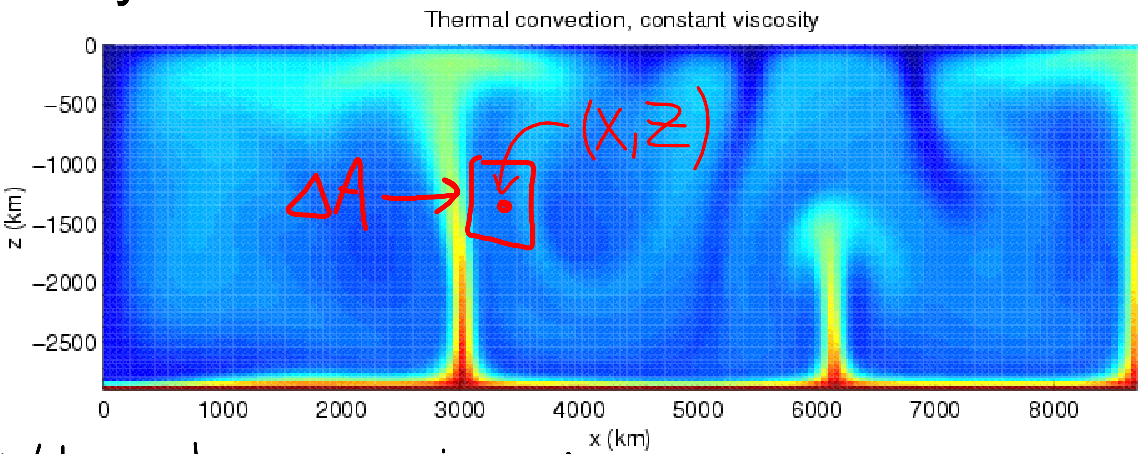
Density



E : thermal energy in region ΔA

$\Rightarrow$ Thermal energy density is $E/\Delta A$ "energy per unit area"

$$\lim_{\Delta A \to 0} \frac{E}{\Delta A} = \rho(x, z) \text{ : "local thermal energy density".}$$

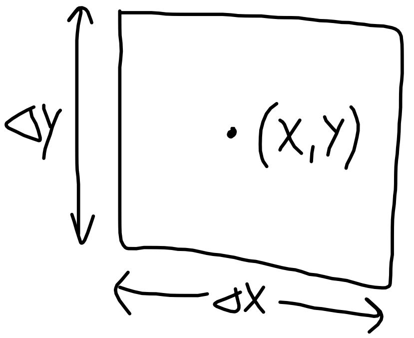
Generally, if $\rho(x,y)$ is a local density function for some quantity, then the total amount of the quantity over an area A is

$$\iint_A \rho(x,y)\,dA.$$

For example, if ρ is mass density, then the total mass, m, is $$m = \iint_A \rho\,dA.$$

NOTE: Point-wise densities are a mathematical tool and MODEL, not a "real" thing. For example, what does mass density at a point mean? There is no physical description of mass at a single point; even sub-atomic particles are considered to take up some non-zero volume of space.

To clarify the point...

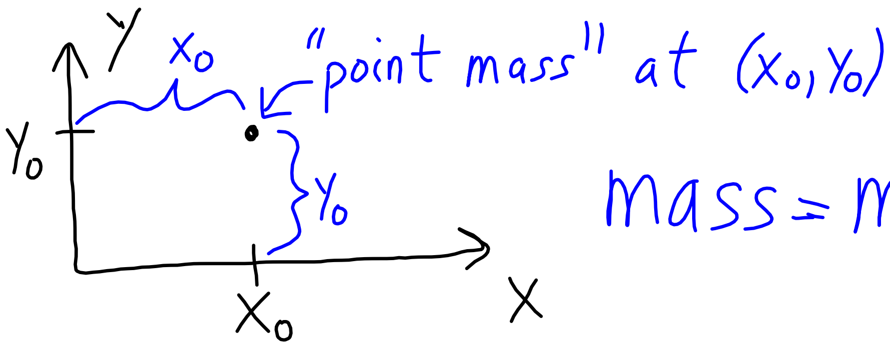


Let Δm be the total mass within a tiny region centered at (x,y).

$$\rho(x,y) = \lim_{\Delta x, \Delta y \to 0} \frac{\Delta m}{\Delta x \cdot \Delta y}$$

ρ is a <u>LIMIT</u> at each point. It just models the behavior of mass <u>near</u> a particular point.

Moments



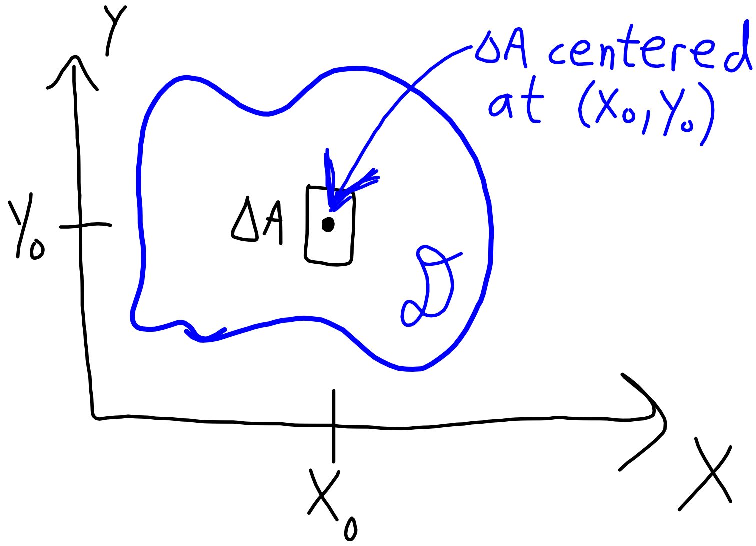
mass $= m$

x-moment: my_0

y-moment: mx_0

Now consider a solid body as a collection/ continuum of particles.

$\rho(x,y)$: local mass density



local moment about x-axis

$$\approx \underbrace{\left(\rho(x_0,y_0)\,\Delta A\right)}_{\text{local mass}} \underbrace{y_0}_{\text{distance to x-axis}}$$

M_x : moment of body about x-axis

Riemann sum... $M_x \approx \sum_{i,j} \rho(x_i^*, y_j^*) \Delta A_{ij} \, y_j^*$

$(\Delta A_{ij} = \Delta A, \text{constant})$ $= \Delta A \sum_{i,j} \rho(x_i^*, y_j^*) \, y_j^*$

$$\Rightarrow M_x = \lim_{\Delta A \to 0} \Delta A \sum_{i,j} \rho(x_i^*, y_j^*) \, y_j^* = \iint_D y \, \rho(x,y) \, dA.$$

$$M_x = \iint_D y\rho \, dA$$ Moment about x-axis

$$M_y = \iint_D x\rho \, dA$$ Moment about y-axis

EX: Let a region $D = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 4\}$ represent a solid object with mass density $\rho(x,y) = 1 + x^2$. Find the moments about the x- & y-axes.

$$M_x = \int_0^2 \int_1^4 y(1+x^2)\,dy\,dx = \frac{1}{2}\int_0^2 (16-1)(1+x^2)\,dx$$

$$= \frac{15}{2}\left[x + \frac{1}{3}x^3\right]_0^2 = \frac{15}{2}\left(2 + \frac{8}{3}\right) = \boxed{35.}$$

$$M_y = \int_0^2 \int_1^4 x(1+x^2)\,dy\,dx$$

$$= 3\int_0^2 x + x^3\,dx = 3\left[\frac{1}{2}x^2 + \frac{1}{4}x^4\right]_0^2$$

$$= 3\left[2+4\right] = \boxed{18.}$$

Center of mass

$$\bar{x} = \frac{M_y}{m}.$$

$$\bar{y} = \frac{M_x}{m}.$$

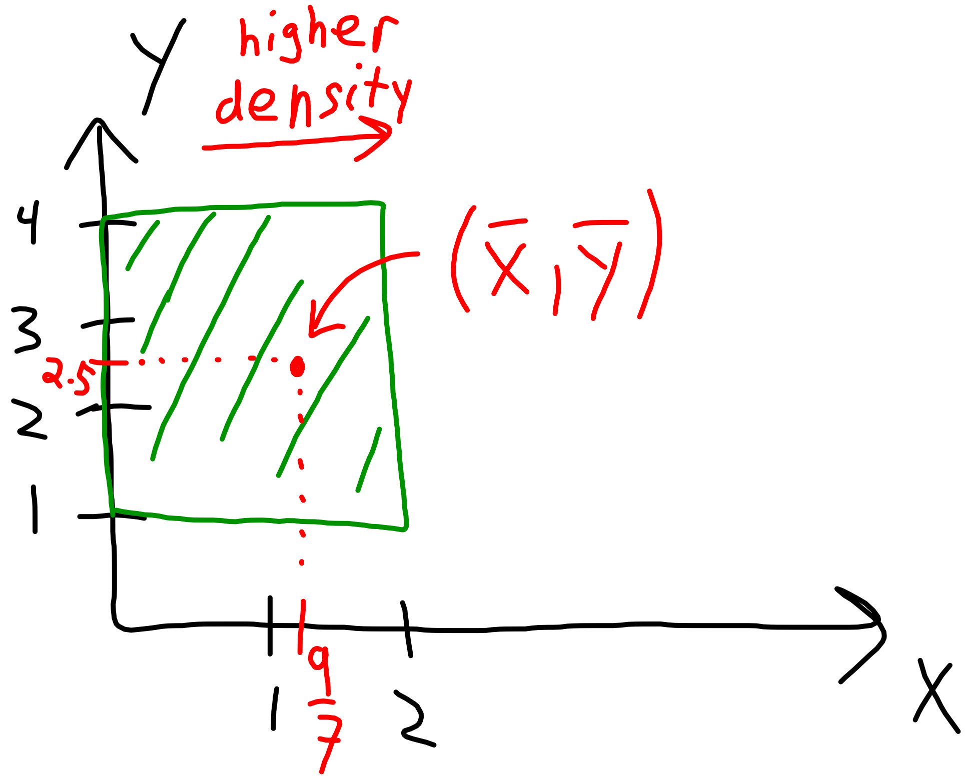
$$m = \iint_D \rho \, dA.$$

EX: For the previous example, find the center of mass.

$$\rho = 1+x^2 \Rightarrow m = \int_0^2 \int_1^4 (1+x^2)\,dy\,dx = 3\int_0^2 1+x^2\,dx$$

$$= 3\left[x+\tfrac{1}{3}x^3\right]_0^2 = 3\left(2+\tfrac{8}{3}\right) = 14.$$

Thus $\bar{x} = \frac{M_y}{m} = \frac{18}{14} = \frac{9}{7}$, $\bar{y} = \frac{35}{14} = \frac{5}{2}$.


y
higher density
4
3
2.5
2
1
$(\bar{x}, \bar{y})$
1
$\frac{9}{7}$
2
x

Moments of inertia: find the torque needed to generate some angular acceleration.

1) $I_x = \iint_D y^2 \rho \, dA$ about x-axis

2) $I_y = \iint_D x^2 \rho \, dA$ about y-axis

3) $I_0 = I_x + I_y = \iint_D (x^2+y^2) \rho \, dA.$ rotation around origin

Comments

* The required torque increases with distance.

* The required torque increases with density, but not as quickly as with distance.

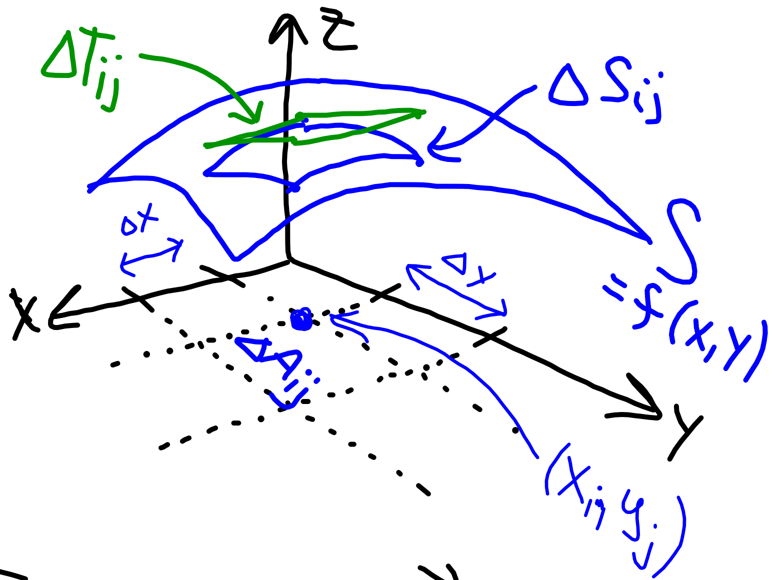
* Think of walking a tight-rope... people hold a long pole rather than something smaller but heavier.

* Abstractly, the n-th moment of f(x,y) with respect to variable ω is

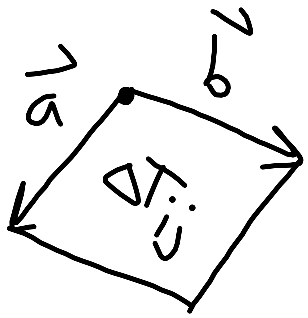
$$\iint_{D} \omega^n f \, dA.$$

Surface area

ΔT_{ij} : piece of tangent plane to f at (x_{ij}, y_{ij}) lying above ΔA_{ij}.



Formula for ΔT_{ij}:



$$\Delta T_{ij} = |\vec{a} \times \vec{b}|$$

Note $\vec{a}$ lies along tangent line in the x-direction & $\vec{b}$ " " in y-direction;

$$\vec{a} = \langle \Delta x, 0, \Delta x \cdot f_x \rangle = \Delta x \langle 1, 0, f_x \rangle$$

$$\vec{b} = \langle 0, \Delta y, \Delta y\, f_y \rangle = \Delta y \langle 0, 1, f_y \rangle$$

$$\Rightarrow \vec{a} \times \vec{b} = \Delta x \cdot \Delta y \langle -f_x, -f_y, 1 \rangle$$

$$|\vec{a} \times \vec{b}| = \Delta T_{ij} = \Delta x \Delta y \sqrt{1 + (f_x)^2 + (f_y)^2}.$$

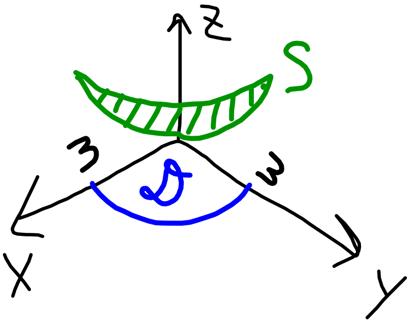
$$\Rightarrow S \approx \sum_{i,j} \Delta T_{ij} = \Delta x \Delta y \sum_{i,j} \sqrt{1 + f_x(x_i, y_j)^2 + f_y(x_i, y_j)^2}$$

Let $\Delta T_{ij} = \Delta x \cdot \Delta y \to 0 \ldots$

Surface area is a double integral

$$\boxed{S = \lim_{\Delta x \Delta y \to 0} \sum_{i,j} \Delta T_{ij} = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA.}$$

Ex: Calculate the surface area of the paraboloid $z = x^2 + y^2$ over the domain $\mathcal{D} = \{(x,y) \mid x^2+y^2 \le 9,\ x \geq 0,\ y \geq 0\}$.



$z_x = 2x \ldots \quad z_y = 2y$

Polar: $0 \le \theta \le \pi/2$

$0 \le r \le 3$

$z_x^2 + z_y^2 = 4(x^2+y^2) = 4r^2$

$$S = \int_0^{\pi/2} \int_0^3 \sqrt{1+4r^2}\, r\, dr\, d\theta \quad \begin{cases} u = 4r^2 \\ \frac{1}{8} du = r\, dr \end{cases}$$

$$S = \frac{1}{8}\int_0^{\pi/2}\int_{u=0}^{u=4\cdot 3^2=36} \sqrt{1+u}\, du\, d\theta$$

$$= \frac{1}{8}\left(\frac{\pi}{2}\right)\left(\frac{2}{3}(1+u)^{3/2}\Big|_0^{36}\right)$$

$$= \frac{\pi}{24}\left(37^{3/2} - 1\right).$$

Practice!

#1 Let a solid disc have radius $r=3$ and density $\rho(x,y) = e^{-x^2-y^2}$. Find (a) the total mass and (b) center of mass.

$u = r^2$

$\frac{1}{2}du = r\,dr$

$$\int_0^{2\pi}\int_0^3 e^{-r^2}\, r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^9 e^{-u}\,du\,d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \left[\int_0^9 e^{-u} \, du \right] d\theta$$

$$= \frac{1}{2} \int_0^9 e^{-u} \, du \cdot \int_0^{2\pi} d\theta$$

$$= \pi \left[-e^{-u} \right]_0^9 = \pi \left(1 - e^{-9} \right)$$

(b) center of mass

$$M_x = \int_0^{2\pi}\int_0^3 r\sin\theta\, e^{-r^2}\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^3 r^2 e^{-r^2}\sin\theta\,dr\,d\theta$$

don't try this, switch order

$$= \int_0^3\int_0^{2\pi} r^2 e^{-r^2}\sin\theta\,d\theta\,dr = 0, \text{ since } \int_0^{2\pi}\sin\theta\,d\theta = 0.$$

Similarly, $M_y = 0$

$$\Rightarrow \bar{x} = \bar{y} = 0.$$

#2) The portion of a cone above the xy-plane is given as

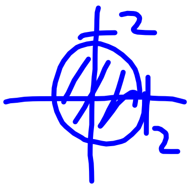
$$z = \frac{1}{2}\sqrt{x^2+y^2}.$$

Find the surface area lying above

$$D = \{(x,y) \mid x^2+y^2 \le 4\}.$$

$$z_x = \frac{2x}{4\sqrt{x^2+y^2}} = \frac{x}{2\sqrt{x^2+y^2}} \quad z_y = \frac{y}{2\sqrt{x^2+y^2}}$$

$$z_x^2 + z_y^2 = \frac{1}{4}$$



$$S = \int_0^{2\pi}\int_0^{2} \sqrt{1+\frac{1}{4}}\; r\,dr\,d\theta$$

$$= \frac{\sqrt{5}}{2}\cdot 2\pi \int_0^2 r\,dr = \pi\sqrt{5}\,\left.\frac{r^2}{2}\right|_0^2$$

$$= \boxed{2\pi\sqrt{5}}\,.$$