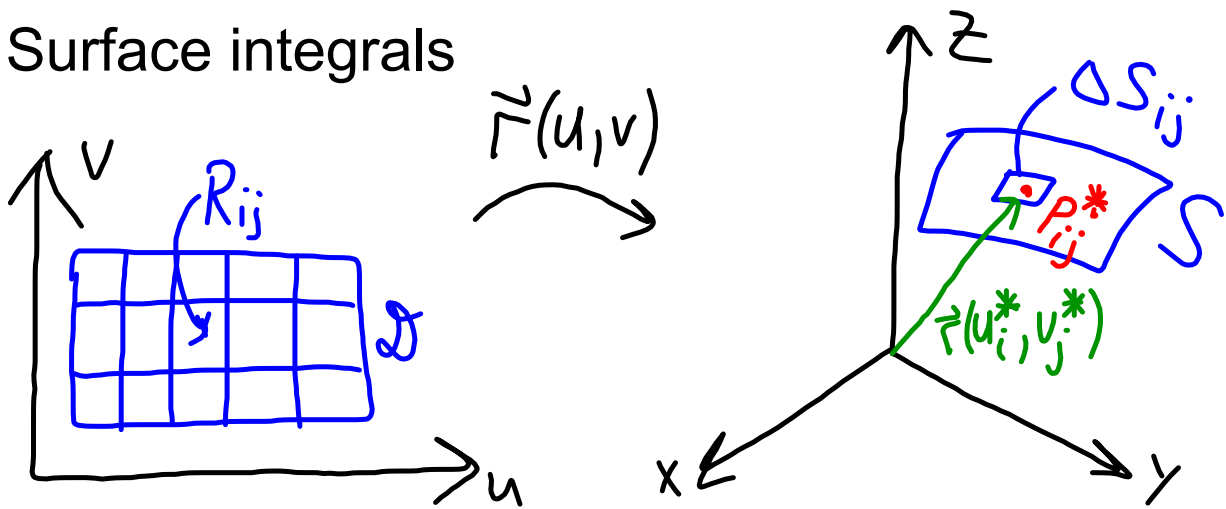


Surface integrals



- $f(x, y, z)$ is defined on S .
- S is parameterized by $\vec{r}(u, v)$.

$$\text{Recall } \Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v| \Delta u \cdot \Delta v$$

$$\Rightarrow \iint_S f(x, y, z) dS = \lim_{N \rightarrow \infty} \sum_{i, j=1}^N f(P_{ij}^*) \Delta S_{ij}$$

$$= \lim_{N \rightarrow \infty} \sum_{i, j=1}^N f(\vec{r}(u_i^*, v_j^*)) |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

$$= \iint_{\mathcal{D}} f(\vec{r}(u, v)) |\vec{r}_u \times \vec{r}_v| du dv.$$

Surface integral for scalar-valued f :

$$\iint_S f dS = \iint_{\mathcal{D}} f(\vec{r}(u,v)) |\vec{r}_u \times \vec{r}_v| du dv.$$

In case $f \equiv 1$ we recover surface area:

$$|S| = \iint_S dS = \iint_{\mathcal{D}} |\vec{r}_u \times \vec{r}_v| du dv.$$

Recall for $\vec{r}(x,y) = \langle x, y, z(x,y) \rangle$

we get

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}, \text{ so}$$

$$\iint_S f dS = \iint_D f(x,y,z(x,y)) \sqrt{1 + (z_x)^2 + (z_y)^2} dx dy.$$

special case we use often.

EX: Calculate $\int_S x^2 + y^2 dS$, where S is the sphere of radius 3 centered at the origin. Recall

$$\begin{aligned}x &= 3 \cos \theta \sin \phi \\y &= 3 \sin \theta \sin \phi \\z &= 3 \cos \phi\end{aligned}$$

Also, $|\vec{r}_\theta \times \vec{r}_\phi| = 3^2 \sin \phi = 9 \sin \phi$, and

$$x^2 + y^2 = r^2 = (3 \sin \phi)^2 = 9 \sin^2 \phi.$$

$$\begin{aligned}
\int_S \int x^2 + y^2 dS &= \int_0^{2\pi} \int_0^\pi 9 \sin^2 \phi \cdot 9 \sin \phi d\phi d\theta \\
&= 81 \cdot 2\pi \cdot \int_0^\pi \sin^3 \phi d\phi = 162\pi \int_0^\pi \sin \phi (1 - \cos^2 \phi) d\phi \\
&= 162\pi \left[\int_0^\pi \sin \phi d\phi - \int_0^\pi \sin \phi \cos^2 \phi d\phi \right] \\
&= 162\pi \left[-\cos \phi \Big|_0^\pi + \frac{1}{3} \cos^3 \phi \Big|_0^\pi \right] = \boxed{216\pi}.
\end{aligned}$$

Ex: Calculate $\iint_S z^2 dS$, where
 S is given by $f(x,y) = 1+x+y$, $0 \leq x \leq 1$
 $0 \leq y \leq 2$.

$$f_x = 1 = f_y \Rightarrow \sqrt{1+f_x^2+f_y^2} = \sqrt{3}$$

$$\iint_S z^2 dS = \int_0^1 \int_0^2 (1+x+y)^2 \sqrt{3} dy dx = \sqrt{3} \int_0^1 \frac{1}{3} (1+x+y)^3 \Big|_0^2 dx$$

$$= \frac{\sqrt{3}}{3} \int_0^1 (3+x)^3 - (1+x)^3 dx = \frac{\sqrt{3}}{3} \left[\frac{1}{4} (3+x)^4 - \frac{1}{4} (1+x)^4 \right] \Big|_0^1$$

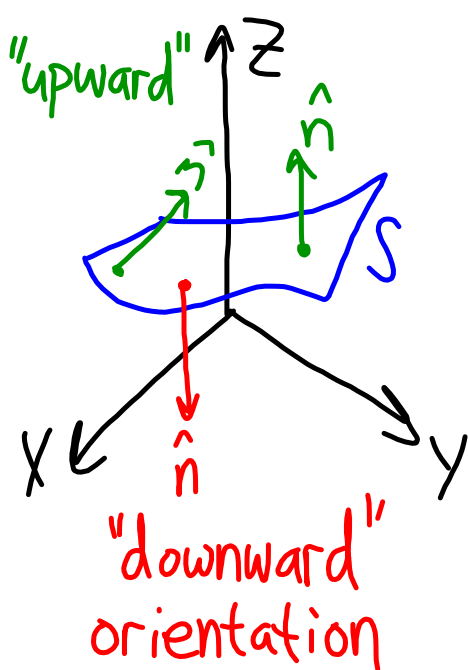
$$= \frac{\sqrt{3}}{12} [4^4 - 3^4 - 2^4 + 1]$$

$$= \frac{\sqrt{3}}{12} [256 - 81 - 16 + 1]$$

$$= \frac{\sqrt{3} \cdot 160}{12} = \boxed{\frac{40\sqrt{3}}{3}}$$

Surface orientation

We will assign a unit normal vector to a surface, depending on where you are on that surface.



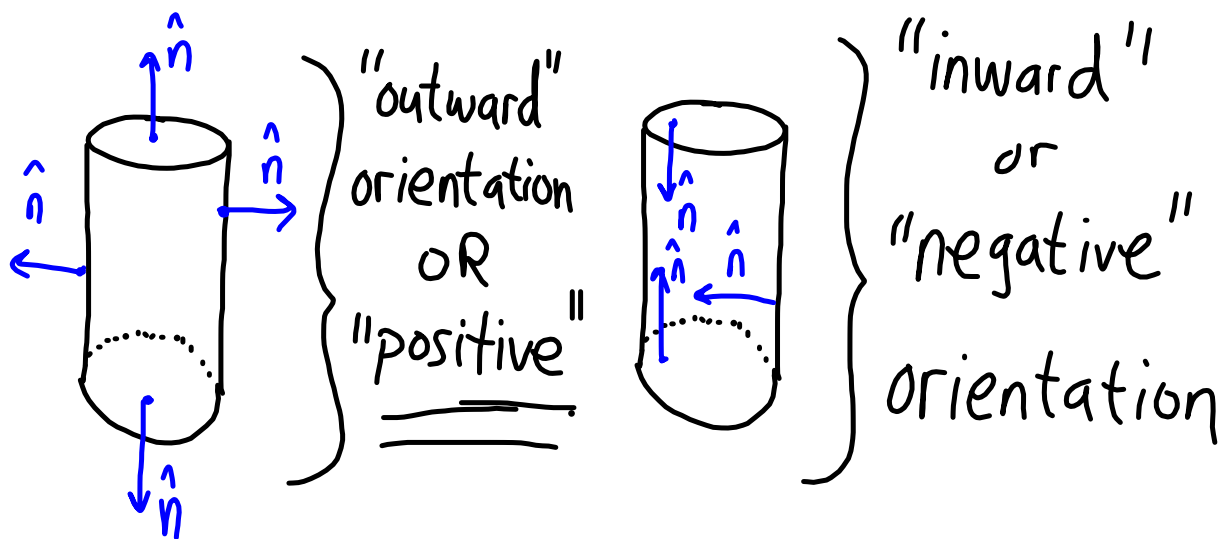
$$\hat{n} = \hat{n}(x, y, z)$$

We only consider surfaces with 2 distinct sides (no Möbius strips, etc.)

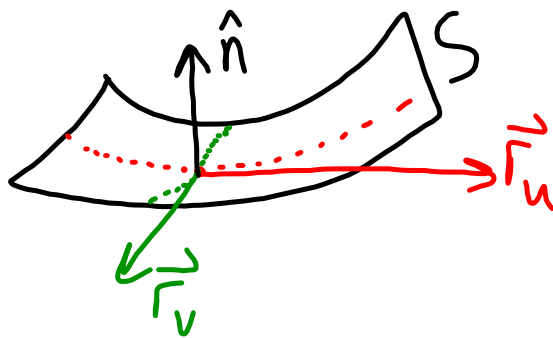
Thus, there are 2 possible "orientations".

More precisely, "upward" means the \vec{k} -component of \hat{n} is positive.

Closed surfaces (enclose a volume):



How to get \hat{n}



$$\vec{n} = \vec{r}_u \times \vec{r}_v$$

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

If you calculate \hat{n} this way and it points the "wrong" way then multiply by -1 !

$$-\hat{n} = \frac{-\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \frac{\vec{r}_v \times \vec{r}_u}{|\vec{r}_v \times \vec{r}_u|}$$

For the natural parameterization,

$$\vec{r}(x,y) = \langle x, y, z(x,y) \rangle \left\{ \begin{array}{l} \vec{r}_x = \langle 1, 0, z_x \rangle \\ \vec{r}_y = \langle 0, 1, z_y \rangle \end{array} \right.$$

$$\vec{r}_x \times \vec{r}_y = \langle -z_x, -z_y, 1 \rangle$$

$$\Rightarrow \hat{n} = \frac{1}{\sqrt{1+(z_x)^2+(z_y)^2}} \langle -z_x, -z_y, 1 \rangle$$

"upward"

Surface integrals for vector fields

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \hat{n}) dS$$

notation \downarrow scalar, so we know what this integral means

Parameterize S by $\vec{r}(u,v)$, domain D , then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \underbrace{\left(\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \right)}_{\hat{n}} \underbrace{|\vec{r}_u \times \vec{r}_v|}_{dS} du dv$$

$$\Rightarrow \int_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) du dv.$$

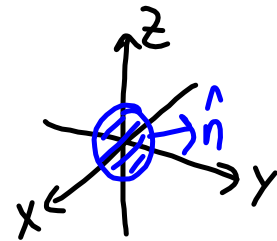
Summary of integrals:

Scalar f	{	$\int_c^b f ds = \int_a^b f(\vec{r}(t)) \vec{r}'(t) dt$		$\iint_S f dS = \iint_D f(\vec{r}(u,v)) \vec{r}_u \times \vec{r}_v du dv$
		LINE	-	SURFACE

Vector \vec{F}	{	$\int_c^b \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$		$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) du dv.$
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EX: Let S be the unit sphere with positive orientation. Find $\iint_S \vec{F} \cdot d\vec{S}$ if $\vec{F} = \langle 0, 0, 1 \rangle$.

We want \hat{n} pointing outward:



$$\left. \begin{aligned} x &= \cos\theta \sin\phi \\ y &= \sin\theta \sin\phi \\ z &= \cos\phi \end{aligned} \right\} \text{ We calculated previously}$$

$$\vec{r}_\theta \times \vec{r}_\phi = -\cos\theta \sin^2\phi \vec{i} - \sin\theta \sin^2\phi \vec{j} - \sin\phi \cos\phi \vec{k}.$$

We don't really need \hat{n} but will look at it for illustration... $|\vec{r}_\theta \times \vec{r}_\phi| = \sin\phi$

$$\hat{n} = \frac{\vec{r}_\theta \times \vec{r}_\phi}{|\vec{r}_\theta \times \vec{r}_\phi|} = \langle -\cos\theta \sin\phi, -\sin\theta \sin\phi, -\cos\phi \rangle$$

$= -\langle x, y, z \rangle$ } points "inward"

So we use $-\vec{r}_\theta \times \vec{r}_\phi = \vec{r}_\phi \times \vec{r}_\theta$ in the integral:

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^\pi \underbrace{\langle 0, 0, 1 \rangle \cdot (\vec{r}_\phi \times \vec{r}_\theta)}_{\sin\phi \cos\phi} d\phi d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \sin\phi \cos\phi d\phi d\theta = 2\pi \int_0^{\pi} \frac{1}{2} \frac{d}{d\phi} \sin^2\phi d\phi$$

$$= \frac{2\pi}{2} \sin^2\phi \Big|_0^{\pi} = \boxed{0}.$$

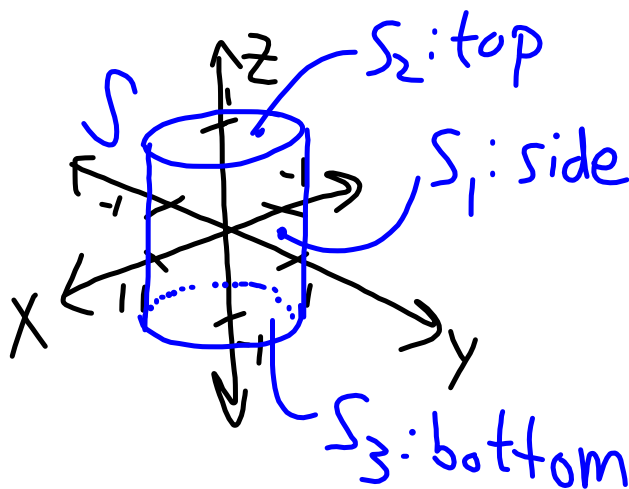
*Subtlety: note we did not stick in $\rho^2 \sin\phi$ here, but applied the surface integral formula.

Using a parameterization, not change of variables formula.

EX: Let S be the surface of the cylinder $\{(x, y, z) \mid x^2 + y^2 \leq 1, -1 \leq z \leq 1\}$, with positive orientation. Find

$$\iint_S \vec{F} \cdot d\vec{S} \text{ if } \vec{F} = \langle x, y, z \rangle.$$

$$S = S_1 \cup S_2 \cup S_3$$



$$\underline{\text{On } S_1 : \vec{r}(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle}$$
$$0 \leq \theta \leq 2\pi, -1 \leq z \leq 1$$

$$\begin{aligned} \vec{r}_\theta &= \langle -\sin\theta, \cos\theta, 0 \rangle \\ \vec{r}_z &= \langle 0, 0, 1 \rangle \end{aligned} \left\{ \begin{aligned} \vec{r}_\theta \times \vec{r}_z &= \langle \cos\theta, \sin\theta, 0 \rangle \\ &= \langle x, y, 0 \rangle, \text{ "outward" } \end{aligned} \right.$$

$$\vec{F}(\vec{r}(\theta, z)) = \langle \cos\theta, \sin\theta, z \rangle$$

$$\vec{F} \cdot (\vec{r}_\theta \times \vec{r}_z) = \cos^2\theta + \sin^2\theta = 1.$$

$$\text{Thus } \iint_{S_1} \vec{F} \cdot d\vec{S}_1 = \int_0^{2\pi} \int_{-1}^1 1 \, dz \, d\theta = \underline{4\pi}$$

On S_2 (top) : $\vec{r}(x,y) = \langle x, y, 1 \rangle$, $\underbrace{x^2 + y^2 \leq 1}_{\mathcal{D}}$

$$\left. \begin{array}{l} \vec{r}_x = \langle 1, 0, 0 \rangle \\ \vec{r}_y = \langle 0, 1, 0 \rangle \end{array} \right\} \vec{r}_x \times \vec{r}_y = \langle 0, 0, 1 \rangle$$

"outward"

$$\iint_{S_2} \vec{F} \cdot d\vec{S}_2 = \iint_{\mathcal{D}} \langle x, y, 1 \rangle \cdot \langle 0, 0, 1 \rangle \, dx \, dy = \iint_{\mathcal{D}} 1 \, dx \, dy = \overbrace{\pi}^{\text{Area of circle}} = \underline{\underline{\pi}}$$

$$\underline{\underline{\text{On } S_3: \vec{r}(x,y) = \langle x, y, -1 \rangle}}$$

$$\vec{r}_x \times \vec{r}_y = \langle 0, 0, 1 \rangle \left. \begin{array}{l} \text{points} \\ \text{wrong way} \end{array} \right\}$$

$$\Rightarrow \iint_{S_3} \vec{F} \cdot d\vec{S}_3 = \iint_{\mathcal{D}} \langle x, y, -1 \rangle \cdot \langle 0, 0, -1 \rangle dx dy$$

$$\mathcal{D} = \iint_{\mathcal{D}} 1 dx dy = \underline{\underline{\pi}}$$

$$\text{Sum over 3 surfaces: } \iint_S \vec{F} \cdot d\vec{S} = 4\pi + \pi + \pi = \boxed{6\pi}$$

Stoke's Theorem: relates the integral of the tangential part of a vector field over a curve C , which bounds a surface S , to the integral of the CURL of the field over the surface.

* Very important result.

* Requires the curve to have positive orientation with respect to the surface.



Theorem :
(Stoke's)

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

Notation: If C is a closed curve and is the boundary of S , a surface, then

$$C = \partial S \leftarrow \text{not a derivative}$$

If a volume V is bounded by surface S ,
 $S = \partial V$.

Stoke's again:

$$\int_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

or

$$\int_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

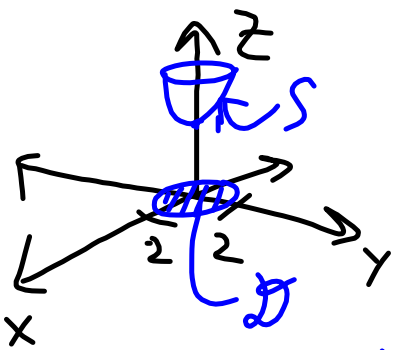
We will pick up here next time.

Practice!

(#1) Let $S = \left\{ (x, y, z) \mid \begin{array}{l} x^2 + y^2 \leq 4, \\ z = 1 + x^2 + y^2 \end{array} \right\},$

with upward orientation.

Find $\iint_S \vec{F} \cdot d\vec{S}$, if $\vec{F} = \langle -x, -y, 0 \rangle$.



$$\vec{r}(x,y) = \langle x, y, 1+x^2+y^2 \rangle$$

$$\vec{r}_x = \langle 1, 0, 2x \rangle$$

$$\vec{r}_y = \langle 0, 1, 2y \rangle$$

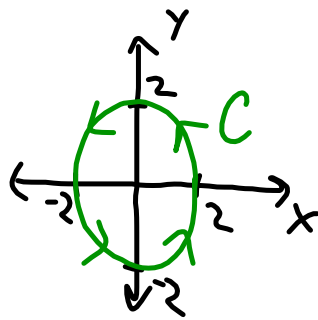
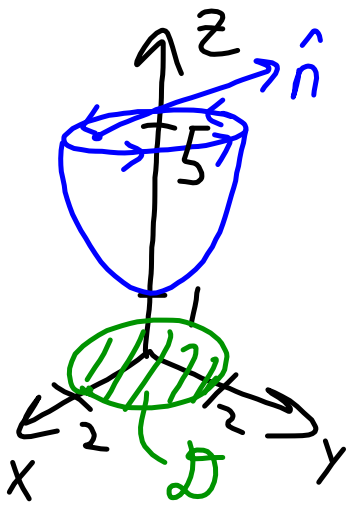
$$\vec{r}_x \times \vec{r}_y = \langle -2x, -2y, 1 \rangle$$

$$\vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = \langle -x, -y, 0 \rangle \cdot \langle -2x, -2y, 1 \rangle$$

$$= 2(x^2 + y^2).$$

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \iint_D 2(x^2 + y^2) dA \\ &= 2 \int_0^{2\pi} \int_0^2 r^2 r dr d\theta = 2 \cdot 2\pi \cdot \frac{1}{4} r^4 \Big|_0^2 \\ &= 2^4 \pi = \boxed{16\pi}\end{aligned}$$

(#2) Parameterize ∂S in the last problem with positive orientation.



$$\left. \begin{array}{l}
 z = 1 + x^2 + y^2 \\
 = 1 + r^2 = 1 + 4 = 5 \\
 x = r \cos \theta = 2 \cos \theta \\
 y = 2 \sin \theta \\
 z = 5
 \end{array} \right\}$$

$$\vec{r}(\theta) = \langle 2 \cos \theta, 2 \sin \theta, 5 \rangle.$$

$$0 \leq \theta \leq 2\pi.$$