Welcome to MATH 2110Q

Multivariable Calculus

Our goal is to extend calculus tools to account for multiple dimensions. For example, the Navier-Stokes equations for an incompressible fluid with 3D in space + 1D in time are:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + \nabla p - \nu\Delta\vec{u} = \vec{f}$$

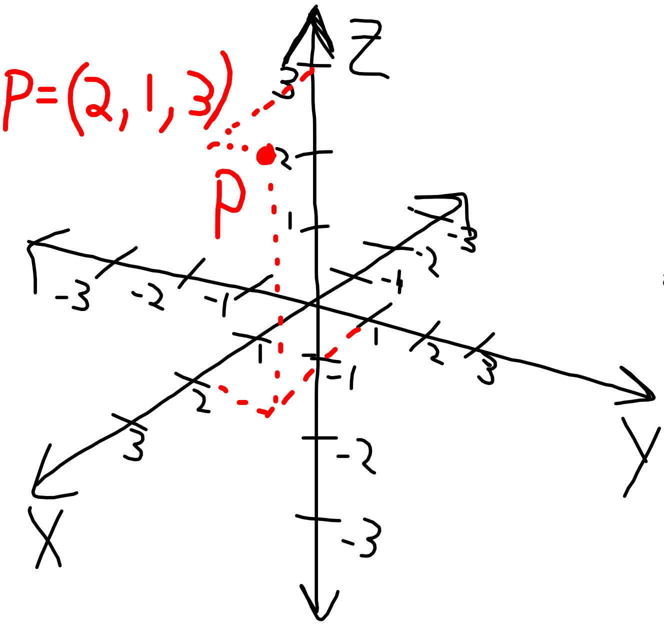
$$\nabla\cdot\vec{u} = 0$$

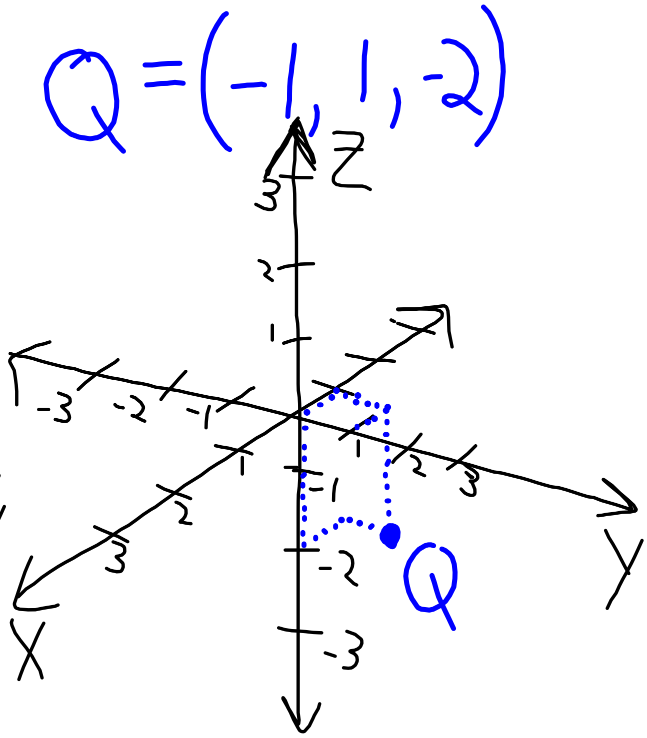
where $\vec{u} = \vec{u}(\underbrace{x,y,z}_{\text{space}},\underbrace{t}_{\text{time}})$ is a "vector".

You should be able to "read" these equations at the end of the course.

Let us stop for a bit and talk about the syllabus and other details of the course.

3D Coordinates

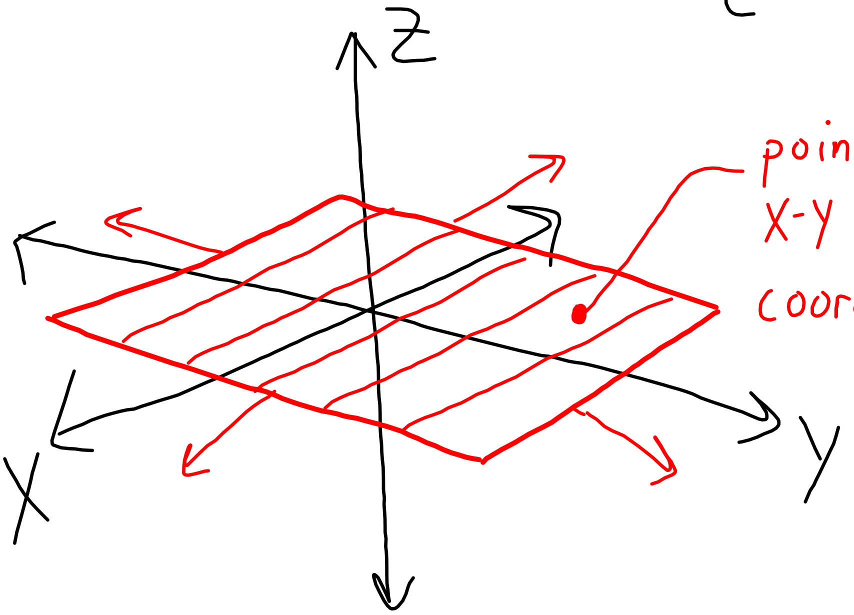




Note coordinates are always (x, y, z)

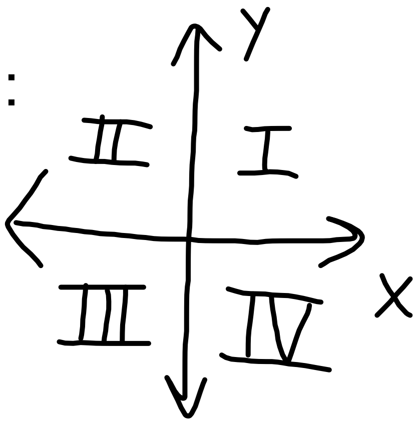
We can look at x-y, y-z, and x-z planes in 3D.

Notation... x-y plane is $\{(x,y,z) \mid z=0\}$.

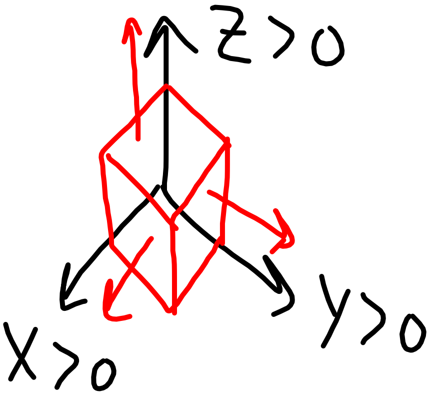


points in the x-y plane have coordinates $(x,y,0)$

Recall the four quadrants in 2D:



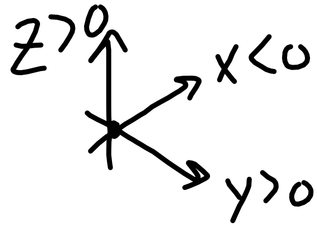
In 3D there are eight "octants". You only need to know the "first octant" by number:

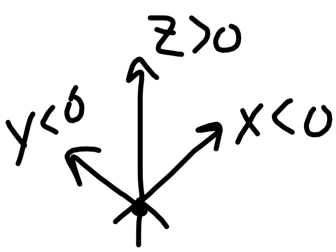


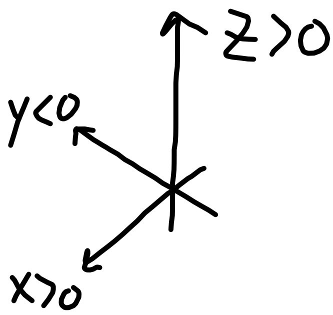
First octant

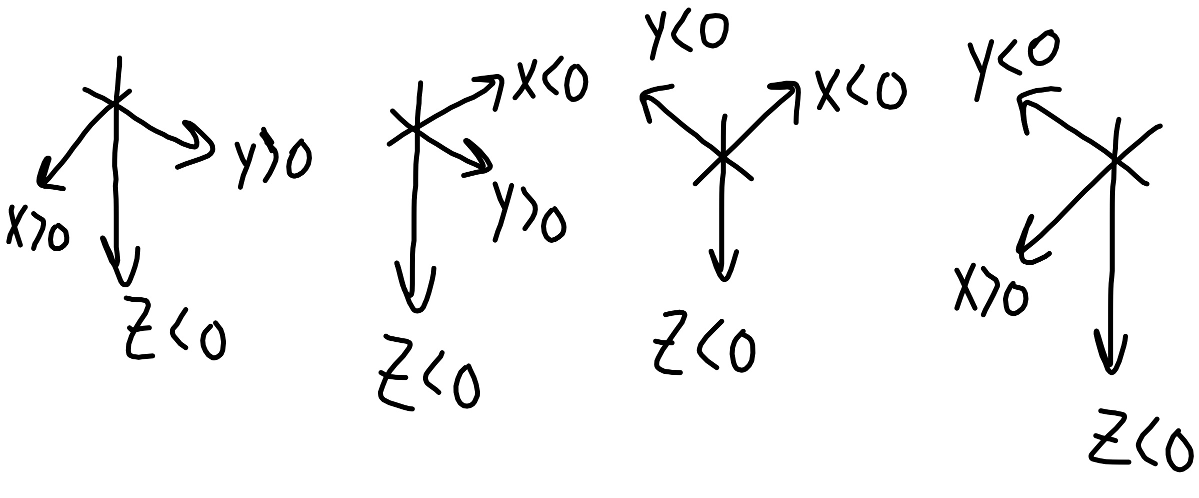
$$\{(x, y, z) \mid x \geq 0, y \geq 0, z \geq 0\}$$

Other octants:



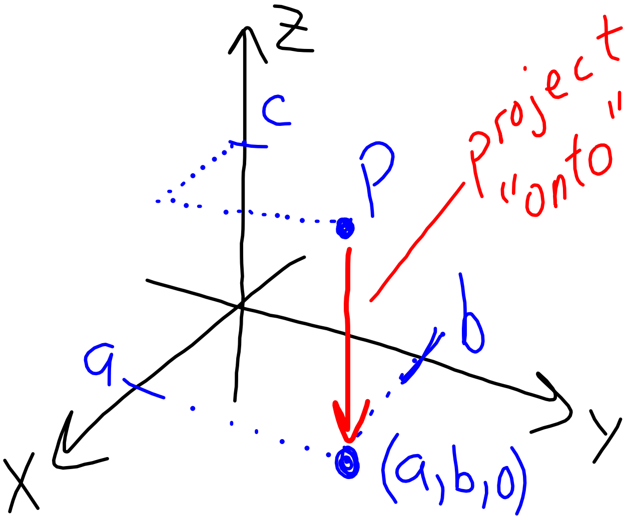






Projections: this concept can be generalized in many ways far outside the scope of this course. We will discuss "point" and "vector" projections.

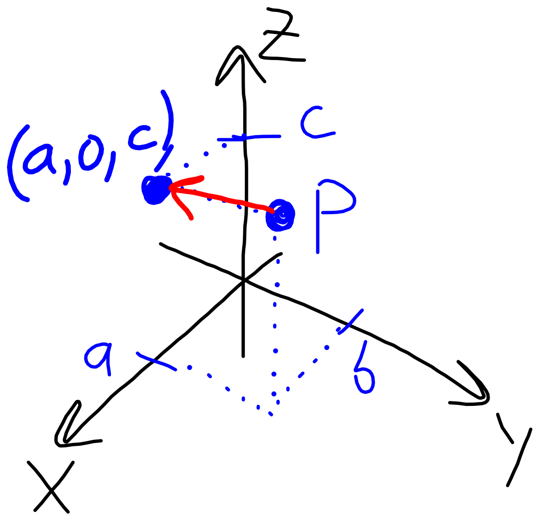
Projection of P=(a,b,c) onto the x-y plane is (a,b,0):



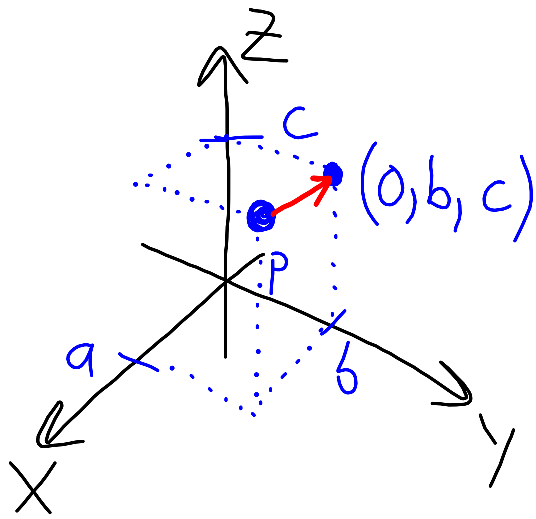
project P "onto" x-y plane.

* (a,b,0) is the point in the x-y plane that is "closest" to P.

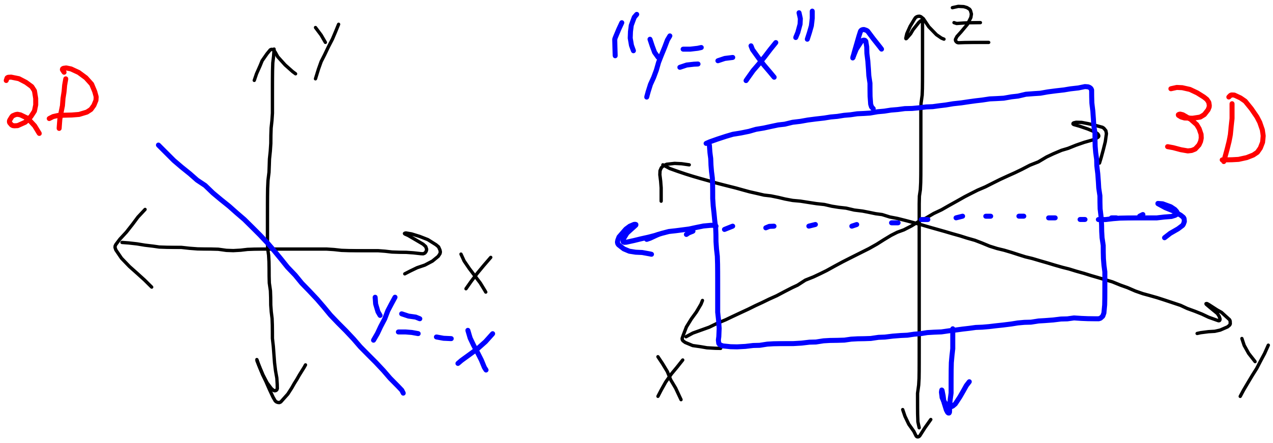
Projection of P onto x-z plane is (a,0,c)



Projection of P onto y-z plane is (0,b,c)

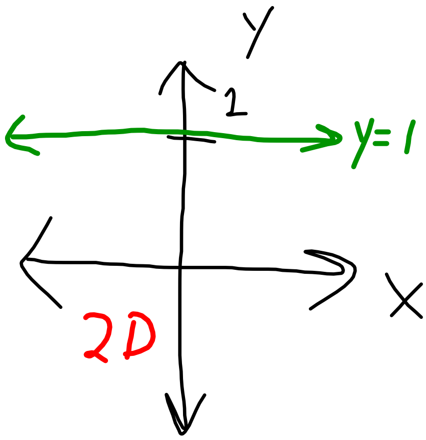


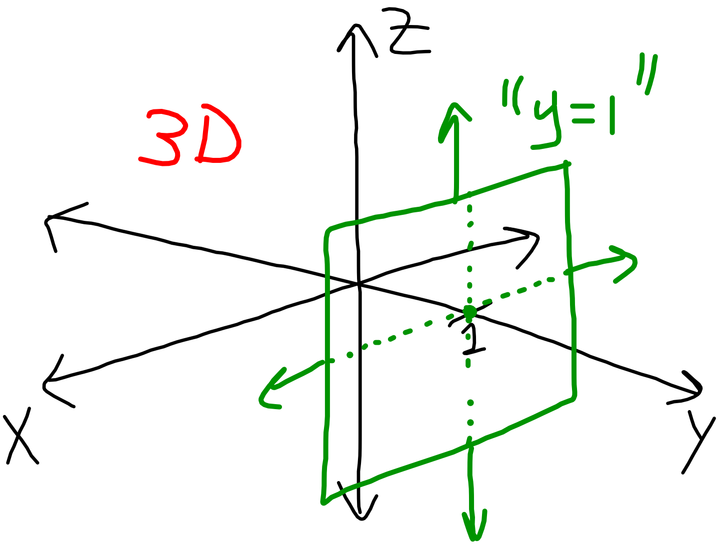
SURFACES. 2D lines become surfaces in 3D.



There is no restriction on z in 3D for the relationship y=-x, so ALL z-values are included: $\{(x,y,z) \mid y=-x\}$.

Another common example would be y=1.

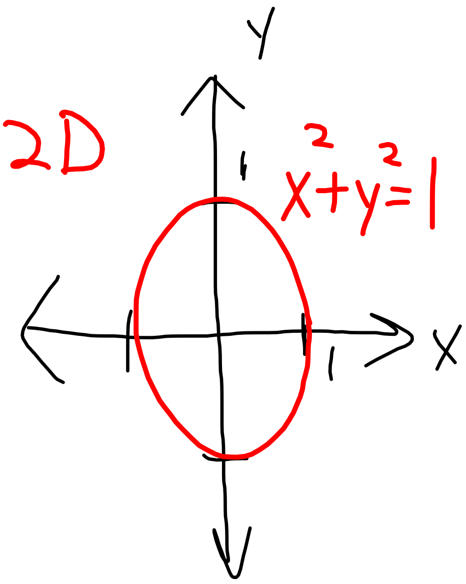


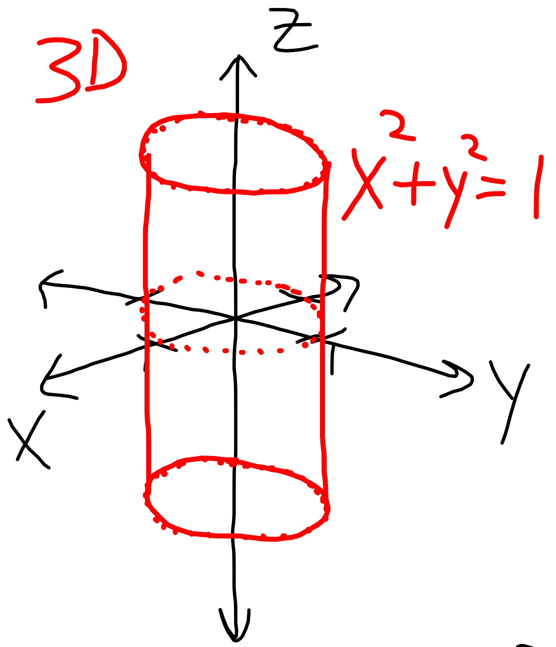


In 3D, the surface y=1 is a plane parallel to the x-z plane but intersecting the y-axis at y=1;

$$\{(x,y,z) \mid y=1\}.$$

A more complicated example is a CYLINDER.

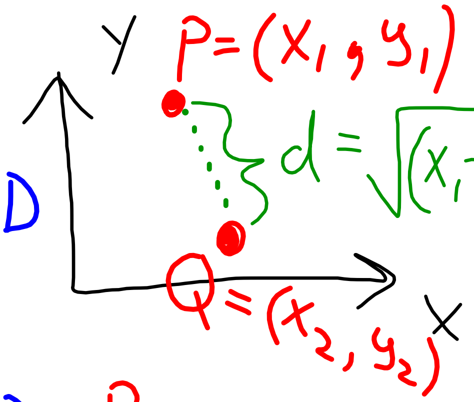




$$\{(x,y,z) \mid x^2+y^2=1\}.$$

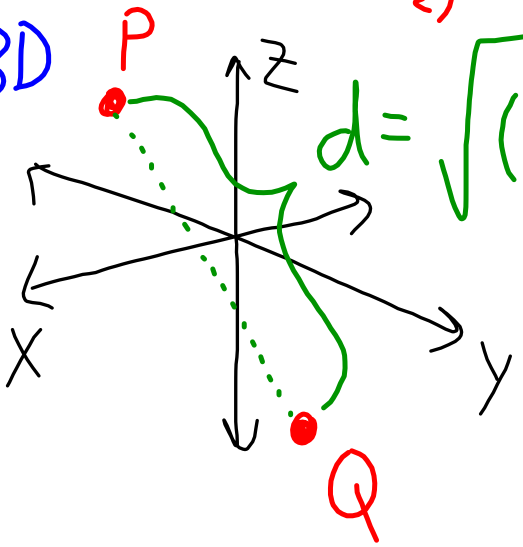
Distance between points.

2D



$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

3D



$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

$$P = (x_1, y_1, z_1)$$

$$Q = (x_2, y_2, z_2)$$

Note: sometimes we say $d = |PQ|$.

Ex. 1: Find $|PQ|$ if $P = (1, 2, 3)$ and $Q = (-1, 3, 4)$.

$$|PQ| = \sqrt{(1-(-1))^2 + (2-3)^2 + (3-4)^2}$$

$$= \sqrt{2^2 + (-1)^2 + (-1)^2} = \underline{\underline{\sqrt{6}}}.$$

Ex. 2: Find $|PQ|$ if $P=(1,2,3)$ & $Q=(1,2,0)$.

$$|PQ| = \sqrt{(1-1)^2+(2-2)^2+(3-0)^2}$$

$$= \sqrt{3^2} = 3.$$

Note Q is the projection of P onto the x-y plane here. The distance is just the size of the z-coordinate of P.

One useful idea is the set of all points that are EQUIDISTANT from a central point.

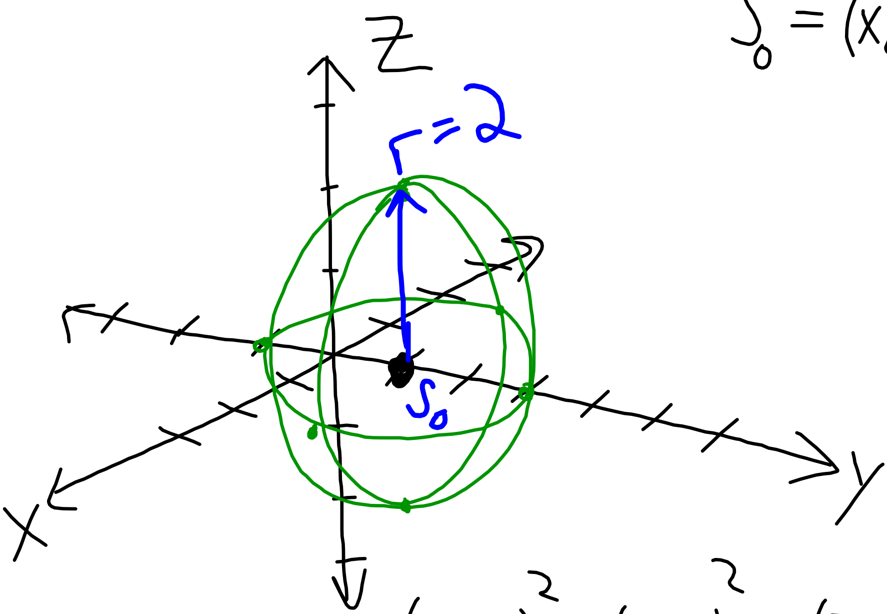
"Sphere of radius r"

S: sphere $\quad S_0 = (x_0, y_0, z_0)$: center

Then $S = \left\{ (x,y,z) \,\middle|\, \underbrace{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2}_{\text{"STANDARD FORM"}} \right\}$

*Note also $S = \left\{ P = (x,y,z) \,\middle|\, |PS_0| = r \right\}$.

$$S_0 = (x_0, y_0, z_0) = (0, 1, 0)$$



$$(x-0)^2 + (y-1)^2 + (z-0)^2 = 2^2 = 4$$

or $$x^2 + (y-1)^2 + z^2 = 4$$

Ex. 3: Put $x^2+y^2+z^2+3x+2z-4y=10$ into standard form.

Remember: "complete the square"...

$$\left(x+\frac{3}{2}\right)^2+(y-2)^2+(z+1)^2=10+\left(\frac{3}{2}\right)^2+(-2)^2+1^2$$

$$=10+\frac{9}{4}+4+1$$

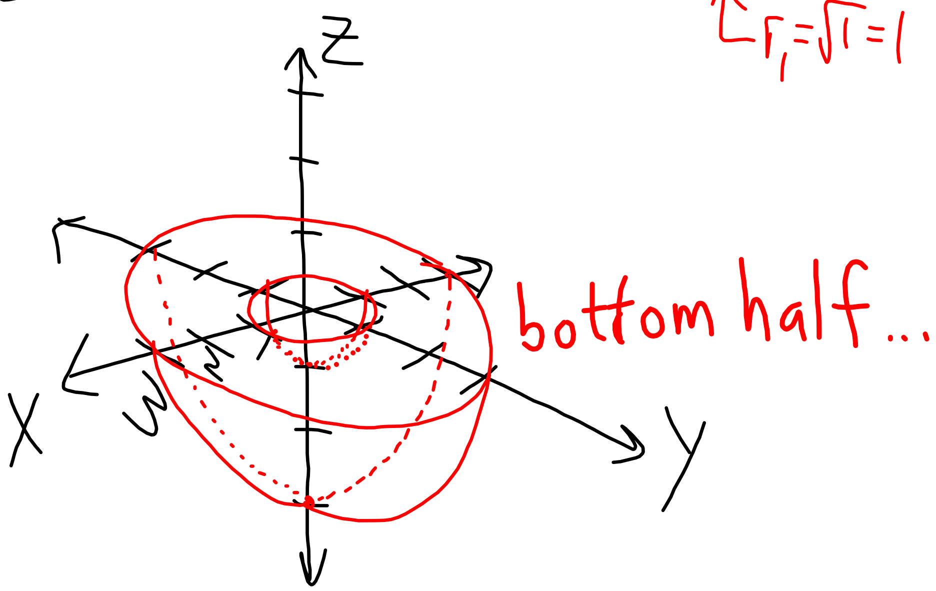
$$=\frac{40+9+16+4}{4}=\frac{69}{4}.$$

We can also envision things like spherical VOLUMES;

$$r_1 \leq \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} \leq r_2$$

Ex. 4: Sketch $V = \{(x,y,z) \mid 1 \leq x^2+y^2+z^2 \leq 9\}$.

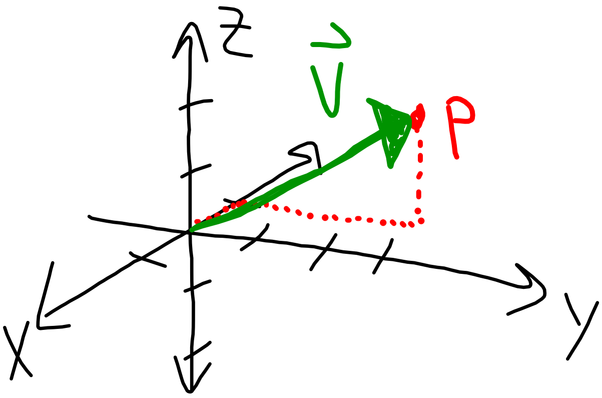
$r_1 = \sqrt{1} = 1$ $\qquad$ $r_2 = \sqrt{9} = 3$



VECTORS. These are defined by precisely two characteristics: (1) length and (2) magnitude.

$\vec{v} = \langle -1, 3, 2 \rangle$ (vector)

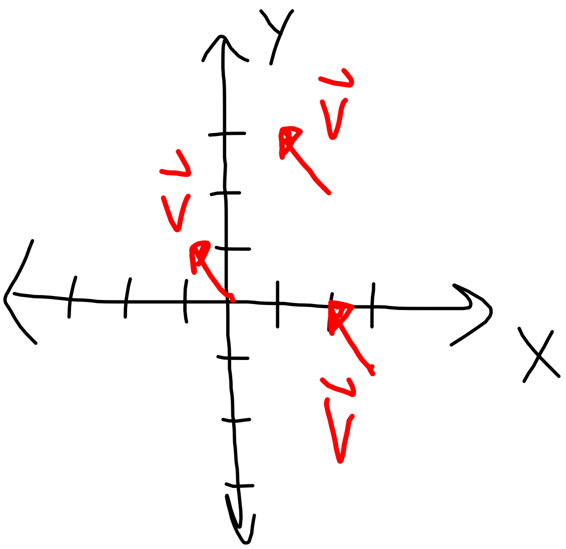
$P = (-1, 3, 2)$ (point)



One way to visualize $\vec{v}$.

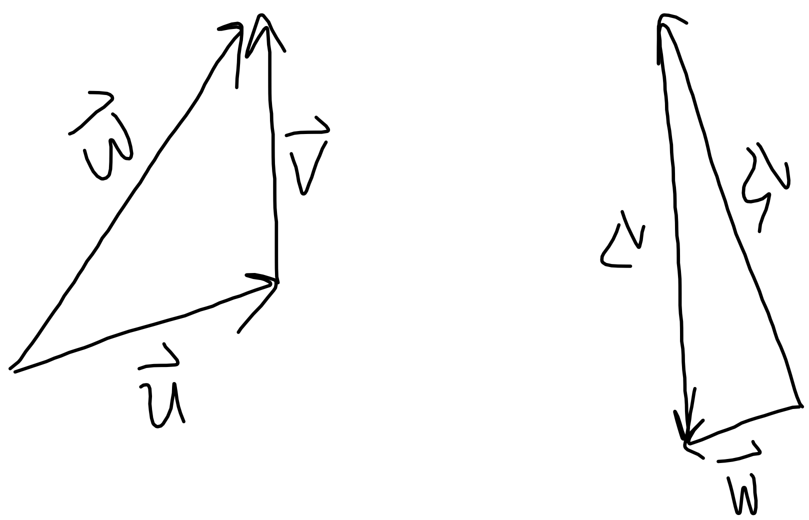
A vector can be VISUALIZED with the tail at any desired point; it is still the same vector.

<u>2D example</u> $\vec{v} = \langle -1, 1 \rangle$

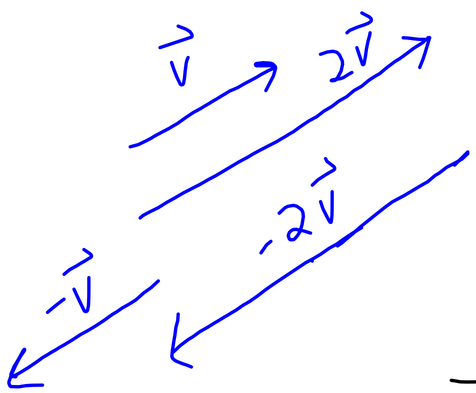


All of these are $\vec{v}$, just placed at 3 different points.

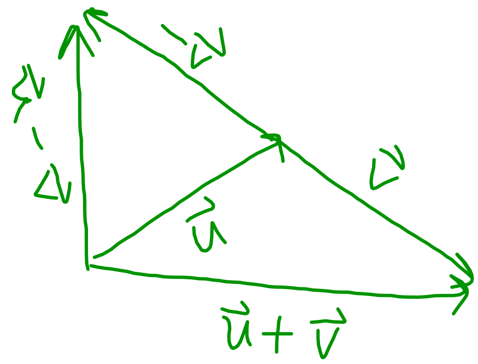
Vector addition. Put tail of $\vec{v}$ at head of $\vec{u}$ to make $\vec{w} = \vec{u} + \vec{v}$:



Multiplication by a scalar. Let $c \in \mathbb{R}$ and $\vec{v}$ a vector. Then $\vec{w} = c\vec{v}$ is also a vector. If c is negative then the direction flips.



This also gives a way to think of vector subtraction:



Here are the algebraic rules; things are simply done "component-wise".

$\vec{u} = \langle u_1, u_2, u_3 \rangle$
$\vec{v} = \langle v_1, v_2, v_3 \rangle$

Addition/subtraction:

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$$

$$\vec{u} - \vec{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$$

Scalar multiplication:

$$c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle.$$

$\underline{\text{Ex. 5}}$: $\vec{v} = \langle 1, 0, -5 \rangle$. Find $4\vec{v}$.

$$4\vec{v} = \langle 4\cdot 1, 4\cdot 0, 4\cdot(-5) \rangle = \langle 4, 0, -20 \rangle.$$

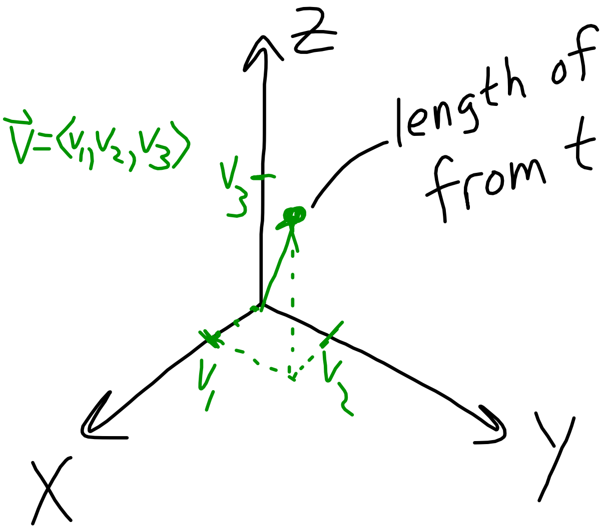
$\underline{\text{Ex. 6}}$: $\vec{u} = \langle -2, 7, -3 \rangle$, $\vec{v} = \langle 1, 2, 0 \rangle$.

Find $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$.

$$\vec{u}+\vec{v} = \langle -2+1, 7+2, -3+0 \rangle = \langle -1, 9, -3 \rangle.$$

$$\vec{u}-\vec{v} = \langle -2-1, 7-2, -3-0 \rangle = \langle -3, 5, -3 \rangle.$$

Magnitude of a vector.



length of $\vec{v}$ is the distance from the origin to point (v_1, v_2, v_3)

$|\vec{v}|$ = magnitude of $\vec{v}$

$$= \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}.$$

Ex. 7: Find $|\vec{v}|$ if $\vec{v} = \langle -1, 1, 2 \rangle$.

$$|\vec{v}| = \sqrt{(-1)^2 + (1)^2 + (2)^2} = \sqrt{6}.$$

Practice #1

Which of the following points is closest to the origin? $P=(1,-2,2)$ or $Q=(3,0,1)$?

For P, $d=\sqrt{1+4+4}=\sqrt{9}$

For Q, $d=\sqrt{9+0+1}=\sqrt{10}$

P is closer.

Practice #2

Find the distance from P=(5, -7, 4) to the y-z plane.

projection is $(0, -7, 4)$

$$d = \sqrt{(5-0)^2 + (-7-(-7))^2 + (4-4)^2} = 5$$

Practice #3.

Find an equation for a sphere with center (0,-1,2) and radius 7.

$$(x-0)^2+(y-(-1))^2+(z-2)^2=7^2$$

$$\Rightarrow x^2+(y+1)^2+(z-2)^2=49.$$

Practice #4.

Find the center and radius of the sphere

$$x^2-2x+y^2+2y+z^2-6z=-10.$$

$$(x-1)^2+(y+1)^2+(z-3)^2=-10+3^2+1^2+1^2$$

$$=-10+9+2=1=r^2$$

So $r=1$, center $(1,-1,3)$.

Practice #5.

If $\vec{u} = \langle 5, -1 \rangle$ and $\vec{v} = \langle 3, 4 \rangle$, find $|\vec{u} - \vec{v}|$.

$$\vec{u} - \vec{v} = \langle 2, -5 \rangle$$

$$\Rightarrow |\vec{u} - \vec{v}| = \sqrt{2^2 + 5^2} = \sqrt{4 + 25} = \sqrt{29}.$$