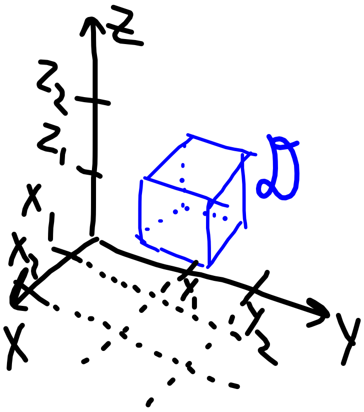
Triple integrals

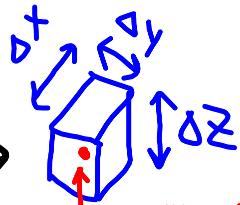
$f = f(x,y,z)$ now has a 3-D domain.

Consider a box domain $\mathcal{D}$,

$$\mathcal{D} = \{(x,y,z) \mid x_1 \le x \le x_2,\ y_1 \le y \le y_2,\ z_1 \le z \le z_2\}.$$



Look at a small volume $\Delta x \Delta y \Delta z = \Delta V$ inside $\mathcal{D}$ →



(x_i^*, y_j^*, z_k^*)

SAMPLE POINT IN BOX (i,j,k)

Form a Riemann sum over the boxes

$$\sum_{i,j,k=1}^{N} f(x_i^*, y_j^*, z_k^*)\, \Delta x \cdot \Delta y \cdot \Delta z$$

Now shrink the boxes and consider the limit as the number of boxes goes to ∞:

$$\iiint_{D} f(x,y,z)\, dx\, dy\, dz = \lim_{N \to \infty} \Delta x \Delta y \Delta z \sum_{i,j,k=1}^{N} f(x_i^*, y_j^*, z_k^*).$$

Calculation of triple integrals: Fubini's applies for continuous integrands.

<u>EX</u>: Let $\mathcal{D} = \{(x, y, z) \mid 0 \leq x \leq 3, 0 \leq y \leq 2, 0 \leq z \leq 1\}$.

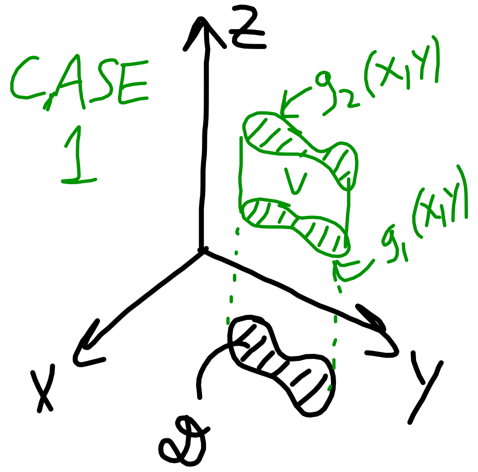
Find $\iiint_{\mathcal{D}} xyz\, dV$.

$$= \int_0^1 \int_0^2 \int_0^3 xyz\, dx\, dy\, dz = \int_0^1 \int_0^2 \left[\frac{1}{2} x^2 yz \right]_{x=0}^{x=3} dy\, dz$$

$$= \int_0^1 \int_0^2 \frac{9}{2} yz\, dy\, dz = \int_0^1 \frac{9}{4} y^2 z \Big|_{y=0}^{y=2} dz$$

$$= \int_0^1 9z\,dz = \frac{9}{2} z^2 \Big|_0^1 = \boxed{\frac{9}{2}}.$$

Integration over domains bounded between surfaces; there are three cases.



- Domain of f is $V \subset \mathbb{R}^3$
- Domain of g_1, g_2 is $\mathcal{D} \subset \mathbb{R}^2$

$$\iiint_V f\,dV = \iint_{\mathcal{D}} \left[\int_{g_1(x,y)}^{g_2(x,y)} f\,dz \right] dx\,dy.$$

EX: Let V be the volume below $z=1+x^2+y^2$ above $\mathcal{D}=\{(x,y)\mid 0\le x\le 1,\ 1\le y\le 2\}$ in the xy-plane. Find $\iiint_V xy\,dV$.

$$=\int_0^1\int_1^2\int_0^{1+x^2+y^2} xy\,dz\,dy\,dx=\int_0^1\int_1^2 xyz\Big|_{z=0}^{z=1+x^2+y^2} dy\,dx$$

$$=\int_0^1\int_1^2 xy(1+x^2+y^2)\,dy\,dx=\int_0^1\int_1^2 xy+x^3y+xy^3\,dy\,dx$$

$$=\int_0^1\left[\frac{1}{2}xy^2+\frac{1}{2}x^3y^2+\frac{1}{4}xy^4\right]_{y=1}^{y=2}dx$$

$$= \int_0^1 \left(\frac{3}{2}x + \frac{3}{2}x^3 + \frac{15}{4}x\right)dx$$

$$= \left[\frac{3}{4}x^2 + \frac{3}{8}x^4 + \frac{15}{8}x^2\right]_0^1$$

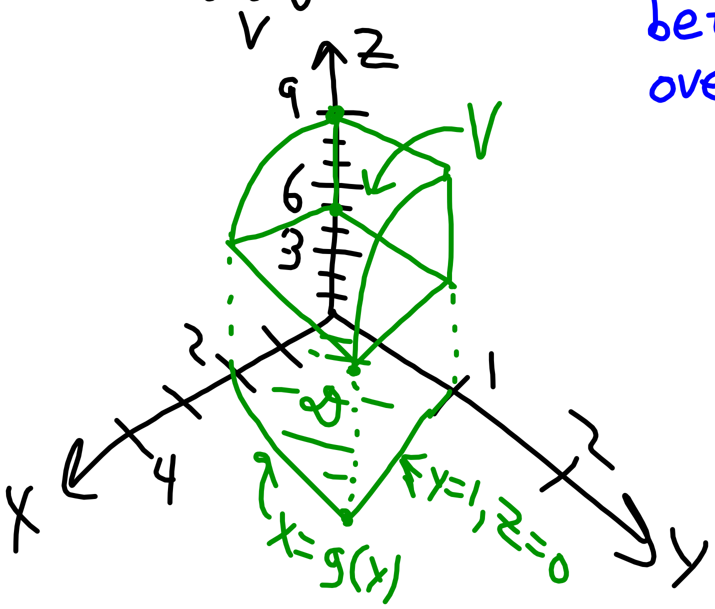
$$= \frac{3}{4} + \frac{3}{8} + \frac{15}{8}$$

$$= \frac{6+3+15}{8} = \frac{24}{8} = \boxed{3}.$$

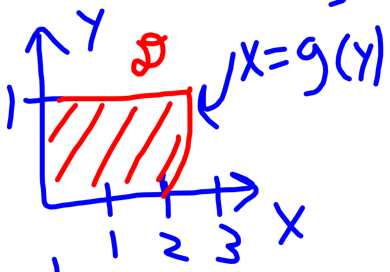
EX: Let V be the region bounded by

$x=0,\ y=0,\ y=1,\ z=5-y,\ z=9-x^2.$

Find $\iiint_V 2\,dV.$



Look at V as bounded between two surfaces over $\mathcal{D}$:



$g(y)$ is where

$5-y=z=9-x^2$

$\Rightarrow x^2=y+9-5=y+4$

$g(y)=\sqrt{y+4}$

$$= 2\int_0^1 \int_0^{\sqrt{y+4}} \int_{5-y}^{9-x^2} dz\,dx\,dy$$

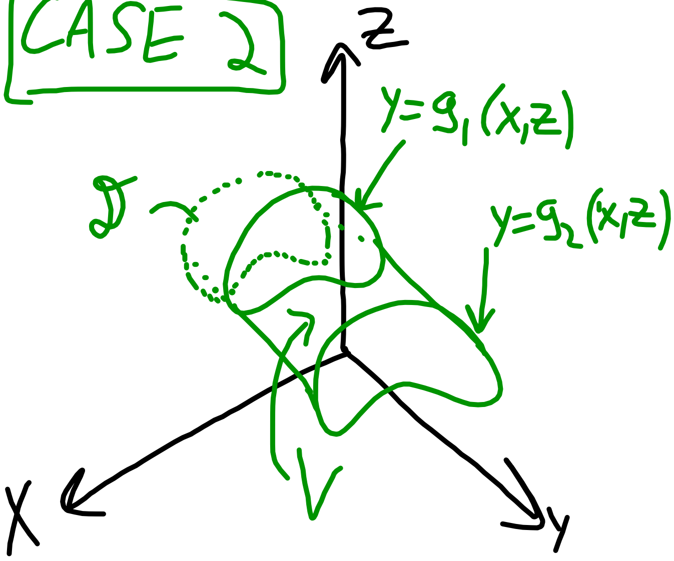
$$= 2\int_0^1 \int_0^{\sqrt{y+4}} 9 - x^2 - 5 + y \, dx\,dy = 2\int_0^1 \int_0^{\sqrt{y+4}} 4 - x^2 + y \, dx\,dy$$

$$= 2\int_0^1 \left[4x - \frac{1}{3}x^3 + xy\right]_{x=0}^{x=\sqrt{y+4}} dy = 2\int_0^1 4\sqrt{y+4} - \frac{1}{3}(y+4)^{3/2} + y\sqrt{y+4}\, dy$$

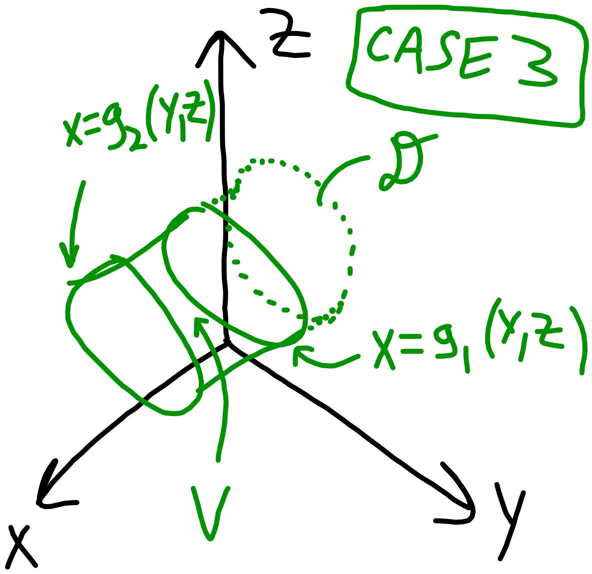
$$= 2\int_0^1 (y+4)\sqrt{y+4} - \frac{1}{3}(y+4)^{3/2} dy = 2\int_0^1 \frac{2}{3}(y+4)^{3/2} dy$$

$$= \frac{4}{3} \cdot \frac{2}{5} \cdot (y+4)^{5/2} \Big|_0^1 = \boxed{\frac{8}{15}\left(5^{5/2} - \underbrace{4^{5/2}}_{2^5 = 32}\right)}$$

CASE 2

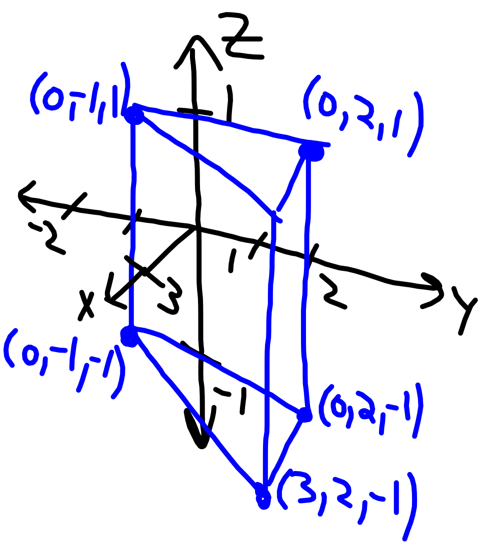


CASE 3

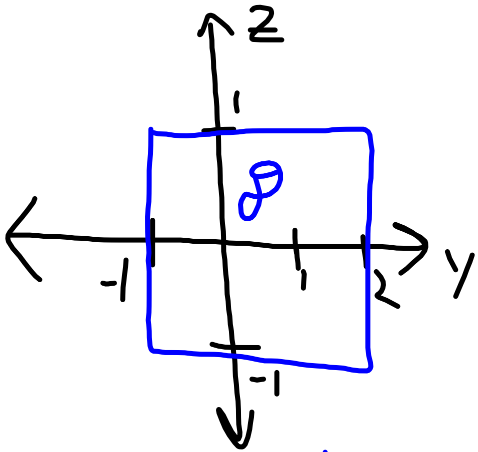


<u>EX</u>: Find $\iiint\limits_V x+y+z\,dV$, where V is the region bounded by $x=0$, $x=1+y$, $y=2$, $z=-1$, $z=1$.

Sketch V...



Let $\mathcal{D} \subset$ yz-plane...



V between $x=0$ & $x=1+y$...

$$\int_{-1}^{2}\int_{-1}^{1}\int_{0}^{1+y}(x+y+z)\,dx\,dz\,dy=\int_{-1}^{2}\int_{-1}^{1}\left[\frac{1}{2}x^2+x(y+z)\right]_{0}^{1+y}dz\,dy$$

$$=\int_{-1}^{2}\int_{-1}^{1}\frac{1}{2}(1+y)^2+(1+y)(y+z)\,dz\,dy$$

$$=\int_{-1}^{2}\left[\frac{1}{2}(1+y)^2z+\frac{1}{2}(1+y)(y+z)^2\right]_{z=-1}^{z=1}dy$$

$$=\int_{-1}^{2}(1+y)^2+\frac{1}{2}(1+y)^3-\frac{1}{2}(1+y)(y-1)^2\,dy$$

$$= \int_{-1}^{2} (1+y)^2 + \frac{1}{2}(1+y)\underbrace{\left[(1+y)^2 - (1-y)^2\right]}\,dy$$

$$a^2 - b^2 = (a-b)(a+b)$$

$$(1+y)^2 - (1-y)^2 = (1+y-1+y)(1+y+1-y)$$
$$= 4y$$

$$= \int_{-1}^{2} (1+y)^2 + 2y(1+y)\,dy = \int_{-1}^{2} (1+y)^2 + 2y + 2y^2\,dy$$

$$= \left[\frac{1}{3}(1+y)^3 + y^2 + \frac{2}{3}y^3\right]_{-1}^{2} = 9 + 4 + \frac{16}{3} - 1 + \frac{2}{3}$$

$$= \boxed{18}$$

Center of mass in 3D

$\rho = \rho(x,y,z)$: density in space

$$m = \iiint_V \rho \, dV \quad : \text{total mass in } V$$

Moments: $M_{xy} = \iiint_V z\rho \, dV$ – about xy-plane

$M_{xz} = \iiint_V y\rho \, dV$ – about xz-plane

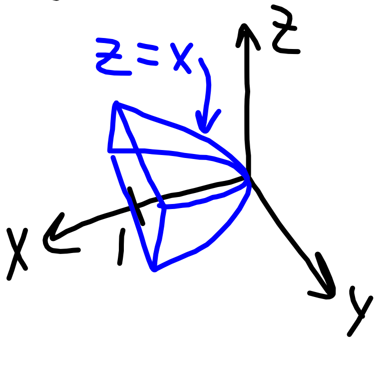
$M_{yz} = \iiint_V x\rho \, dV$ – about yz-plane center of mass is $(\bar{x}, \bar{y}, \bar{z})$

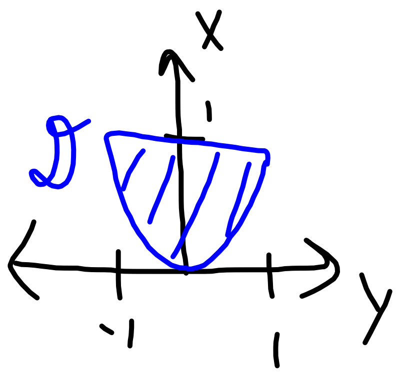
$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}$$

<u>EX</u>: Let $\rho \equiv$ constant on V,

$$V = \{(x, y, z) \mid -1 \le y \le 1,\ y^2 \le x \le 1,\ 0 \le z \le x\}.$$

Find the center of mass.





Mass: $m = \iiint_V \rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} \rho \, dz \, dx \, dy$

$= \rho \int_{-1}^{1} \int_{y^2}^{1} x \, dx \, dy = \frac{1}{2} \rho \int_{-1}^{1} x^2 \Big|_{y^2}^{1} dy = \frac{\rho}{2} \int_{-1}^{1} 1 - y^4 \, dy$

$\Rightarrow m = \frac{\rho}{2} \left(y - \frac{1}{5} y^5 \right) \Big|_{-1}^{1} = \frac{\rho}{2} \left(2 - \frac{2}{5} \right) = \underbrace{\boxed{\frac{4\rho}{5}}}_{m}$

Moments:

$M_{xz} = 0$ (ρ constant, V symmectric in y, so $\bar{y} = 0 \Rightarrow M_{xz} = 0$).

$$M_{xy} = \rho \int_{-1}^{1} \int_{y^2}^{1} \int_{0}^{x} z \, dz \, dx \, dy$$

$$= \rho \int_{-1}^{1} \int_{y^2}^{1} \tfrac{1}{2} x^2 \, dx \, dy = \frac{\rho}{6} \int_{-1}^{1} x^3 \Big|_{y^2}^{1} dy$$

$$= \frac{\rho}{6} \int_{-1}^{1} (1 - y^6) \, dy = \frac{\rho}{6} \left(y - \tfrac{1}{7} y^7 \right) \Big|_{-1}^{1}$$

$$= \frac{\rho}{6} \left(2 - {}^{2}/_{7} \right) = \boxed{\frac{2\rho}{7}} .$$

M_{xy}

$$M_{yz} = \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} x\rho \, dz\,dx\,dy = \frac{4\rho}{7}.$$

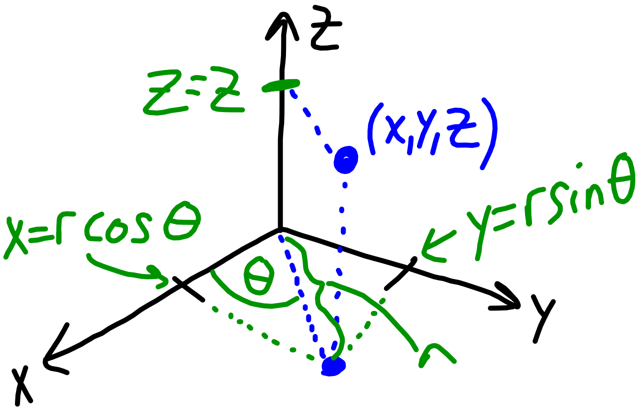
(Exercise)

$$\bar{x} = \frac{M_{yz}}{m} = \frac{4\rho}{7}\cdot\frac{5}{4\rho} = \frac{5}{7}$$

$$\bar{y} = 0$$

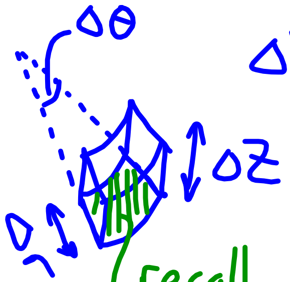
$$\bar{z} = \frac{M_{xy}}{m} = \frac{2\rho}{7}\cdot\frac{5}{4\rho} = \frac{5}{14}.$$

Cylindrical coordinates



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$
$$x^2 + y^2 = r^2$$
$$\tan\theta = y/x$$

Differential volume?



recall $\Delta A = r\Delta r\Delta\theta$

$\Delta V = r\Delta r\Delta\theta\Delta z$

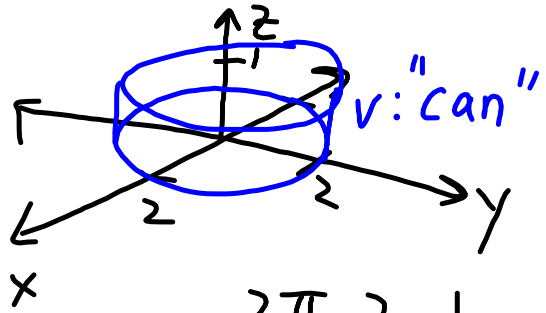
"$dV = r\,dr\,d\theta\,dz$"

Integration formula:

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V f(r,\theta,z)\,r\,dr\,d\theta\,dz.$$

<u>EX</u>: Note that volume of V, $|V|$, is $|V| = \iiint_V 1\,dV$. Find the volume of

$$V = \left\{(x,y,z) \,\middle|\, x^2+y^2 \le 4,\ 0 \le z \le 1\right\}.$$

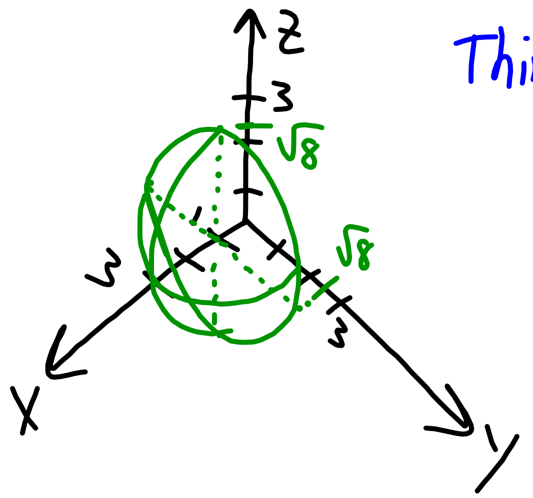


$$0 \le \theta \le 2\pi$$
$$0 \le r \le 2$$
$$0 \le z \le 1$$

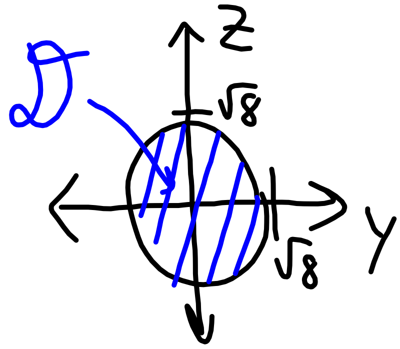
$$|V| = \int_0^{2\pi}\int_0^{2}\int_0^{1} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2}r^2\Big|_0^2\,d\theta$$
$$= 2\cdot\int_0^{2\pi} d\theta = 4\pi.$$

*Check: $V = \pi r^2 h = \pi(2)^2(1) = 4\pi.$

EX: Find the volume of the spherical cap between $x^2+y^2+z^2=9$ and $x=1$ (so $1 \le x$).



Think of V between $x=1$, $x=\sqrt{9-y^2-z^2}$



$y = r\cos\theta \qquad x = x$

$z = r\sin\theta \qquad 0 \le r \le \sqrt{8}$

$0 \le \theta \le 2\pi$

$$|V| = \int_0^{2\pi}\int_0^{\sqrt{8}}\int_1^{x=\sqrt{9-y^2-z^2}} r\,dx\,dr\,d\theta$$

$$y^2+z^2=r^2 \Rightarrow |V| = \int_0^{2\pi}\int_0^{\sqrt{8}}\int_1^{\sqrt{9-r^2}} r\,dx\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{8}} r\sqrt{9-r^2} - r\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{8}} r\sqrt{9-r^2}\,dr\,d\theta$$

$u = 9-r^2$

$du = -2r\,dr$

$-\frac{1}{2}du = r\,dr$

$$-\int_0^{2\pi}\int_0^{\sqrt{8}} r\,dr\,d\theta$$

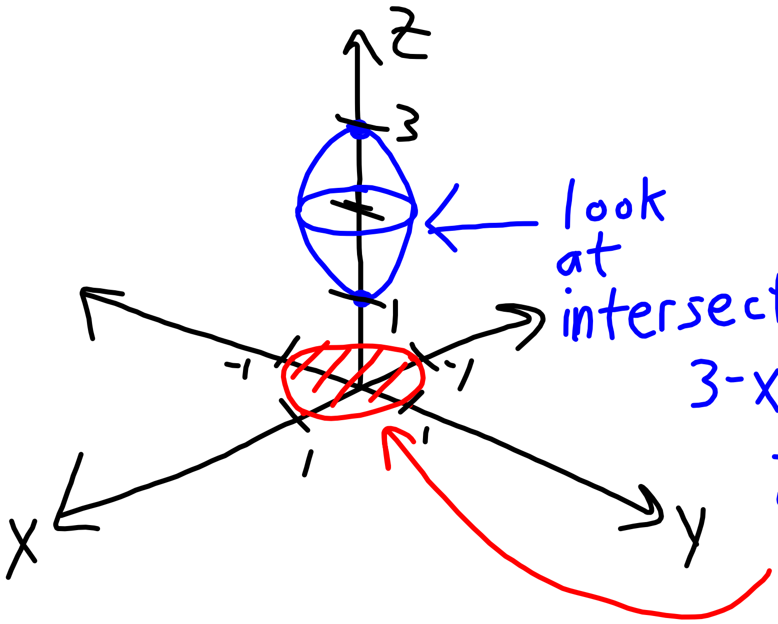
$$|V| = \int_0^{2\pi}\int_{u=9}^{u=1} -\tfrac{1}{2}\sqrt{u}\,du\,d\theta - \int_0^{2\pi} \tfrac{1}{2}r^2\Big|_0^{\sqrt{8}} d\theta$$

$$= \int_0^{2\pi}\int_1^9 \tfrac{1}{2}\sqrt{u}\,du\,d\theta - 4\cdot 2\pi$$

$$= \int_0^{2\pi}\left[\tfrac{1}{3}u^{3/2}\right]_1^9 d\theta - 8\pi = \frac{2\pi}{3}\left(3^3-1\right) - 8\pi$$

$$= \frac{(52-24)\pi}{3} = \boxed{\frac{28\pi}{3}}$$

Practice Problem : Find the volume of the region V bounded by $z = 3 - x^2 - y^2$ above and $z = 1 + x^2 + y^2$ below.



look at intersection

$$3 - x^2 - y^2 = z = 1 + x^2 + y^2$$

$$2 = 2(x^2 + y^2)$$

$$1 = x^2 + y^2$$

The volume is bounded between two surfaces over the domain $x^2+y^2 \le 1$.

Cylindrical: $z = 3 - x^2 - y^2 = 3 - r^2$

$z = 1 + x^2 + y^2 = 1 + r^2$

$$\Rightarrow V = \int_0^{2\pi}\int_0^1\int_{1+r^2}^{3-r^2} r \, dz\, dr\, d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^1 r(3-r^2-1-r^2)\, dr\right)$$

$$\Rightarrow V = 2\pi \int_0^1 2r - 2r^3 \, dr = 2\pi\left[r^2 - \frac{1}{2}r^4\right]_0^1 = \frac{2\pi}{2} = \boxed{\pi}.$$