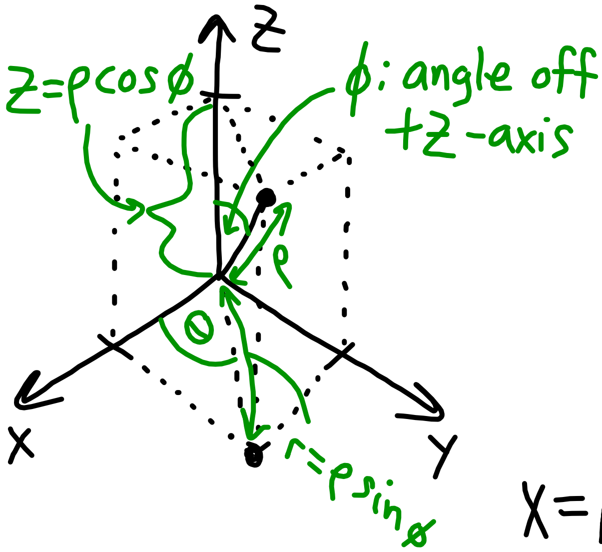
Spherical coordinates



ρ: distance to origin

θ: angle off +x-axis in the xy-plane

ϕ: angle off +z-axis

$0\le\rho,\ 0\le\theta\le2\pi,\ 0\le\phi\le\pi$

$$x=r\cos\theta=\rho\sin\phi\cos\theta$$
$$y=r\sin\theta=\rho\sin\phi\sin\theta$$
$$z=\rho\cos\phi.$$

Claim: $x^2+y^2+z^2=\rho^2$.

Proof: $x^2+y^2+z^2=\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta$
$+\rho^2\cos^2\phi$

$=\rho^2\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right)+\rho^2\cos^2\phi$

$=\rho^2\sin^2\phi+\rho^2\cos^2\phi$

$=\rho^2\left(\sin^2\phi+\cos^2\phi\right)=\rho^2$.

$\square$

EX: Given $\left(-\sqrt{3/2}, -\sqrt{3/2}, -1\right)$ in Cartesian coordinates, find the spherical coordinates.

$$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{\frac{3}{2}+\frac{3}{2}+1} = \sqrt{4} = \boxed{2 = \rho}$$

$$z = \rho\cos\phi = 2\cos\phi = -1 \Rightarrow \cos\phi = -\frac{1}{2} \Rightarrow \boxed{\phi = \frac{2\pi}{3}}.$$

$$x = \rho\cos\theta\sin\phi = 2\cos\theta\sin\left(\frac{2\pi}{3}\right) = \cos\theta\sqrt{3} = \frac{-\sqrt{3}}{\sqrt{2}}$$

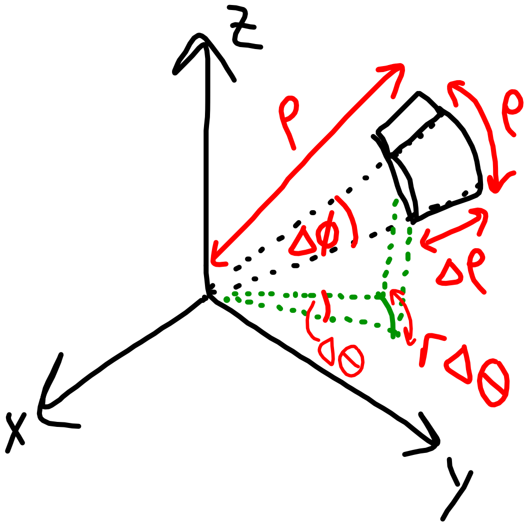
$$\text{so } \cos\theta = \frac{-1}{\sqrt{2}} \Rightarrow \theta = \frac{3\pi}{4} \text{ or } \theta = \frac{5\pi}{4}$$

$$y = -\frac{\sqrt{3}}{\sqrt{2}} = \rho \sin\phi \sin\theta = 2\,\frac{\sqrt{3}}{2}\,\sin\theta$$

$$\Rightarrow -\frac{1}{\sqrt{2}} = \sin\theta, \text{ thus } \boxed{\theta = \frac{5\pi}{4}}$$

$$\text{since } \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

Differential volume: think about a small change in each of the coordinates $\Delta\rho, \Delta\theta, \Delta\phi$



$$\Delta V = (\rho\Delta\phi)(\Delta\rho)(r\Delta\theta)$$
$$= (\rho\Delta\phi)(\Delta\rho)(\rho\sin\phi\,\Delta\theta)$$
$$= \rho^2 \sin\phi\,\Delta\rho\,\Delta\theta\,\Delta\phi$$

$$dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

Integration formula:

$$\iiint_V f(x,y,z)\,dV = \iiint_V f(\rho,\theta,\phi)\,\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi .$$

Ex: Derive the formula for the volume of a sphere of radius $\rho = r$.

$$|V| = \iiint_V dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{r} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\Rightarrow |V| = \int_0^{2\pi} \int_0^{\pi} \frac{1}{3}\rho^3 \sin\phi \Big|_{\rho=0}^{\rho=r} d\phi\, d\theta$$

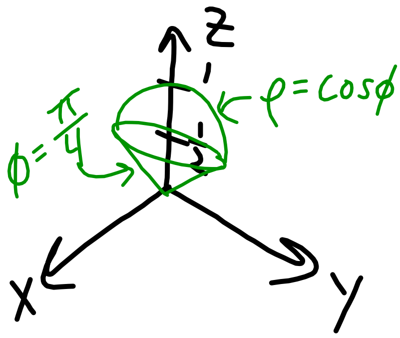
$$= \int_0^{2\pi} \int_0^{\pi} \frac{1}{3} r^3 \sin\phi\, d\phi\, d\theta$$

$$= \frac{1}{3} r^3 \int_0^{2\pi} \left(-\cos\phi\right)\Big|_0^{\pi} d\theta$$

$$= \frac{1}{3} r^3 \int_0^{2\pi} 2\, d\theta = \boxed{\frac{4\pi r^3}{3}}.$$

EX: Find the volume of a solid that lies above the cone $z=\sqrt{x^2+y^2}$ & below $x^2+y^2+(z-\frac{1}{2})^2=\frac{1}{4}$.

radius $\frac{1}{2}$



Think "spherical"

Cone: $z=\sqrt{\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)}$

$=\rho\sin\phi=\rho\cos\phi$

$\Rightarrow \sin\phi=\cos\phi \Rightarrow \underbrace{\phi=\frac{\pi}{4}}_{\text{"cone"}}$

Upper boundary...

$x^2+y^2+z^2-z+\frac{1}{4}=\frac{1}{4} \Rightarrow \rho^2=z=\rho\cos\phi$

$\Rightarrow \rho=\cos\phi$

Limits of integration: $0 \leq \theta \leq 2\pi$

$0 \leq \rho \leq \cos\phi$

$0 \leq \phi \leq \pi/4$

$$|V| = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}\cos^3\phi \sin\phi \, d\phi \, d\theta$$

$$\int_0^{2\pi}\int_0^{\pi/4} -\frac{1}{12}\frac{\partial}{\partial\phi}\cos^4\phi \, d\phi \, d\theta = \int_0^{2\pi} -\frac{1}{12}\left[\left(\frac{1}{\sqrt{2}}\right)^4 - 1\right] d\theta$$

$$\Rightarrow |V| = 2\pi\left(-\frac{1}{12}\right)\left(-\frac{3}{4}\right) = \frac{3\pi}{24} = \boxed{\frac{\pi}{8}}.$$

Change of variables

Recall $\int_a^b f(x)\,dx$ is sometimes evaluated

(Chain Rule)

via a change $x = g(u) \Rightarrow dx = g'(u)\,du$

If g^{-1} exists,

$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))\,g'(u)\,du.$$

Extend to multiple variables?

Transformations

We want some $T: \mathbb{R}^2 \to \mathbb{R}^2$,

$T(u,v) = (x,y)$

$$\left.\begin{aligned} x &= g(u,v) \\ y &= h(u,v) \end{aligned}\right\} \text{we often write} \quad \begin{aligned} x &= x(u,v) \\ y &= y(u,v). \end{aligned}$$

T : "transformation"

We can talk about the partial derivatives

$$\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$$

When these are all CTS, we say

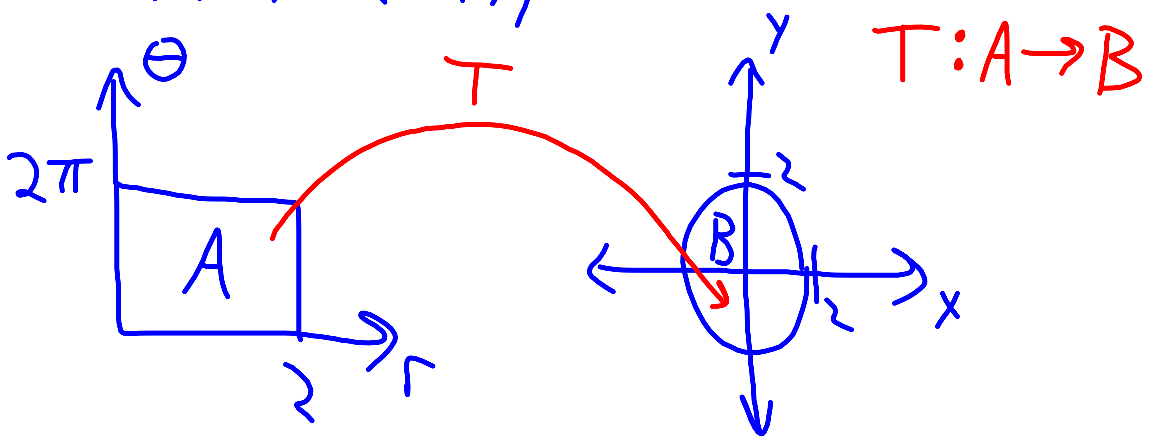
"T is a C^1-transformation".

We will work with C^1-transformations.

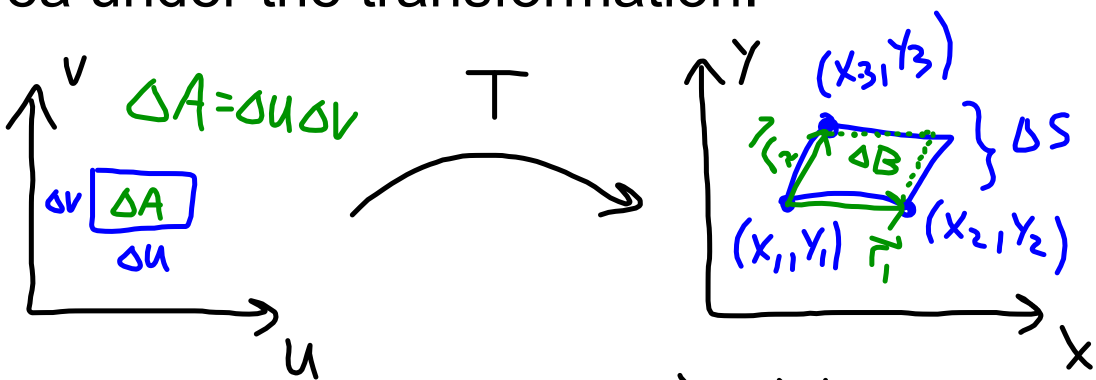
EX: Find a transformation T from a rectangle to $B = \{(x,y) \mid x^2+y^2 \le 4\}$.

Take $x = r\cos\theta$, $y = r\sin\theta$

$T(r,\theta) = (x,y)$



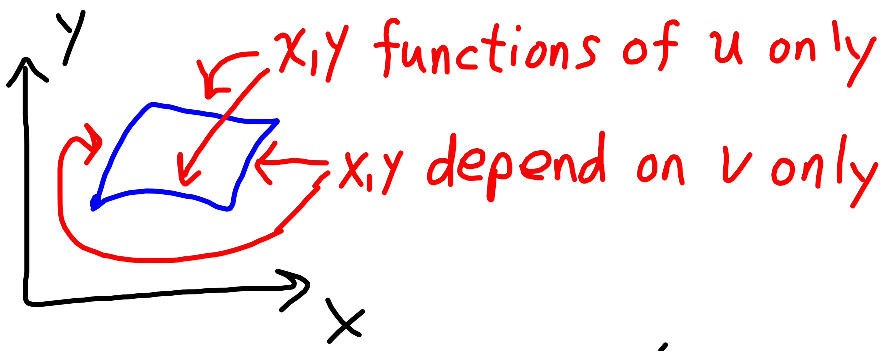
Let's sketch out a derivation for the differential of area under the transformation.



The region $\Delta S \approx \Delta B = |\vec{r}_1 \times \vec{r}_2|$

$$\vec{r}_1 = \langle x_2 - x_1, y_2 - y_1 \rangle$$

$$\vec{r}_2 = \langle x_3 - x_1, y_3 - y_1 \rangle$$



So, e.g., $x_2 - x_1 \approx \frac{\partial x}{\partial u} \cdot \Delta u$ (Mean Value Thm.)

$$y_2 - y_1 \approx \frac{\partial y}{\partial u} \cdot \Delta u$$

$$x_3 - x_1 \approx \frac{\partial x}{\partial v} \cdot \Delta v$$

$$y_3 - y_1 \approx \frac{\partial y}{\partial v} \cdot \Delta v$$

$$\Rightarrow \vec{r}_1 = \Delta u \langle x_u, y_u \rangle$$

$$\vec{r}_2 = \Delta v \langle x_v, y_v \rangle$$

$$\vec{r}_1 \times \vec{r}_2 = \Delta u \Delta v \underbrace{\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}}_{\text{2D-cross product}}$$

$*$ Notation: $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = x_u y_v - x_v y_u$

is called the "Jacobian of T"

Thus, $\Delta S \approx \Delta B = |\vec{r}_1 \times \vec{r}_2| = \Delta u \Delta v \left| \dfrac{\partial(x,y)}{\partial(u,v)} \right|$

$$\Rightarrow \Delta B = \Delta A \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

$(\Delta A = \Delta u \cdot \Delta v)$

$$\Rightarrow dx\,dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$\boxed{\iint_S f(x,y)\,dx\,dy = \iint_A f\big(x(u,v), y(u,v)\big) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.}$$

$S \xleftarrow{T : A \to S} A$

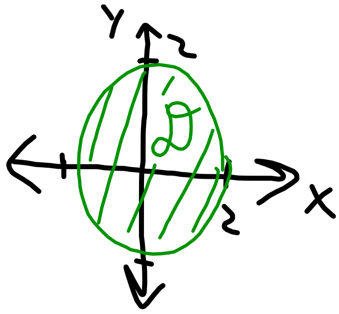
"Change of variables formula"

As usual, there are restrictions. Some conditions to look out for:

* f is CTS.
* T is C'
* Jacobian $\neq 0$

Two uses for a change of variables: 1) work with an easier domain of integration and 2) simplify the complexity of the integrand.

EX: Find $\iint_D x^2+y^2\,dA$,



Polar: $x = r\cos\theta$
$y = r\sin\theta$

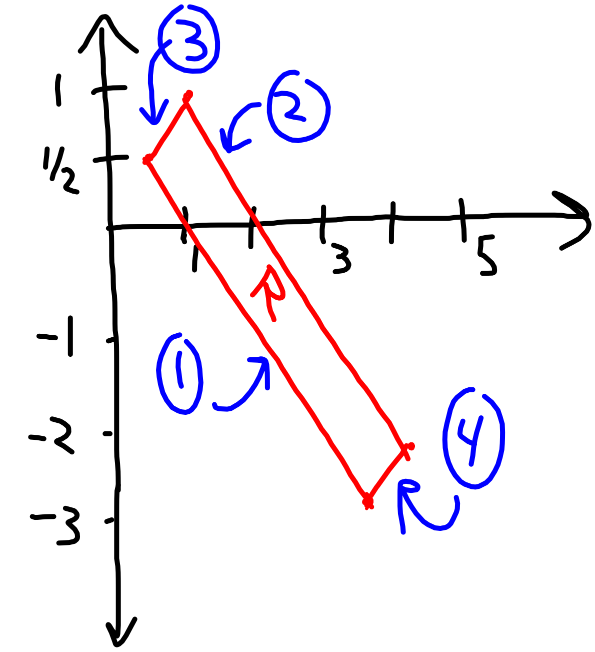
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

$$\Rightarrow \iint_D x^2+y^2\,dx\,dy = \int_0^{2\pi}\int_0^2 r^2 \cdot r\,dr\,d\theta = 2\pi \cdot \frac{1}{4} r^4 \Big|_0^2 = \boxed{8\pi.}$$

EX: Let R be the rectangle in the xy-plane with corners $\left(\frac{1}{2},\frac{1}{2}\right)$, $(1,1)$, $\left(\frac{1}{2}+\pi,\frac{1}{2}-\pi\right)$, $(1+\pi,1-\pi)$. Find $\displaystyle\iint_R \frac{x-y}{\underbrace{x+y}_{}}\,dA$.

$x+y\neq 0$ over R



Idea: $u = x-y$
$v = x+y$

1 $y = 1-x$ 2 $y = 2-x$

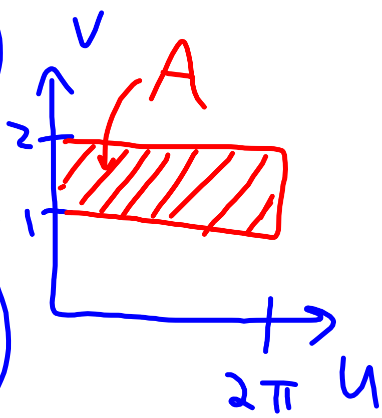
3 $y = x$ 4 $y = x-2\pi$

Note ① $\Rightarrow$ $x+y=1=v$

② $\Rightarrow$ $x+y=2=v$

③ $\Rightarrow$ $x-y=0=u$

④ $\Rightarrow$ $x-y=2\pi=u$



$T(u,v)=(x,y) \Rightarrow T: A \to R.$

Need $x=x(u,v)$ $y=y(u,v)$ ... "invert" the equations

$$\begin{aligned} u&=x-y \\ v&=x+y \end{aligned} \Rightarrow \begin{aligned} u-v&=-2y \\ u+v&=2x \end{aligned} \Rightarrow \begin{aligned} x&=\frac{u+v}{2} \\ y&=\frac{v-u}{2} \end{aligned}$$

Jacobian: $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

(Note $\dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$, T is C'...)

$$\iint_R \frac{x-y}{x+y}\, dA = \int_1^2 \int_0^{2\pi} \frac{u}{v} \cdot \frac{1}{2} \cdot du\, dv = \frac{1}{2} \int_1^2 \frac{1}{2} \frac{u^2}{v} \Big|_{u=0}^{u=2\pi} dv$$

$$= \frac{1}{4} \int_1^2 \frac{1}{v} (4\pi^2)\, dv = \boxed{\pi^2 \ln(2).}$$

Change of variables for triple integrals

Let $T: S \to R$ in $\mathbb{R}^3$; $T(u,v,w) = (x,y,z)$.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(u,v,w) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw.$$

Spherical coordinates...

$$x=\rho\cos\theta\sin\phi \quad y=\rho\sin\theta\sin\phi \quad z=\rho\cos\phi$$

$$x_\rho=\cos\theta\sin\phi \quad x_\theta=-\rho\sin\theta\sin\phi \quad x_\phi=\rho\cos\theta\cos\phi$$

$$y_\rho=\sin\theta\sin\phi \quad y_\theta=\rho\cos\theta\sin\phi \quad y_\phi=\rho\sin\theta\cos\phi$$

$$z_\rho=\cos\phi \quad z_\theta=0 \quad z_\phi=-\rho\sin\phi$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=\cos\theta\sin\phi\begin{vmatrix}\rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi\\ 0 & -\rho\sin\phi\end{vmatrix}$$

$$+\rho\sin\theta\sin\phi\begin{vmatrix}\sin\theta\sin\phi & \rho\sin\theta\cos\phi\\ \cos\phi & -\rho\sin\phi\end{vmatrix}+\rho\cos\theta\cos\phi\begin{vmatrix}\sin\theta\sin\phi & \rho\cos\theta\sin\phi\\ \cos\phi & 0\end{vmatrix}$$

$$
\begin{aligned}
&= \cos\theta \sin\phi\left(-\rho^2 \cos\theta \sin^2\phi\right) \\
&\quad + \rho \sin\theta \sin\phi\left(-\rho \sin\theta \sin^2\phi - \rho \sin\theta \cos^2\phi\right) \\
&\quad + \rho \cos\theta \cos\phi\left(-\rho \cos\theta \sin\phi \cos\phi\right) \\
&= -\rho^2 \sin^3\phi \cos^2\theta - \rho^2 \sin^2\theta \sin\phi \\
&\quad - \rho^2 \cos^2\theta \sin\phi \cos^2\phi \\
&= -\rho^2 \sin\phi\left[\sin^2\phi \cos^2\theta + \sin^2\theta + \cos^2\theta \cos^2\phi\right]
\end{aligned}
$$

$$= -\rho^2 \sin\phi (\cos^2\theta + \sin^2\theta)$$

$$= -\rho^2 \sin\phi$$

Thus $\left| \dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} \right| = \left| -\rho^2 \sin\phi \right| = \rho^2 \sin\phi$

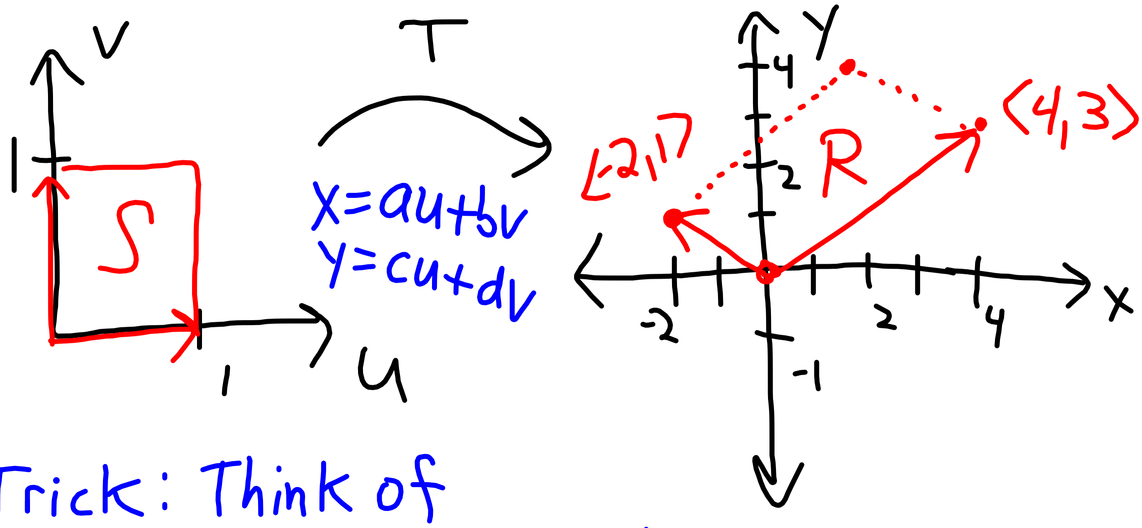
$(\sin\phi > 0)$

$$\Rightarrow \iiint_V f(x,y,z)\,dV = \iiint_V f(\rho,\theta,\phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi .$$

Practice!

#1 Let R be a parallelogram with vertices $(0,0)$, $(4,3)$, $(2,4)$, $(-2,1)$.

Find a transformation $T: S \to R$ where S is a rectangular region in the uv-plane.



Trick: Think of mapping $\langle 1,0\rangle \xrightarrow[T]{} \langle 4,3\rangle$ & $\langle 0,1\rangle \xrightarrow[T]{} \langle -2,1\rangle$.

(Idea from linear algebra.)

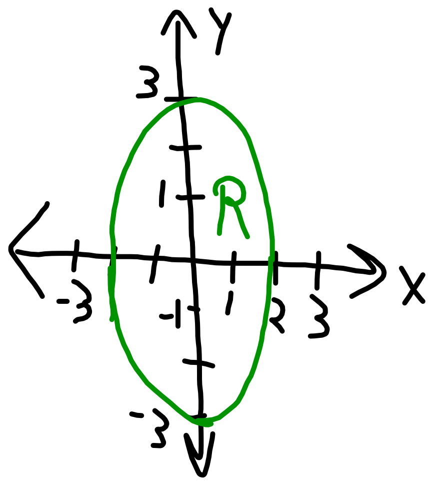
$$T(1,0) = (a(1)+b(0),\ c(1)+d(0))$$
$$= (a,c) = (4,3)$$
$$T(0,1) = (a(0)+b(1),\ c(0)+d(1))$$
$$= (b,d) = (-2,1).$$

Thus $T(u,v) = (4u-2v,\ 3u+v)$.

*Check: $T(0,0)=(0,0)$, $T(1,0)=(4,3)$, $T(0,1)=(-2,1)$

$T(1,1)=(2,4)$. ✓

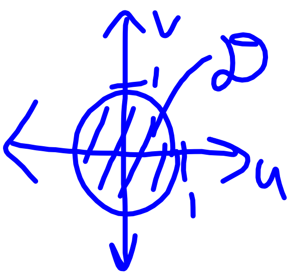
#2 Find $\iint_R x^2\,dA$... R bounded by the ellipse $9x^2+4y^2=36$. Use $\begin{cases} x=2u \\ y=3v \end{cases}$.



Plug $x=2u$, $y=3v$ into ellipse-eq'n...

$$9(2u)^2+4(3v)^2=36 \Rightarrow 36u^2+36v^2=36$$

$$\Rightarrow u^2+v^2=1 \quad \text{(unit circle).}$$



$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 2\cdot 3 = 6 \neq 0$$

$$\iint_R x^2\,dA = \iint_D 4u^2\cdot 6\cdot du\,dv = 24\iint_D u^2\,du\,dv \leftarrow \text{polar}$$

$u = r\cos\theta$

$v = r\sin\theta$

$$\Rightarrow \iint_R x^2\,dA = 24\int_0^{2\pi}\int_0^{1} r^2\cos^2\theta\, r\,dr\,d\theta$$

$$= 24\int_0^{2\pi} \cos^2\theta \, \frac{1}{4} r^4\Big|_0^1 \, d\theta = \frac{24}{4}\underbrace{\int_0^{2\pi}\cos^2\theta\,d\theta}_{=\pi}$$

$$= \boxed{6\pi}$$

Trick to remember $\int_0^{2\pi} \cos^2\theta \, d\theta = \pi$:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\int_0^{2\pi} \sin^2\theta \, d\theta + \int_0^{2\pi} \cos^2\theta \, d\theta = \int_0^{2\pi} 1 \, d\theta = 2\pi$$

same

area under curves

$$\Rightarrow 2\int_0^{2\pi} \cos^2\theta \, d\theta = 2\pi \Rightarrow \int_0^{2\pi} \cos^2\theta \, d\theta = \pi.$$

$$= \int_0^{2\pi} \sin^2\theta \, d\theta$$