Maximums and minimums

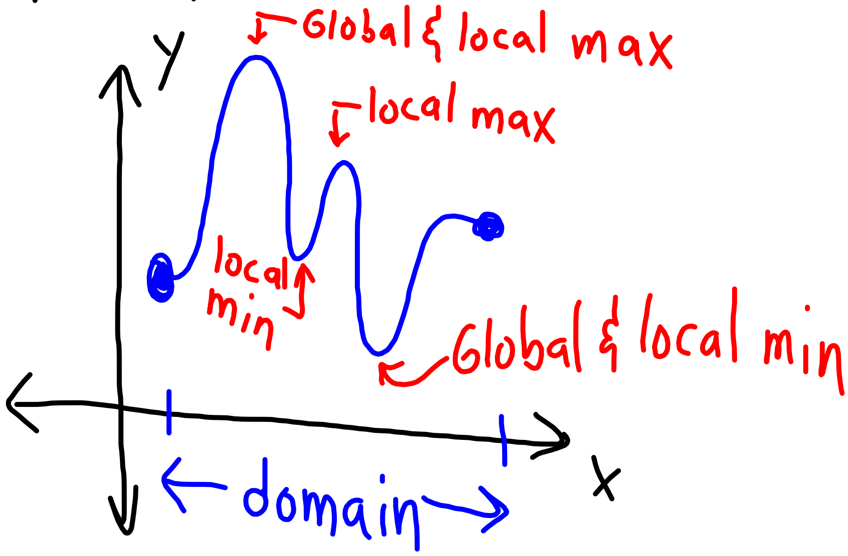
Let $f(x,y)$ be defined on $D \subset \mathbb{R}^2$.

Then for (a,b) in D,

- Local max @ (a,b) $\Leftrightarrow$ $f(x,y) \leq f(a,b)$ holds for all (x,y) near (a,b).

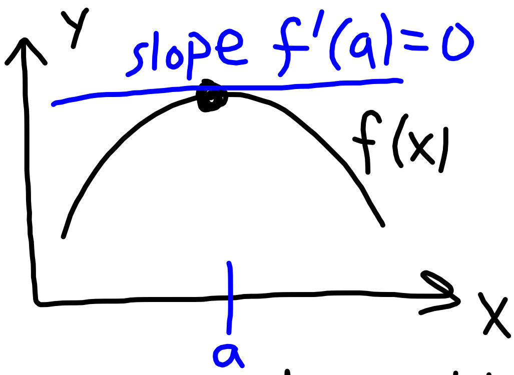
- Global max/absolute max $\Leftrightarrow$ $f(x,y) \leq f(a,b)$ for all (x,y) in D.

- Local min $\iff f(x,y) \geq f(a,b)$ holds for all (x,y) near (a,b)
- Global/absolute min if this holds for all (x,y) in D.

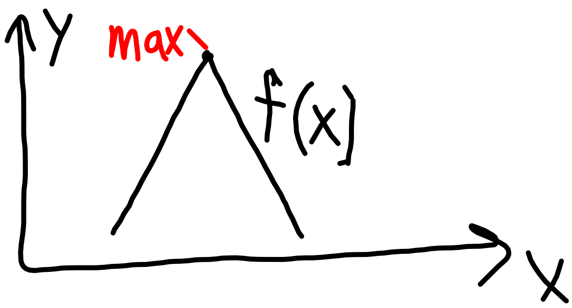


When derivatives exist at a max/min, they must be zero.

Case 1 : derivative $= 0$



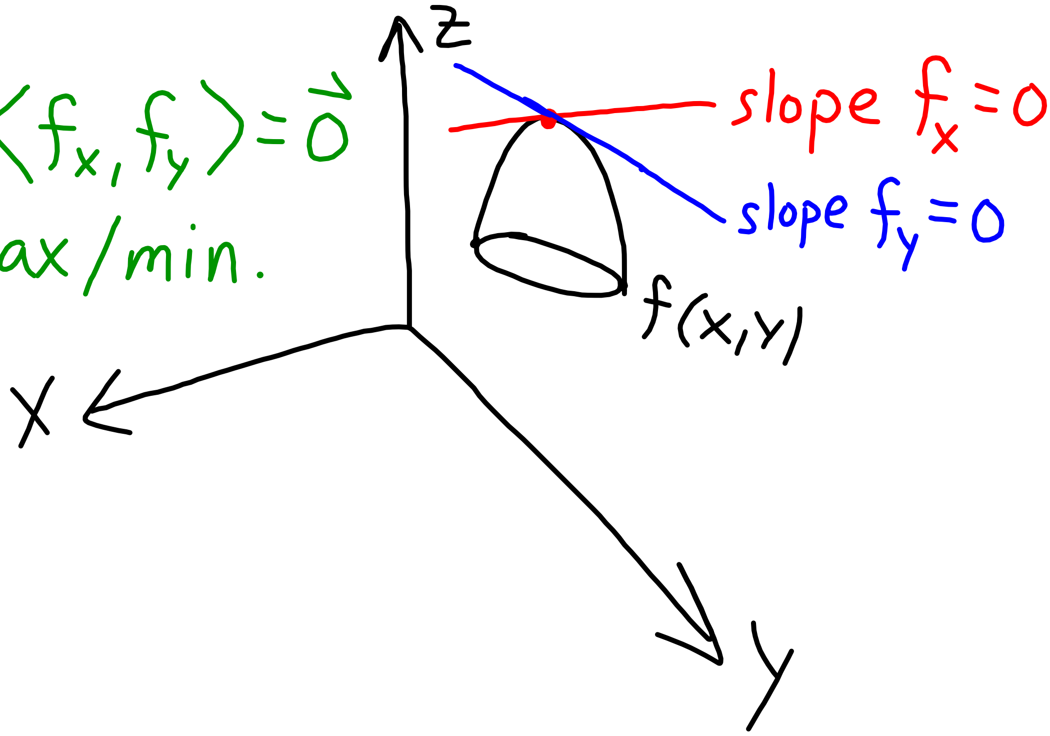
Case 2: no derivative



With multiple variables, the partial derivatives will be zero when they exist at max/min points.

$\nabla f = \langle f_x, f_y \rangle = \vec{0}$

at max/min.



... similarly for more variables

$f(x,y,z)$ has MAX/MIN at $(a,b,c) \Rightarrow \nabla f = \langle f_x, f_y, f_z \rangle$

$= \langle 0,0,0 \rangle$ at (a,b,c).

Since ∇f may not exist at MAX/MIN points we check all CRITICAL POINTS, i.e. where $\nabla f = \vec{0}$ or $\nabla f =$ D.N.E.

EX: Find all critical points of

$$f(x,y) = x^2 + xy - \frac{1}{4}y^2 .$$

$$f_x = 2x + y = 0$$

$$f_y = x - \frac{1}{2}y = 0$$

(0,0) only critical pt.

$$f_x - 2f_y = 2y = 0 \Rightarrow \underbrace{y = 0 \Rightarrow x = 0.}$$

*Note f decreases in y-direction but increases in x-direction at (0,0)

$\Rightarrow$ SADDLE POINT.

EX: Find the maxima of

$$f(x,y) = -x^2+2x-y^2-4y-5.$$

$$\left.\begin{aligned} f_x &= -2x+2 = 0 \quad (x=1) \\ f_y &= -2y-4=0 \quad (y=-2) \end{aligned}\right\} \text{CRIT. PT. } (1,-2)$$

Complete the square...

$$f(x,y) = -(x-1)^2-(y+2)^2 \leq 0 = f(1,-2)$$

holds for all $(x,y) \Rightarrow$ Global max @ $(1,-2)$.

The Second Derivative Test gives a more convenient way to classify critical points.

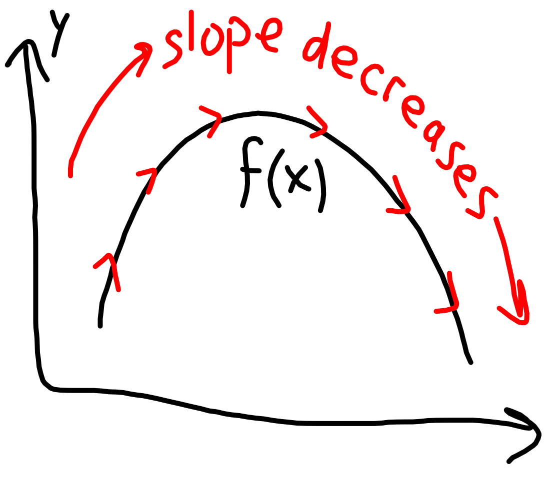
$$D = f_{xx} f_{yy} - (f_{xy})^2$$

(1) $D > 0,\ f_{xx} > 0 \Rightarrow$ Local min.

(2) $D > 0,\ f_{xx} < 0 \Rightarrow$ Local max.

(3) $D < 0 \Rightarrow$ Saddle point.

f_x increasing
$\Rightarrow f_{xx} > 0$ at MIN.



... and $f_{xx} < 0$
at MAX.

EX: Classify all critical points of

$$f(x,y) = x^4 + y^4 - 4xy + 1.$$

$$f_x = 4x^3 - 4y = 0 \qquad f_y = 4y^3 - 4x = 0$$

$$\Rightarrow y = x^3 \longrightarrow x^9 - x = 0$$

$$\Rightarrow x(x^8 - 1) = 0$$

$$\Rightarrow x = 0, \pm 1.$$

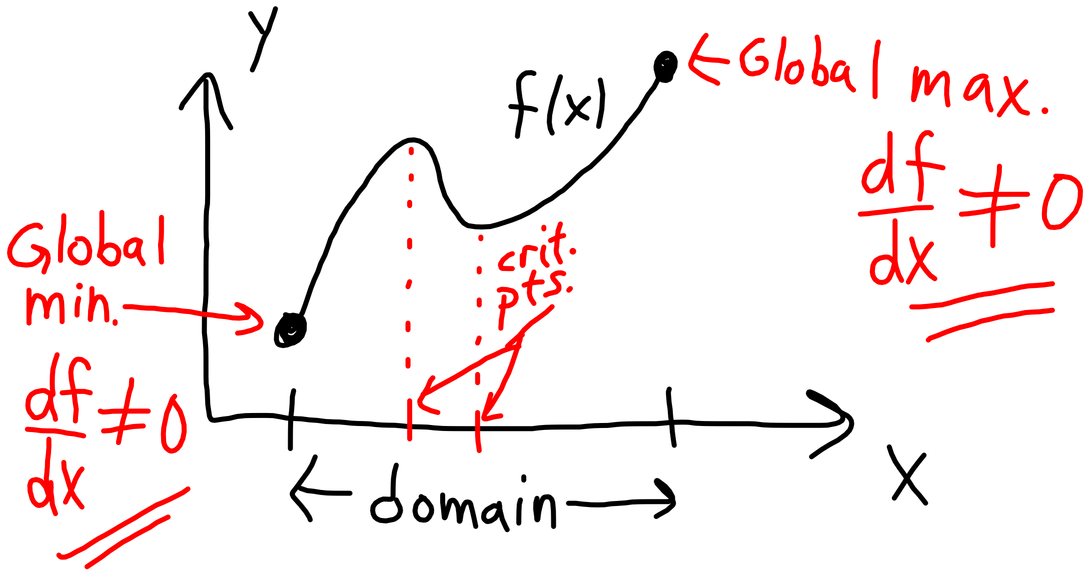
CRIT. PTS. $\{(0,0), (-1,-1), (1,1)\}$.

$$\left.\begin{aligned} f_{xx} &= 12x^2 \\ f_{yy} &= 12y^2 \\ f_{xy} &= -4 \end{aligned}\right\} \begin{aligned} &D = f_{xx} f_{yy} - (f_{xy})^2 \\ &D(0,0) = -16 < 0 \\ &D(-1,-1) = 144 - 16 > 0, \quad f_{xx} > 0. \\ &D(1,1) = 144 - 16 > 0, \quad f_{xx} > 0. \end{aligned}$$

$(0,0) \Rightarrow$ SADDLE PT.

$(1,1)$ & $(-1,-1) \Rightarrow$ LOCAL MIN.

Looking at critical points gives us local extrema and sometimes global extrema; not generally on a closed and bounded domain.



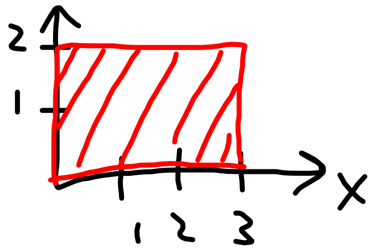
Identifying GLOBAL extrema takes work:

(1) Find f-values at all critical points inside the domain.

(2) Find max/min f-values along the domain boundary.

(3) Take MAX/MIN of all f-values from steps (1) & (2).

EX: Find the global extrema of $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $0 \leq x \leq 3$, $0 \leq y \leq 2$.



(1) Find f @ CRIT. PTS.

$f_x = 0 = 2x - 2y \quad (x = y)$

$f_y = 0 = 2 - 2x \quad (x = 1 \Rightarrow y = 1)$

$f(1,1) = 1 - 2 + 2 = \boxed{1}$

$f = x^2 - 2xy + 2y$

(2) Max/min along boundary.

- $x=0 \Rightarrow f=2y, \quad 0\le y\le 2 \Rightarrow 0\le f\le 4$
- $x=3 \Rightarrow f=9-6y+2y=9-4y \Rightarrow 1\le f\le 9$
- $y=0 \Rightarrow f=x^2, \quad 0\le x\le 3 \Rightarrow 0\le f\le 9$
- $y=2 \Rightarrow f=x^2-4x+4=(x-2)^2 \Rightarrow 0\le f\le 4$

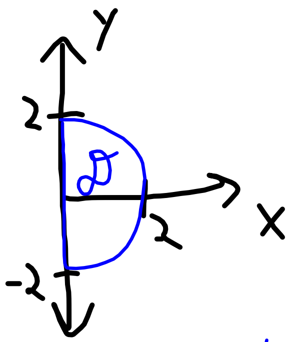
(3)

$f=1$ from step (1) $\Rightarrow$ $\boxed{\begin{array}{l} MIN=0 \\ MAX=9 \end{array}}$.

Ex: Find the global max/min for

$f(x,y) = 8x - 4x^2 - y^2$ on $\mathcal{D} = \{(x,y) \mid x^2+y^2 \le 4,\ x \ge 0\}$.

Domain is a bit trickier now...



$f_x = 8 - 8x = 0 \iff x = 1$

$f_y = -2y = 0 \iff y = 0$

$f(1,0) = 8 - 4 = 4 \leftarrow$ candidate for max/min

Two boundaries to check... on $x=0$, $-2 \le y \le 2$

$\Rightarrow f(0,y) = -y^2$ so $-4 \le f \le 0$, then $-4 \leq f \leq 4$ so far. To check the curved boundary, parameterize $f(x,y)$ somehow... note $x^2+y^2=4$ on this part of the boundary, so $y^2=4-x^2$

$$\Rightarrow f = 8x-4x^2-(4-x^2) = 8x-3x^2-4$$

$$\Rightarrow f'(x) = 8-6x = 0 \Leftrightarrow x = 4/3$$

Check $f\left(\frac{4}{3}\right) = 8\cdot\frac{4}{3}-3\left(\frac{4}{3}\right)^2-4 = \frac{32-16-12}{3} = 4/3.$

$$\boxed{\text{MAX}=4,\ \text{MIN}=-4}$$

Lagrange Multipliers: used to maximize or minimize a function under constraints.

-- Maximize the volume of a wooden crate given a fixed amount of wood to use.

-- Find the minimum distance to a planet along a fixed, elliptical orbit.

-- Analogous problems abound in engineering applications.

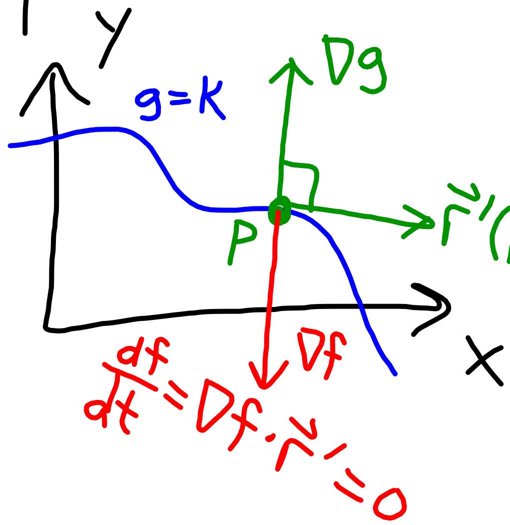
Abstractly, we consider

$f(x,y)$: A function to minimize or maximize

$g(x,y)=K$: "Constraint equation" restricts (x,y) allowed (think "design parameters")

*First step: identify these for your problem.

A picture for the idea...



Let $f(P)$ be a MAX/MIN of f along curve $g(x,y)=K$.

Since $g=K \Rightarrow$ level curve we know $\underline{\nabla g \perp \vec{r}'}$, where $\vec{r}(t)$ is a parameterization of the curve.

Along the curve, $f = f(x(t), y(t)) \Rightarrow \frac{df}{dt}(P) = 0$.

Chain Rule... $f_x x' + f_y \cdot y' = \nabla f \cdot \vec{r}' = 0$.

In summary, at an extreme value of f <u>along</u> $g=K$, we have

$$\boxed{\nabla f = \lambda \nabla g}$$ "$\nabla f \parallel \nabla g$"

scalar, "Lagrange multiplier"

$g(x,y)=K$ } Solve this with $\nabla f = \lambda \nabla g$.

Let $f = f(x,y,z)$... $g(x,y,z) = K$ constraint.

To find the MAX or MIN of f, solve

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} \\ \frac{\partial f}{\partial z} &= \lambda \frac{\partial g}{\partial z} \\ g(x,y,z) &= K \end{aligned}\right\}$$

Solving for x, y, z, λ

4 variables

4 equations

EX: Find the minimum distance from the origin to $\frac{x^2}{4}+\frac{y^2}{9}=1$.

- We are minimizing $d=\sqrt{x^2+y^2}$.
- Equivalently, use $f(x,y)=x^2+y^2$ (easier).
- Constraint: $g(x,y)=\frac{x^2}{4}+\frac{y^2}{9}=1$.

Now apply the method; solve

$$f_x = 2x = \lambda \tfrac{1}{2} x = \lambda g_x \quad (x=0 \text{ or } \lambda=4)$$

$$f_y = 2y = \lambda \tfrac{2}{9} y = \lambda g_y \quad (y=0 \text{ or } \lambda=9)$$

$$\left. g = \frac{x^2}{4} + \frac{y^2}{9} = 1 \right\} \quad x, y \text{ not both } 0$$

- $x=0 \Rightarrow \lambda=9$, $y^2=9$ so $y=\pm 3$
  $d(0,\pm 3) = 3$

MIN

- $y=0 \Rightarrow \lambda=4$, $x^2=4 \Rightarrow x=\pm 2$, $d(\pm 2, 0) = \boxed{2}$.

EX: Find the minimum distance from $(0,0,1)$ to the hyperboloid $x^2+\frac{1}{4}y^2-z^2=1$.

use $f(x,y,z)=x^2+y^2+(z-1)^2$

and $g=x^2+\frac{1}{4}y^2-z^2=1$.

Solve: $\begin{cases} 2x=2x\lambda \to x=0 \text{ or } \lambda=1 \\ 2y=\frac{1}{2}y\lambda \to y=0 \text{ or } \lambda=4 \\ 2(z-1)=-2z\lambda \to (1+\lambda)z=1 \\ x^2+\frac{1}{4}y^2-z^2=1 \to x,y \text{ not } \underline{\text{both}} \text{ zero...} \end{cases}$

Solve: $\begin{cases} 2x = 2x\lambda \rightarrow x=0 \text{ or } \lambda=1 \\ 2y = \frac{1}{2}y\lambda \rightarrow y=0 \text{ or } \lambda=4 \\ 2(z-1) = -2z\lambda \rightarrow (1+\lambda)z = 1 \\ x^2+\frac{1}{4}y^2-z^2=1 \rightarrow x,y \text{ not } \underline{\text{both}} \text{ zero...} \end{cases}$

• $x=0 \Rightarrow \lambda=4 \Rightarrow z=\frac{1}{5} \Rightarrow \frac{1}{4}y^2-\frac{1}{25}=1 \Rightarrow y^2=\frac{4\cdot 26}{25}$

$\Rightarrow d=\sqrt{0^2+\frac{4\cdot 26}{25}+\left(\frac{1}{5}-1\right)^2}=\sqrt{\frac{120}{25}}=\frac{2}{5}\sqrt{30}$.

• $y=0 \Rightarrow \lambda=1 \Rightarrow z=\frac{1}{2} \Rightarrow x^2-\frac{1}{4}=1 \Rightarrow x^2=5/4$.

Thus $d = \sqrt{\frac{5}{4} + 0^2 + \left(\frac{1}{2} - 1\right)^2} = \frac{\sqrt{6}}{2} \ldots$

$$\frac{\sqrt{6}}{2} < \frac{2}{5}\sqrt{30} \Rightarrow \boxed{MIN = \frac{\sqrt{6}}{2}}.$$

Practice!

#1 Find, classify all critical points of

$$f(x,y)=\frac{1}{2}x^4+xy+y^2+1.$$

$f_x = 2x^3 + y = 0$

$f_y = x + 2y = 0 \Rightarrow x = -2y$

$-16y^3 + y = 0 = y(1-16y^2)$

$\Rightarrow y = 0, \frac{1}{4}, -\frac{1}{4}$ $\quad (0,0), \left(-\frac{1}{2}, \frac{1}{4}\right), \left(\frac{1}{2}, -\frac{1}{4}\right)$

$$\left.\begin{aligned} f_{xx} &= 6x^2 \\ f_{yy} &= 2 \\ f_{xy} &= 1 \end{aligned}\right\} \quad D = 12x^2 - 1$$

$D(0,0) = -1 < 0$ (SADDLE)

$D\left(\frac{-1}{2}, \frac{1}{4}\right) = 2 > 0,\ f_{xx} > 0 \Rightarrow$ (LOCAL MIN)

$D\left(\frac{1}{2}, \frac{-1}{4}\right) = 2 > 0,\ f_{xx} > 0$ (LOCAL MIN)

#2 Find the global MAX/MIN of

$f(x,y) = x + xy - \frac{1}{3}y^3$ with $-2 \le x \le 2$,
$-2 \le y \le 2$.

$f_x = 1 + y = 0 \Rightarrow y = -1$

$f_y = x - y^2 = 0 \Rightarrow x = 1$

$f(1,-1) = 1 - 1 + \frac{1}{3} = \frac{1}{3}$ (CANDIDATE)

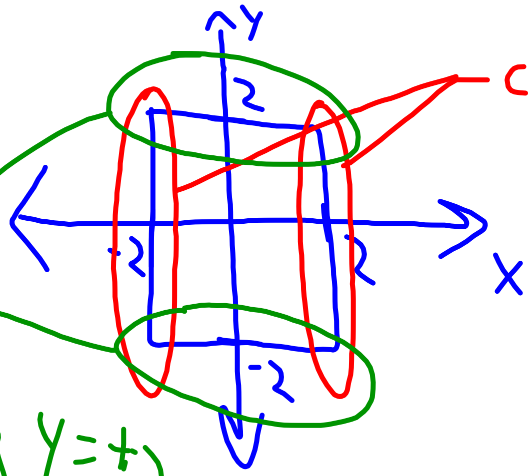
- Check boundary...

Corners:

$f(-2,-2) = \frac{14}{3}$, $f(-2,2) = -26/3$

$f(2,-2) = 2/3$, $f(2,2) = 10/3$

Edges:



check where $\frac{\partial f}{\partial y} = 0$ along $x = \pm 2$

check where $\frac{\partial f}{\partial x} = 0$ along $y = \pm 2$

- $f_x(x,-2) = -1 \neq 0$
- $f_x(x,2) = 3 \neq 0$
- $f_y(-2,y) = -2-y^2 \neq 0$

so nothing to check along 3 edges

- $f_y(2,y) = 2-y^2 = 0$ if $y = \pm\sqrt{2}$

Check $f(2,-\sqrt{2}) = \dfrac{6-4\sqrt{2}}{3}$

$f(2,\sqrt{2}) = \dfrac{6+4\sqrt{2}}{3}$.

Compare with all other f-values now...

- Global min is $f(-2,2) = \frac{-26}{3}$
- Global max is $f(-2,-2) = 14/3$.

#3 Let $f(x,y,z)=x+2y-4z$. Minimize this over the unit sphere

$$\underbrace{x^2+y^2+z^2}_{g(x,y,z)}=1.$$

$$f_x = 1 = \lambda g_x = \lambda 2x \Rightarrow x = \frac{1}{2\lambda}$$

$$f_y = \lambda g_y \Rightarrow 2 = \lambda 2y \Rightarrow y = \frac{1}{\lambda}$$

$$f_z = \lambda g_z \Rightarrow -4 = 2\lambda z \Rightarrow z = \frac{-2}{\lambda}$$

$$\frac{1}{4\lambda^2}+\frac{1}{\lambda^2}+\frac{4}{\lambda^2}=1$$

$$\frac{1}{4}+1+4=\lambda^2=\frac{21}{4}$$

$$\lambda=\frac{\pm\sqrt{21}}{2}$$

$$f\left(\frac{1}{\sqrt{21}},\frac{2}{\sqrt{21}},\frac{-4}{\sqrt{21}}\right)=\sqrt{21}$$

$$f\left(\frac{-1}{\sqrt{21}},\frac{-2}{\sqrt{21}},\frac{4}{\sqrt{21}}\right)=-\sqrt{21}\quad (\text{MIN})$$