \\ x = \sin(t) \\ \vec{r}(t) = \langle \sin(t), 0, \cos(t)\rangle \\ \vec{r}\,'(t) = \langle \cos(t), 0, -\sin(t)\rangle. \end{cases}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \underbrace{\langle 0, \cos(t), \sin(t)\rangle}_{\langle y,z,x\rangle = \vec{F}} \cdot \langle \cos(t), 0, -\sin(t)\rangle \, dt$$

$$= \int_0^{2\pi} -\sin^2(t)\, dt = \boxed{-\pi}\,.$$

Recall $\int_0^{2\pi} \cos^2(t)\,dt = \int_0^{2\pi} \sin^2(t)\,dt = \pi$

We have verified Stoke's Theorem.

Divergence Theorem

We previously derived another version of Green's Theorem: $\int_C (\vec{F}\cdot\hat{n})\,ds = \iint_D (\nabla\cdot\vec{F})\,dA$.

There is an extension to 3D:

$$\boxed{\iint_{S=\partial V} (\vec{F}\cdot\hat{n})\,dS = \iiint_V (\nabla\cdot\vec{F})\,dV.}$$

DIVERGENCE THEOREM

$\hat{n}$ "outward"

EX: Find the flux of $\vec{F} = \langle yz, 4y, x^2+1 \rangle$ outward through the surface of the unit sphere.

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$$

$$= \iiint_V 4\,dV = 4\left(\frac{4}{3}\pi(1)^3\right) = \boxed{\frac{16\pi}{3}}.$$

Ex: Find $\iint_S \vec{F}\cdot\hat{n}\, dS$ if S is the surface of the box $-2 \le x \le 1,\ 0 \le y \le 1,\ 2 \le z \le 4$, and $\vec{F} = \langle x^2, yz, xz \rangle$.

$$\nabla \cdot \vec{F} = 2x + z + x = z + 3x$$

$$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_V z + 3x \, dV = \int_{-2}^{1}\int_{0}^{1}\int_{2}^{4} z+3x \, dz\, dy\, dx$$

$$= \int_{-2}^{1} \left[\frac{1}{2}z^2 + 3xz\right]_2^4 dx$$

$$= \int_{-2}^{1} \frac{1}{2}(16-4) + 3x(4-2)\,dx = \int_{-2}^{1} 6+6x\,dx$$

$$= 3\left[(1+x)^2\right]_{-2}^{1} = 3(4-1) = \boxed{9.}$$

Some commonly-used formulas resulting from the Divergence Theorem

$$\iiint_V \frac{\partial f}{\partial x}\, dV = \iiint_V \nabla \cdot \langle f, 0, 0\rangle\, dV$$

$$= \iint_{\partial V} \langle f, 0, 0\rangle \cdot \hat{n}\, dS = \iint_S f(\vec{i} \cdot \hat{n})\, dS.$$

Also,

$$\iiint_V \frac{\partial f}{\partial y}\, dV = \iint_S f(\vec{j} \cdot \hat{n})\, dS, \quad \iiint_V \frac{\partial f}{\partial z}\, dV = \iint_S f(\vec{k} \cdot \hat{n})\, dS.$$

<u>EX</u>: Let V be the unit "ball" enclosed by the unit sphere ∂V. Write

$$\iiint_V x^2 y \, dV$$ as a surface integral over ∂V.

There are many solutions. For example,

$$x^2 y = \frac{\partial}{\partial z}\left(x^2 y z\right)$$

$$\Rightarrow \iiint_V x^2 y \, dV = \iiint_V \frac{\partial}{\partial z}\left(x^2 y z\right) dV = \iint_{\partial V} x^2 y z \left(\hat{k} \cdot \hat{n}\right) dS.$$

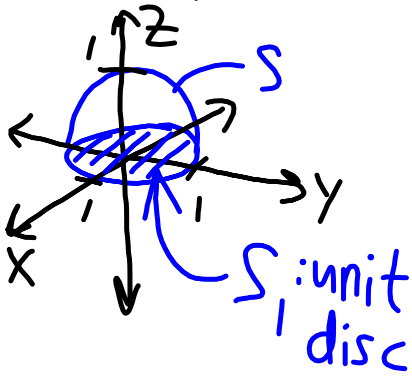
$\underline{\text{EX}}$: Show that $\iiint\limits_V x \, dV = 0$, if V is the unit ball.

Intuitively, this should be true. We can show directly by noting $\iiint\limits_V x \, dV = \iiint\limits_V \frac{\partial}{\partial x}(1-x^2-y^2-z^2)\, dV$

$$= \iint\limits_{\partial V} (1-x^2-y^2-z^2)(\vec{i} \cdot \hat{n})\, dS$$

0, $x^2+y^2+z^2=\rho^2=1$ on ∂V.

<u>EX</u>: Let S be the upper half $(z \geq 0)$ of the unit sphere. Calculate $\iint_S \vec{F} \cdot \hat{n}\, dS$ if $\vec{F} = \langle y, x, z \rangle$.



S & S_1 together enclose volume V

$$\Rightarrow \iiint_V \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \hat{n}\, dS + \iint_{S_1} \vec{F} \cdot \hat{n}\, dS$$

calculate

On S_1, $\vec{F}\cdot\hat{n} = \vec{F}\cdot\langle 0,0,-1\rangle$

$= -z = 0$, since $z=0$ on S_1.

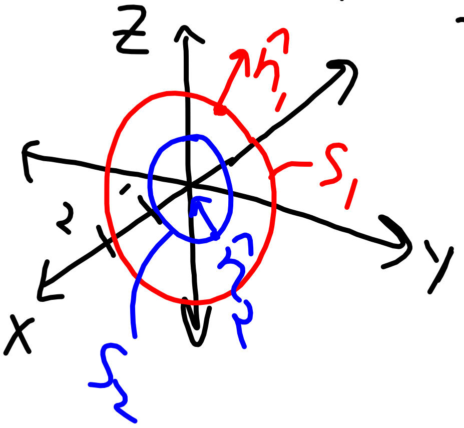
Therefore,

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\langle y,x,z\rangle\,dV$$

$$= \iiint_V 1\,dV = \frac{1}{2}\left(\frac{4}{3}\pi(1)^3\right) = \boxed{\frac{2\pi}{3}}.$$

Extension of the Divergence Theorem

We could have a volume with a "cavity", e.g. $V = \{(x,y,z) : 1 \le x^2+y^2+z^2 \le 2\}$.



Let S_1 bound V_1, S_2 bound V_2. Then

$$\iiint_V \nabla\cdot\vec{F}\,dV = \iiint_{V_1} \nabla\cdot\vec{F}\,dV - \iiint_{V_2} \nabla\cdot\vec{F}\,dV$$

$$= \iint_{S_1} \vec{F}\cdot\hat{n}_1\,dS - \iint_{S_2} \vec{F}\cdot(-\hat{n}_2)\,dS$$

$$\Rightarrow \iiint_V \nabla\cdot\vec{F}\,dV = \iint_{S_1} \vec{F}\cdot\hat{n}_1\,dS + \iint_{S_2} \vec{F}\cdot\hat{n}_2\,dS$$

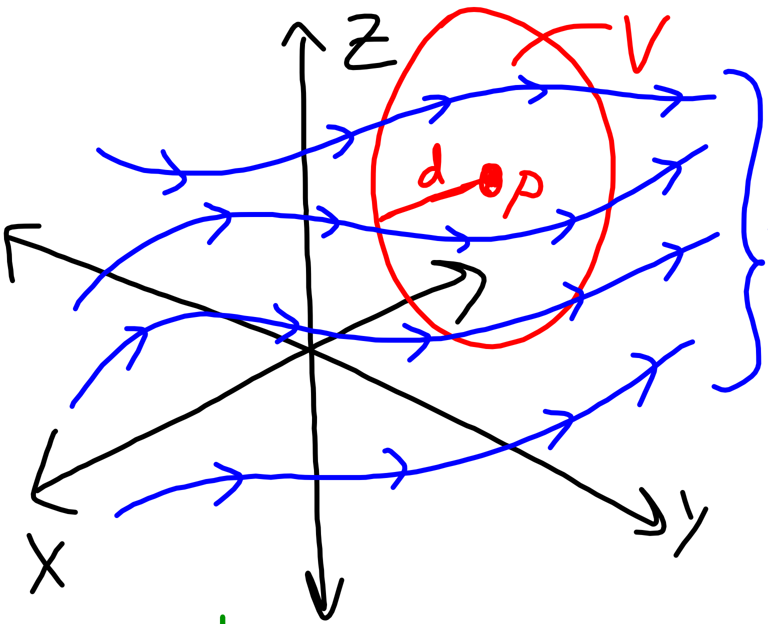
$$= \iint_S \vec{F}\cdot\hat{n}\,dS .$$

Here, $S = S_1 \cup S_2$

$$\hat{n} = \begin{cases} \hat{n}_1, & \text{on } S_1 \\ \hat{n}_2, & \text{on } S_2 . \end{cases}$$

Application: Conservation laws in fluid dynamics are derived partially by using the Divergence Theorem.

Students are not responsible for this section of material.



particle streamlines

$P = (x, y, z)$

d: radius of ball around P

$V = V(d)$: volume (ball) containing P.

ρ: density of some quantity transported by the fluid.

$\vec{u}$: fluid velocity.

The total amount of the transported quantity within V is $\displaystyle\iiint_{V(d)} \rho \, dV$.

The rate of change (in time) of this is determined by the flux $\rho \vec{u} \cdot \hat{n}$ across ∂V:

$$\frac{d}{dt} \iiint_{V(d)} \rho \, dV = - \iint_{\partial V} \rho \vec{u} \cdot \hat{n} \, dS .$$

Divergence Theorem:

$$\frac{d}{dt}\iiint_{V(d)} \rho \, dV = -\iiint_{V(d)} \nabla\cdot(\rho\vec{u})\, dV$$

$$\Rightarrow \iiint_{V(d)} \left\{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u})\right\} dV = 0,$$

for <u>all</u> $d>0$.

In the limit as $d \to 0$,

$$\lim_{d \to 0} \underbrace{\frac{1}{|V(d)|} \iiint_{V(d)} \{\rho_t - \nabla \cdot (\rho \vec{u})\} \, dV}_{\text{average value of } \rho_t - \nabla\cdot(\rho\vec{u}) \text{ on } V \text{ converges to value at center } P \text{ as } d \to 0} = 0$$

$$\Rightarrow \quad \boxed{\rho_t - \nabla \cdot (\rho \vec{u}) = 0 .}$$

"Transport Equation"

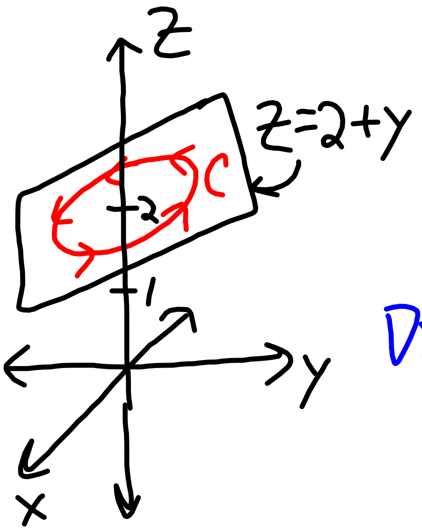
*This is a "partial differential equation" or PDE.

*You need to specify things called "initial conditions" and "boundary conditions".

*You would need to know what the velocity is everywhere to solve for the density. Usually you don't have this and need to solve more complicated equations (in fact, we usually can't solve them directly).

Practice!

#1 Let $\vec{F} = \langle y, \frac{1}{2}z^2 y, z y^2 \rangle$. Find $\int_C \vec{F} \cdot d\vec{r}$ if C is the intersection of $x^2+y^2=1$ and $z=2+y$, oriented counter-clockwise as viewed from "above" meaning high on +z-axis, looking in $-\vec{k}$ direction.



Let's use Stoke's.

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\, dS.$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & \frac{1}{2}z^2 y & zy^2 \end{vmatrix} = \langle zy, 0, -1\rangle.$$

Parameterize S: $x = r\cos\theta$

$y = r\sin\theta$

$z = 2+y = 2+r\sin\theta$

$$\vec{r}_r(r,\theta) = \langle \cos\theta, \sin\theta, \sin\theta \rangle$$

$$\vec{r}_\theta = \langle -r\sin\theta, r\cos\theta, r\cos\theta \rangle$$

$$\vec{r}_r \times \vec{r}_\theta = \langle 0, -r, r \rangle \Bigg\}$$ points up $\Rightarrow$ correct orientation

$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\,dS = \int_0^{2\pi}\int_0^1 \langle 2y, 0, -1\rangle \cdot \langle 0, -r, r\rangle\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 -r\,dr\,d\theta = -2\pi\cdot\frac{1}{2}r^2\Big|_0^1 = \boxed{-\pi}.$$

#2 Let S be the surface of the cube $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$. Calculate $\iint_S \vec{F} \cdot \hat{n} \, dS$ if

$$\vec{F} = \left\langle \cos(x+y)\sin(x+y),\ -\cos(x+y)\sin(x+y),\ z^2+1 \right\rangle,$$

with outward orientation.

Divergence Theorem:

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$$

$$\nabla\cdot\vec{F} = \left[-\sin^2(x+y)+\cos^2(x+y)+\sin^2(x+y)-\cos^2(x+y) + 2z\right]$$

$$= 1-1+2z = 2z$$

$$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,dS = \int_0^1\int_0^1\int_0^1 2z\,dz\,dx\,dy = 2\int_0^1 z\,dz = z^2\Big|_0^1 = \boxed{1.}$$