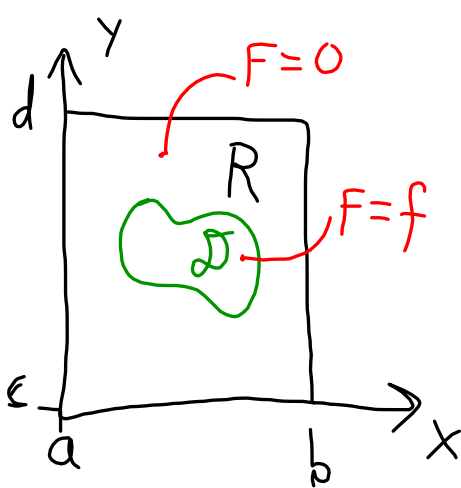


Integration over general regions.

Let $f(x,y)$ have domain \mathcal{D} and define



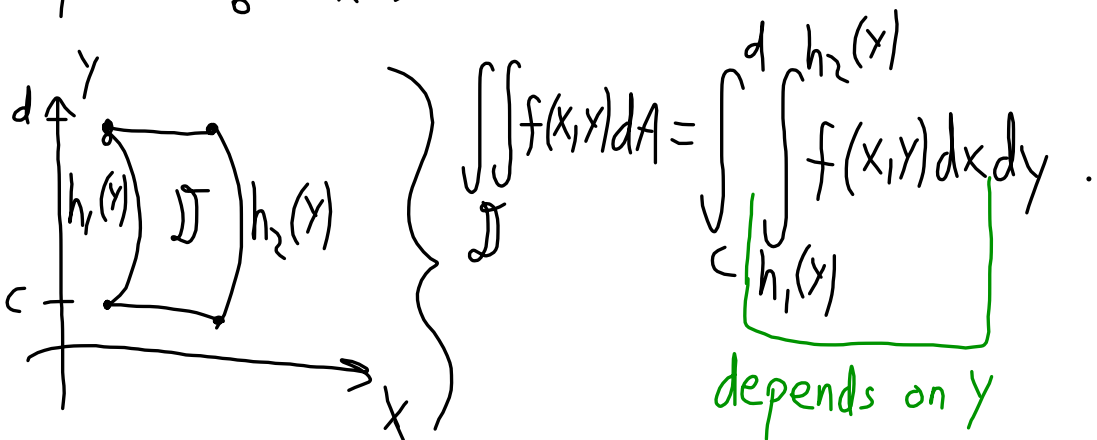
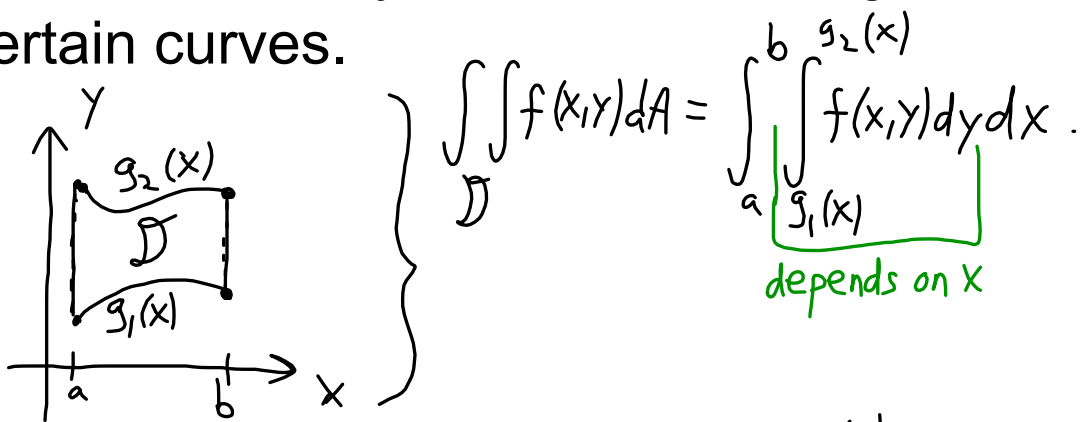
$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ in } \mathcal{D} \\ 0, & \text{otherwise.} \end{cases}$$

Then

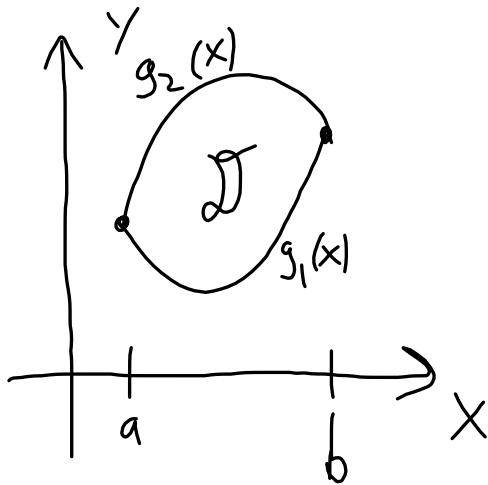
$$\iint_{\mathcal{D}} f(x,y) dA = \iint_R F(x,y) dA.$$

This can be defined using double Riemann sums, same as before.

We often identify domains as being bounded by certain curves.



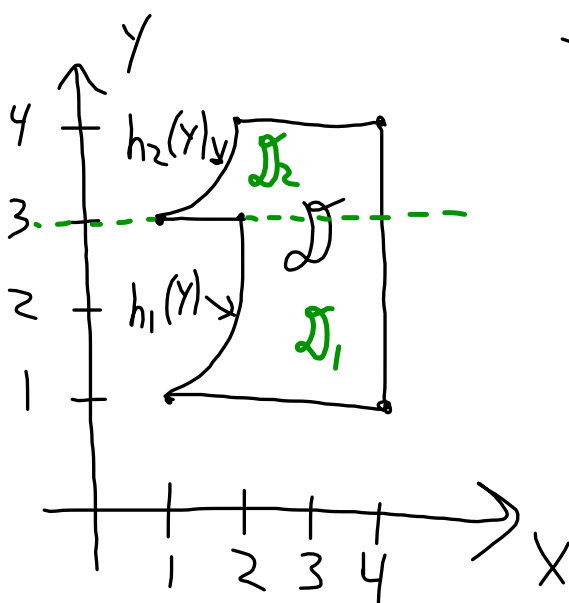
Often you can choose the order of integration; this is a matter of preference usually.



$$\iint_D f dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) dy dx$$

Integration in the opposite order may be possible, but perhaps trickier.

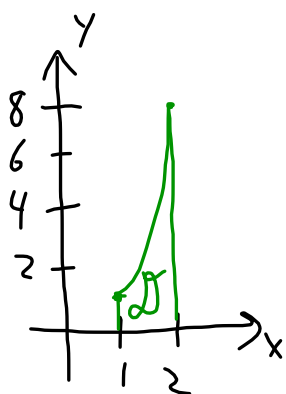
Sometimes it makes sense to break the integral up by breaking up the domain.



$$\begin{aligned}
 \iint_D f dA &= \iint_{D_1} f dA + \iint_{D_2} f dA \\
 &= \int_1^3 \int_{h_1(y)}^4 f(x,y) dx dy \\
 &\quad + \int_3^4 \int_{h_2(y)}^4 f(x,y) dx dy .
 \end{aligned}$$

EX: Let $f(x,y) = e^{y/x}$ on $D = \{(x,y) \mid 1 \leq x \leq 2, 0 \leq y \leq x^3\}$.

Find $\iint_D f(x,y) dA$.



$$= \int_1^2 \int_0^{x^3} e^{y/x} dy dx = \int_1^2 \left[x e^{y/x} \right]_{y=0}^{y=x^3} dx$$

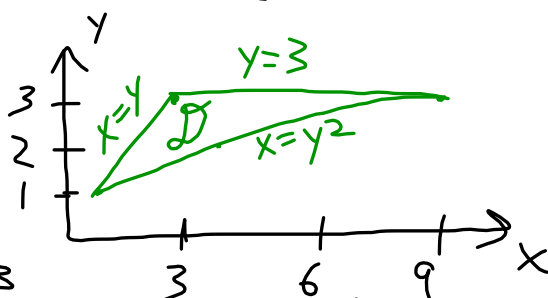
$$= \int_1^2 (x e^{x^2} - x) dx = \int_1^2 x e^{x^2} dx - \int_1^2 x dx$$

$u = x^2$
 $du = 2x dx$

$$= \int_{u=1}^{u=4} \frac{1}{2} e^u du - \int_1^2 x dx = \frac{1}{2} (e^4 - e) - \frac{1}{2} (4 - 1)$$

EX: Let $f(x,y) = xy$ on $D = \{(x,y) \mid y \leq x \leq y^2, 1 \leq y \leq 3\}$.

Find $\iint_D f(x,y) dA$.

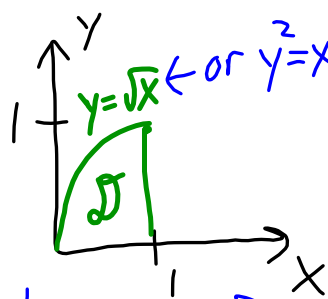


$$= \int_1^3 \int_y^{y^2} xy \, dx \, dy = \frac{1}{2} \int_1^3 (x^2 y) \Big|_{x=y}^{x=y^2} dy = \frac{1}{2} \int_1^3 (y^5 - y^3) dy$$

$$= \frac{1}{2} \left[\frac{1}{6} y^6 - \frac{1}{4} y^4 \right]_1^3 = \frac{1}{2} \left[\frac{1}{6} (3^6 - 1) - \frac{1}{4} (3^4 - 1) \right].$$

EX: Let $f(x,y) = ye^x$ on $D = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x}\}$.

Find $\iint_D f \, dA$.



we'll show 2 ways...

$$\textcircled{1} \int_0^1 \int_0^{\sqrt{x}} ye^x \, dy \, dx = \int_0^1 \left[\frac{1}{2} ye^x \right]_{y=0}^{y=\sqrt{x}} dx = \frac{1}{2} \int_0^1 (xe^x) dx$$

Integrate by parts...

$$= \frac{1}{2} \int_0^1 x \frac{d}{dx}(e^x) dx = \frac{1}{2} \int_0^1 \left(\frac{d}{dx}(xe^x) - \left[\frac{d}{dx}(x) \right] e^x \right) dx$$

$$= \frac{1}{2} [xe^x]_0^1 - \int_0^1 e^x dx = \frac{1}{2} (e - (e-1)) = \boxed{\frac{1}{2}}.$$

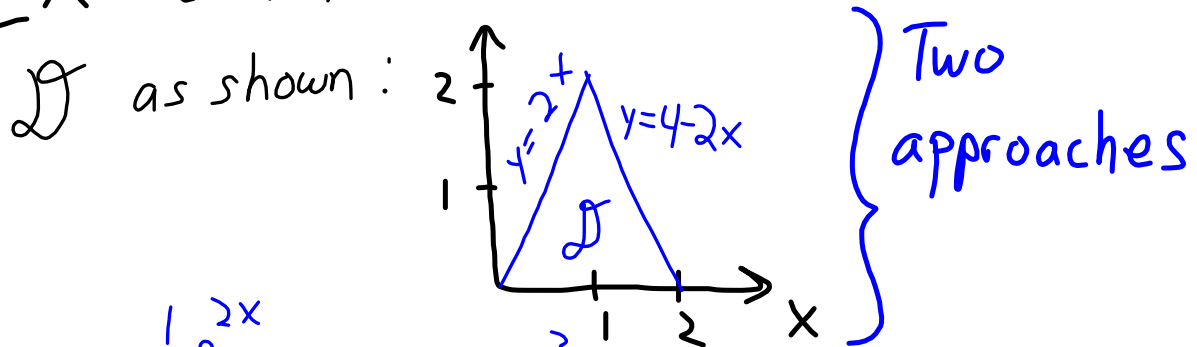
$$\textcircled{2} = \int_0^1 \int_{x=y^2}^{x=1} ye^x dx dy = \int_0^1 [ye^x]_{x=y^2}^{x=1} dy$$

$$= \int_0^1 (ey - ye^{y^2}) dy = e \int_0^1 y dy - \int_0^1 ye^{y^2} dy$$

$$= \frac{e}{2} - \int_0^1 \frac{1}{2} e^u du = \frac{e}{2} - \left(\frac{e}{2} - \frac{1}{2} \right) = \boxed{\frac{1}{2}}.$$

$u=y^2$
 $\frac{1}{2} du = y dy$

EX: Let $f(x,y) = 2x - 4y$. Integrate this over



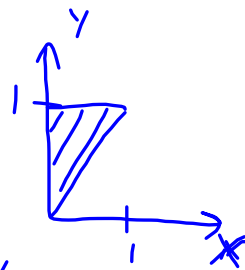
$$(1) \int_0^1 \int_0^{2x} (2x-4y) dy dx + \int_1^2 \int_0^{4-2x} (2x-4y) dy dx$$

$$(2) \int_0^2 \int_{y/2}^{\frac{4-y}{2}} (2x-4y) dx dy = \int_0^2 \left[x^2 - 4xy \right]_{x=y/2}^{x=\frac{4-y}{2}} dy$$

$$\begin{aligned}
&= \int_0^2 \left[\frac{1}{4}(4-y)^2 - \frac{1}{4}y^2 - 2y(4-y) + 2y^2 \right] dy \\
&= \left[\frac{-1}{4 \cdot 3}(4-y)^3 - \frac{1}{12}y^3 - 4y^2 + \frac{2}{3}y^3 + \frac{2}{3}y^3 \right]_0^2 \\
&= -\frac{2}{3} + \frac{16}{3} - \frac{8}{12} - 16 + \frac{32}{3} = -12 + \frac{32}{3} = \frac{32-36}{3} = \boxed{-\frac{4}{3}}.
\end{aligned}$$

EX: Find $\int_0^1 \int_x^1 \sin(y^2) dy dx = ?$

$\int \sin(y^2) dy = ?$



Switch order of integration...

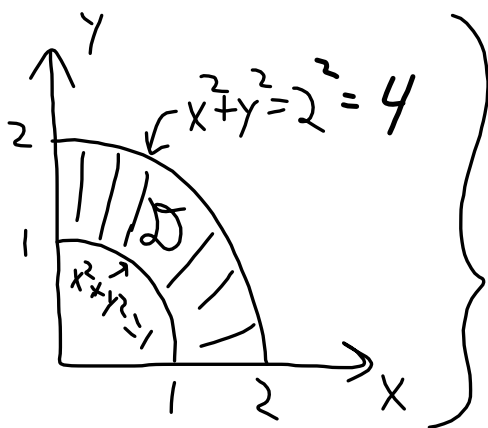
$$= \int_0^1 \int_0^y \sin(y^2) dx dy = \int_0^1 \left[x \sin(y^2) \right]_{x=0}^{x=y} dy$$

$$= \int_0^1 \left[y \sin(y^2) - 0 \right] dy = \frac{1}{2} \int_0^1 \sin(u) du = \frac{1}{2} \left[-\cos(1) + \cos(0) \right]$$

$u = y^2$
 $\frac{1}{2} du = y dy$

$$= \frac{1}{2} (1 - \cos(1))$$

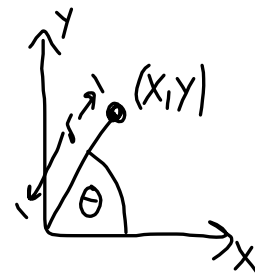
Polar coordinates



This should be handled using polar coordinates:

$$x = r \cos \theta$$

$$y = r \sin \theta$$



$$r=2 \quad \theta = \pi/2$$

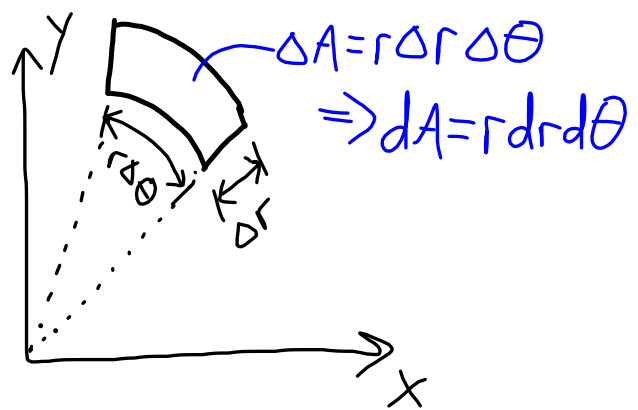
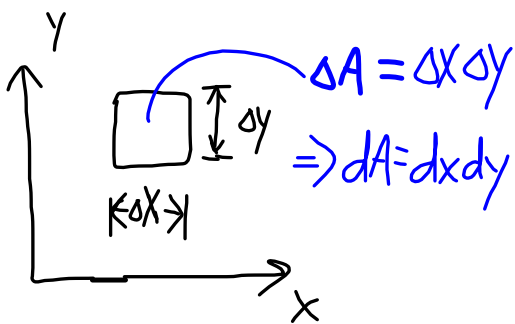
$$\int_{r=1}^2 \int_{\theta=0}^{\pi/2} f(x,y) dA = ?$$

need to discuss this further

$$x = r \cos \theta$$

$$y = r \sin \theta$$

Differential area in polar coordinates



Back to the integral...

$$\int_{r=1}^{r=2} \int_{\theta=0}^{\theta=\pi/2} f(r \cos \theta, r \sin \theta) \underbrace{r}_{\text{don't forget!}} dr d\theta = \iint_{\mathcal{D}} f(x, y) dA.$$

EX: Find $\iint_D f(x,y) dA$ if $D = \{(r,\theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{2}\}$,

$$f = x - 4y^2 \quad f(r,\theta) = r \cos \theta - 4r^2 \sin^2 \theta$$

$$\begin{aligned} \int_0^{\pi/2} \int_1^2 (r \cos \theta - 4r^2 \sin^2 \theta) r dr d\theta &= \int_0^{\pi/2} \int_1^2 (r^2 \cos \theta - 4r^3 \sin^2 \theta) dr d\theta \\ &= \int_0^{\pi/2} \left[\frac{1}{3} r^3 \cos \theta - r^4 \sin^2 \theta \right]_1^2 d\theta = \int_0^{\pi/2} \left[\frac{1}{3} (8-1) \cos \theta - (16-1) \sin^2 \theta \right] d\theta \\ &= \int_0^{\pi/2} \left(\frac{7}{3} \cos \theta - 15 \sin^2 \theta \right) d\theta \end{aligned}$$

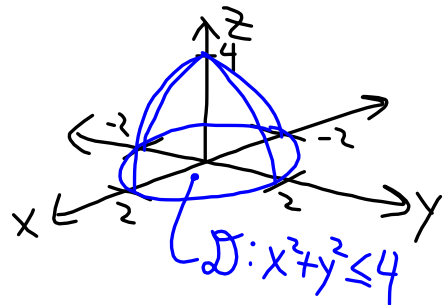
Use $\sin^2 \theta = \frac{1}{2}(1 - \cos(2\theta))$

$$\Rightarrow \int_0^{\pi/2} \left[\frac{7}{3} \cos \theta - \frac{15}{2}(1 - \cos(2\theta)) \right] d\theta$$

$$= \frac{7}{3} \sin \theta \Big|_0^{\pi/2} - \frac{15}{2} \cdot \frac{\pi}{2} + \frac{15}{2} \left(\frac{1}{2} \sin(2\theta) \right) \Big|_0^{\pi/2}$$

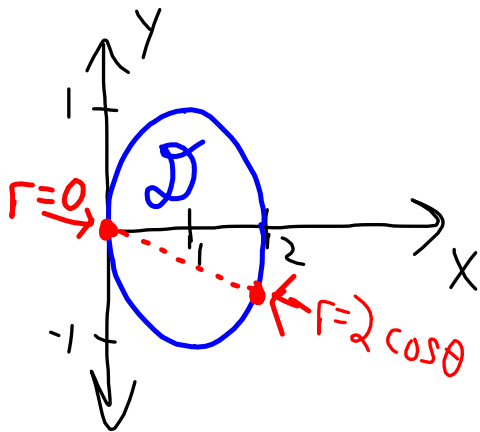
$$= \frac{7}{3} - \frac{15\pi}{4}$$

EX: Find the volume under the surface
 $z = 4 - x^2 - y^2$ & above $z = 0$.



$$\begin{aligned}
 V &= \int_{\theta=0}^{2\pi} \int_{r=0}^2 (4 - r^2 \cos^2 \theta - r^2 \sin^2 \theta) r dr d\theta \\
 &= \int_0^{2\pi} \int_0^2 (4r - r^3) dr d\theta = \int_0^{2\pi} \left(2r^2 - \frac{1}{4}r^4 \right) \Big|_0^2 d\theta \\
 &= \int_0^{2\pi} (8 - 4) d\theta = 4 \cdot 2\pi = 8\pi.
 \end{aligned}$$

EX: Let \mathcal{D} be the region inside the circle $(x-1)^2 + y^2 = 1$. Find $\iint_{\mathcal{D}} xy \, dA$.



On boundary, we have

$$(r \cos \theta - 1)^2 + r^2 \sin^2 \theta = 1$$

$$r^2 \cos^2 \theta - 2r \cos \theta + 1 + r^2 \sin^2 \theta = 1$$

$$r^2 - 2r \cos \theta + 1 = 1$$

$$r(r - 2 \cos \theta) = 0$$

$$\Rightarrow r = 0 \text{ or } r = 2 \cos \theta.$$

$$0 \leq r \leq 2 \cos \theta$$

$$\iint_D xy \, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r \cos\theta \, r \sin\theta \, r \, dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^3 \sin\theta \cdot \cos\theta \, dr \, d\theta$$

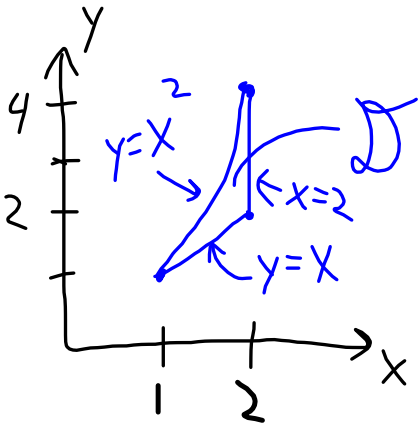
$$= \int_{-\pi/2}^{\pi/2} \sin\theta \cos\theta \left. \frac{1}{4} r^4 \right|_0^{2\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2} 4 \sin\theta \cos^5\theta \, d\theta$$

$$= 4 \int_{-\pi/2}^{\pi/2} -\frac{1}{6} \frac{d}{d\theta} \cos^6\theta \, d\theta = -\frac{2}{3} \left(\cos^6\frac{\pi}{2} - \cos^6\left(-\frac{\pi}{2}\right) \right)$$

= 0.
0
0

Practice!

(#1) Find $\iint_{\mathcal{D}} (x^2 + y) dA$, with \mathcal{D} given in the diagram:



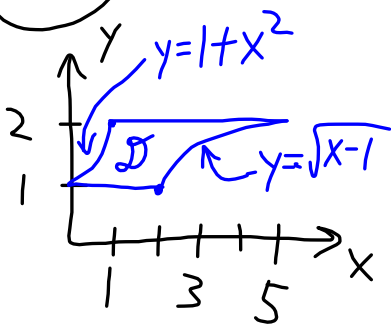
$$= \int_1^2 \int_{x^2}^x (x^2 + y) dy dx$$

$$= \int_1^2 \left[x^2 y + \frac{1}{2} y^2 \right]_{y=x^2}^{y=x} dx$$

$$= \int_1^2 \left(x^4 - x^3 + \frac{1}{2} x^4 - \frac{1}{2} x^2 \right) dx$$

$$\begin{aligned} &= \int_1^2 \left(\frac{3}{2}x^4 - x^3 - \frac{1}{2}x^2 \right) dx \\ &= \left[\frac{3}{10}x^5 - \frac{1}{4}x^4 - \frac{1}{6}x^3 \right]_1^2 \\ &= \frac{3}{10}(32-1) - \frac{1}{4}(16-1) - \frac{1}{6}(8-1) \end{aligned}$$

#2 Let D be as in the diagram; find $\iint_D x \, dA$.



$$1 \leq y \leq 2$$

$$\sqrt{y-1} \leq x \leq y^2+1$$

$$\int_1^2 \int_{\sqrt{y-1}}^{y^2+1} x \, dx \, dy$$

$$= \frac{1}{2} \int_1^2 x^2 \Big|_{\sqrt{y-1}}^{y^2+1} dy$$

$$= \frac{1}{2} \int_1^2 (y^2+1)^2 - (y-1) dy = \frac{1}{2} \int_1^2 (y^4 + 2y^2 + 2 - y) dy$$

$$= \frac{1}{2} \left[\frac{1}{5} y^5 + \frac{2}{3} y^3 + 2y - \frac{1}{2} y^2 \right]_1^2$$

$$= \frac{1}{2} \left(\frac{32-1}{5} + \frac{2(8-1)}{3} + 2(2-1) - \frac{1}{2}(4-1) \right)$$

$$= \frac{1}{2} \left(\frac{31}{5} + \frac{14}{3} + 2 - \frac{3}{2} \right) \dots$$

#3 Find $\iint_D x^2 + y^2 dA$; $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

$$x^2 + y^2 = r^2 \cos^2 \theta + r^2 \sin^2 \theta$$

$$= r^2 (\cos^2 \theta + \sin^2 \theta) = r^2$$



$$\int_0^{2\pi} \int_0^1 r^2 r dr d\theta = \int_0^{2\pi} \frac{1}{4} r^4 \Big|_0^1 d\theta = \frac{\pi}{2}$$

#4 Find the volume V under $z = 1 + y + x$, above $D = \{(x, y) \mid x^2 + y^2 \leq 4\}$ in the xy -plane.

$$\begin{aligned} |V| &= \int_0^{2\pi} \int_0^2 (1 + r\cos\theta + r\sin\theta) r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 (r + r^2(\cos\theta + \sin\theta)) \, dr \, d\theta \end{aligned}$$

$$= \int_0^{2\pi} \left[\frac{1}{2}r^2 + \frac{1}{3}r^3(\cos\theta + \sin\theta) \right]_0^2 d\theta$$

$$= \int_0^{2\pi} \left(2 + \frac{8}{3}(\cos\theta + \sin\theta) \right) d\theta$$

$$= 2 \cdot 2\pi + 0 = 4\pi$$