MATH 2110 - Review for Exam 4

Line integrals

$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

$$\int_C \vec{F}\cdot d\vec{s} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$\vec{F} = \langle P, Q, R\rangle \Rightarrow \int_C \vec{F}\cdot d\vec{s} = \int_C P\,dx + Q\,dy + R\,dz.$$

Ex: Let C be the curve $y = x^5/5$, with initial point $(0,0)$ and end point $(1, 1/5)$. Calculate $\int_C x^2 y \, ds$.

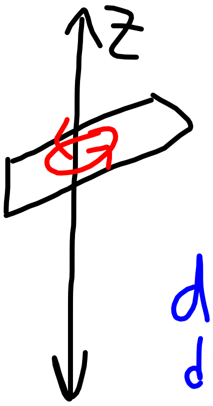
Use the natural parameterization $y = y(x)$ with $0 \le x \le 1$. Our formula is

$$\int_C x^2 y \, ds = \int_0^1 x^2 \cdot \frac{x^5}{5} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \frac{1}{5} \int_0^1 x^7 \sqrt{1 + x^8} \, dx$$

$$= \frac{1}{5} \int_0^1 \frac{d}{dx}\left\{\frac{1}{12}\left(1+x^8\right)^{3/2}\right\} dx$$

$$= \frac{1}{5} \cdot \frac{1}{12} \cdot \left(2\sqrt{2}-1\right) = \boxed{\frac{2\sqrt{2}-1}{60}}.$$

EX: Find $\int_C 2dx + ydy - dz$, if C is the intersection of the cylinder $x^2+y^2=4$ and the plane $z=1+x+y$, oriented counter-clockwise when viewed from "above" (+z-axis).



$x = 2\cos(t)$

$y = 2\sin(t)$

$z = 1 + x + y = 1 + 2(\cos(t) + \sin(t))$

$dx = -2\sin(t)\,dt$

$dy = 2\cos(t)\,dt$

$dz = 2(\cos(t) - \sin(t))\,dt$

$$\int_C 2\,dx + y\,dy - dz$$

$$= \int_0^{2\pi} \Big[\underbrace{-4\cancel{\sin(t)}}_{0} + \underbrace{4\cos(t)\sin(t)} \overbrace{- 2\cancel{\cos(t)} + 2\cancel{\sin(t)}}^{0} \Big]\, dt$$

$u = \sin(t)$

$du = \cos(t)\,dt$

$$= 4 \int_{u=0}^{u=\sin(2\pi)=0} u\,du = \boxed{0}.$$

Well, this was a conservative field and a closed curve...

Conservative vector fields

* Check if $\vec{F} = \nabla f$ by verifying $\nabla \times \vec{F} = 0$ (3D) or $\vec{F} = \langle P, Q \rangle \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.

* Know the procedure to find f.

* "Fundamental Theorem"

$$\vec{F} = \nabla f \Rightarrow \int_C \vec{F} \cdot d\vec{s} = f(B) - f(A).$$

B: end-point on C

A: starting point on C.

EX: Let $\vec{F} = \langle ye^{yx} - 1 + \frac{3}{2}y^2\sqrt{x},\ y + xe^{yx} + 2yx^{3/2} \rangle$.
Is $\vec{F}$ conservative? If so, find the potential function $f(x, y)$.

$P = ye^{yx} - 1 + \frac{3}{2}y^2\sqrt{x}$, $Q = y + xe^{yx} + 2yx^{3/2}$

$$\left.\begin{aligned} Q_x &= e^{yx} + xye^{yx} + 3y\sqrt{x} \\ P_y &= e^{yx} + xye^{yx} + 3y\sqrt{x} \end{aligned}\right\} \begin{aligned} &Q_x = P_y \\ &\Rightarrow \text{conservative.} \end{aligned}$$

$$\int P\,dx = \int ye^{yx} - 1 + \frac{3}{2}y^2\sqrt{x}\,dx$$
$$= e^{yx} - x + y^2 x^{3/2} + g(y) = f(x,y).$$

Set y-derivative = $\vec{j}$-component, Q

$$Q = y + xe^{yx} + 2yx^{3/2} = \frac{\partial}{\partial y}\left(e^{yx} - x + y^2 x^{3/2} + g(y)\right)$$
$$= xe^{yx} + 2yx^{3/2} + g'(y)$$

So $y = g'(y) \Rightarrow \frac{1}{2}y^2 = g(y)$. Plug in above to get

$$\boxed{f(x,y) = e^{yx} - x + y^2 x^{3/2} + \frac{1}{2}y^2}.$$

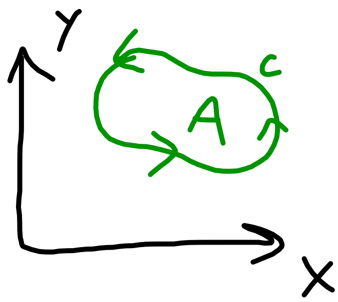
EX: Given $f(x,y,z) = x - y^2 + zy$, find $\int_C \nabla f \cdot d\vec{r}$, where $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ for $0 \le t \le \pi/2$ parameterizes C.

Note $\vec{r}(0) = \langle 1, 0, 0 \rangle$ & $\vec{r}(\frac{\pi}{2}) = \langle 0, 1, \pi/2 \rangle$.

Thus (Fundamental Theorem)

$$\int_C \nabla f \cdot d\vec{r} = f(0,1,\tfrac{\pi}{2}) - f(1,0,0) = -1 + \tfrac{\pi}{2} - 1 = \boxed{\frac{\pi}{2} - 2}.$$

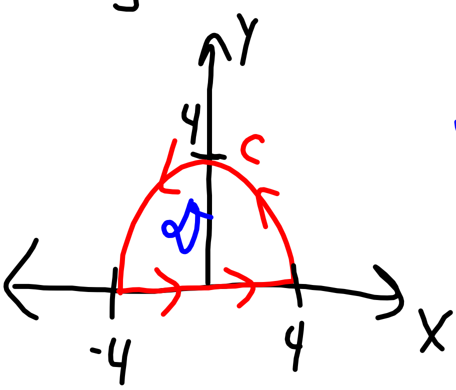
Green's Theorem



$$\int_C P\,dx + Q\,dy = \int_C \vec{F}\cdot d\vec{s}$$

$$= \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

EX: Let C be the boundary of the region $\{(x,y) \mid x^2+y^2 \le 16,\ y \geq 0\}$, with positive orientation. Find $\int_C y\,dx + xy\,dy$, using Green's Theorem.



$$\int_C P\,dx + Q\,dy = \iint_D Q_x - P_y\,dA$$

$$= \iint_D y - 1\,dA = \int_0^{\pi}\int_0^4 (r\sin\theta - 1)\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{4} r^2 \sin\theta - r \, dr \, d\theta = \int_0^{\pi} \frac{1}{3}\cdot 4^3 \sin\theta - \frac{1}{2}\cdot 4^2 \, d\theta$$

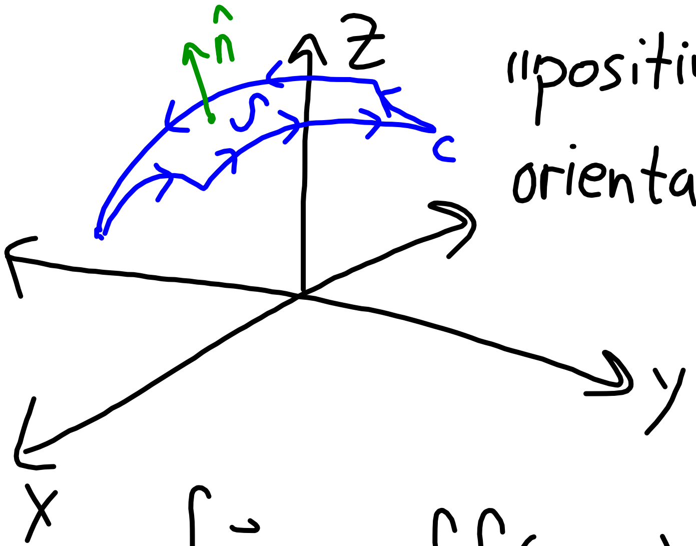
$$= \frac{64}{3}\left(-\cos\theta\right)\Big|_0^{\pi} - 8\pi$$

$$= \boxed{\frac{128}{3} - 8\pi}.$$

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\left|\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right| du\,dv$$

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_D \vec{F}(\vec{r}(u,v))\cdot\left(\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right) du\,dv$$

Stoke's Theorem



"positive" orientation

$$\int_{C=\partial S} \vec{F}\cdot d\vec{s} = \iint_S (\nabla \times \vec{F})\cdot \hat{n}\, dS .$$

<u>Ex</u>: Let S be the section of the surface $z = 1 + y^2$ that lies over $D = \{(x, y) \mid -1 \le x \le 1, -1 \le y \le 1\}$.
Given $\vec{F} = \langle y, yz^2, z^4 y^2 \rangle$, find $\int \vec{F} \cdot d\vec{s}$ if ∂S has positive orientation with respect to the upward-pointing normal on S.

Line integral requires 4 pieces, so let's apply Stoke's.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & yz^2 & y^2z^4 \end{vmatrix} = \langle 2yz^4 - 2yz, 0, -1 \rangle.$$

Recall $\vec{n} = \langle -z_x, -z_y, 1 \rangle = \langle 0, -2y, 1 \rangle = \vec{r}_x \times \vec{r}_y$

if we choose $\vec{r}(x,y) = \langle x, y, 1+y^2 \rangle$.

We get

$$\int_C \vec{F}\cdot d\vec{s} = \int_{-1}^{1}\int_{-1}^{1} (\nabla\times\vec{F})\cdot(\vec{r}_x \times \vec{r}_y)\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} \langle 2yz^4 - 2yz, 0, -1\rangle \cdot \langle 0, -2y, 1\rangle\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} -1\,dy\,dx = \boxed{-4}.$$

Divergence Theorem

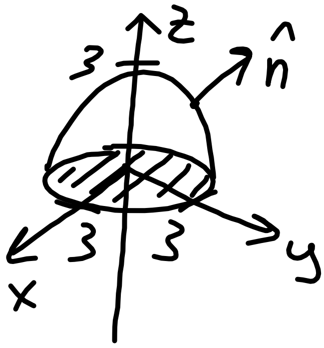
$$\iint_{\partial V} \vec{F}\cdot\hat{n}\, dS = \iiint_V \nabla\cdot\vec{F}\, dV$$

where $\hat{n}$ points "outward".

Ex: Let S be the surface of the region

$V = \{(x,y,z) \mid x^2+y^2+z^2 \le 9,\ z \geq 0\}$, $\hat{n}$ outward,

$\vec{F} = \langle x, y, z^2 \rangle$. Find $\iint_{\partial V} \vec{F}\cdot\hat{n}\, dS$.



Surface integral requires two pieces and is tedious. Use Divergence Theorem:

$$\iint_{\partial V} \vec{F}\cdot\hat{n}\, dS = \int\int_V\int \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z^2)\, dV$$

$$= \iiint_V 2(1+z)\,dV = 2\underbrace{|V|}_{\frac{4}{3}\pi\cdot 3^3} + 2\underbrace{\iiint_V z\,dV}_{\text{use spherical coordinates}}.$$

$$= 36\pi + 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^3 \rho\cos\phi\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 36\pi + 2\int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi\cos\phi\, d\phi \int_0^3 \rho^3\, d\rho$$

$$= 36\pi + 4\pi \left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/2} \left[\frac{1}{4}\rho^4\right]_0^3$$

$$= 36\pi + 4\pi \frac{1}{2}\cdot\frac{1}{4}\cdot 3^4 = 36\pi + \frac{81\pi}{2}$$

$$= \boxed{\frac{153\pi}{2}}.$$