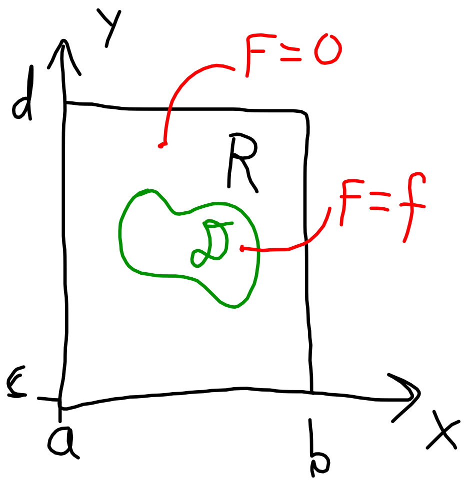
MATH 2110Q Exam 3 Review

Integration over general regions.

Let $f(x,y)$ have domain $\mathcal{D}$ and define



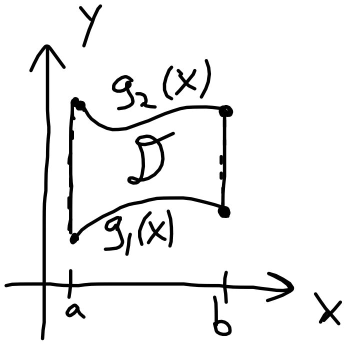
$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ in } \mathcal{D} \\ 0, & \text{otherwise} \end{cases}.$$

Then

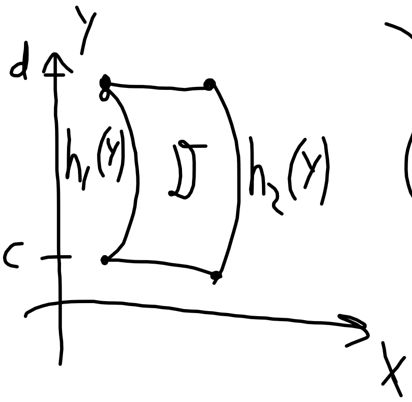
$$\iint_{\mathcal{D}} f(x,y)\,dA = \underbrace{\iint_{R} F(x,y)\,dA}.$$

This can be defined using double Riemann sums, same as before.

We often identify domains as being bounded by certain curves.

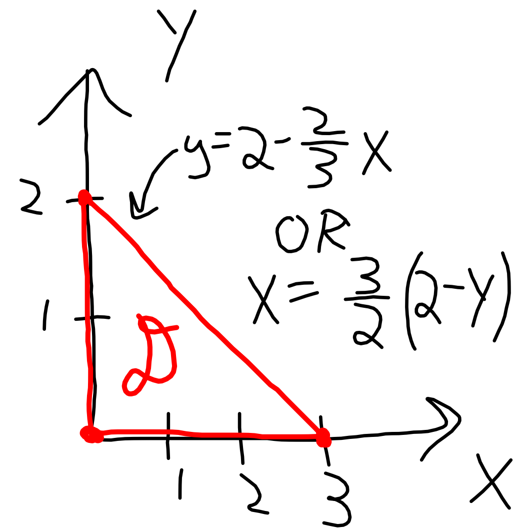


$$\iint_D f(x,y)\,dA = \int_a^b \underbrace{\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy}_{\text{depends on } x}\,dx .$$



$$\iint_D f(x,y)\,dA = \int_c^d \underbrace{\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx}_{\text{depends on } y}\,dy .$$

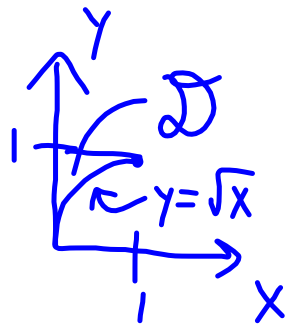
Sometimes we want to switch the order of integration (Fubini)



$$\iint_D f(x,y)\,dA = \int_0^3 \int_0^{2-\frac{2}{3}x} f(x,y)\,dy\,dx$$

$$= \int_0^2 \int_0^{\frac{3}{2}(2-y)} f(x,y)\,dx\,dy$$

Ex: Find the volume of the region bounded by the following surfaces: $x=0$, $y=\sqrt{x}$, $y=1$, $z=0$, $z=\sqrt{x+y^2}$.



Surface $z=\sqrt{x+y^2}$ lies above $\mathcal{D}$.

$$V=\int_0^1\int_{\sqrt{x}}^1 \sqrt{x+y^2}\,dy\,dx$$

? switch order...

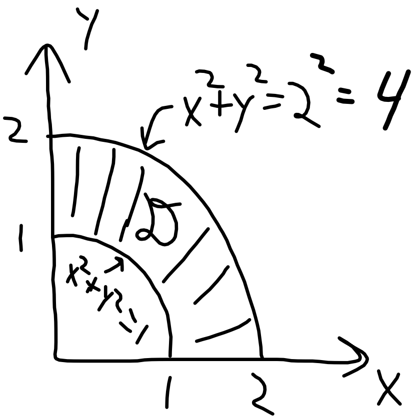
$$V=\int_0^1\int_0^{y^2}\sqrt{x+y^2}\,dx\,dy=\int_0^1\int_0^{y^2}\frac{\partial}{\partial x}\left(\frac{2}{3}(x+y^2)^{3/2}\right)dx\,dy$$

$$=\int_0^1 \frac{2}{3}(x+y^2)^{3/2}\Big|_{x=0}^{x=y^2}\,dy$$

$$=\frac{2}{3}\int_0^1 2^{3/2}(y^2)^{3/2}-(y^2)^{3/2}\,dy=\frac{2}{3}\int_0^1\left(2^{3/2}-1\right)y^3\,dy$$

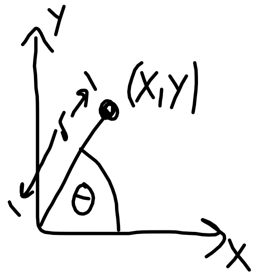
$$=\frac{2}{3}\left(2^{3/2}-1\right)\frac{1}{4}y^4\Big|_0^1=\boxed{\frac{1}{6}\left(2^{3/2}-1\right)}.$$

Polar coordinates



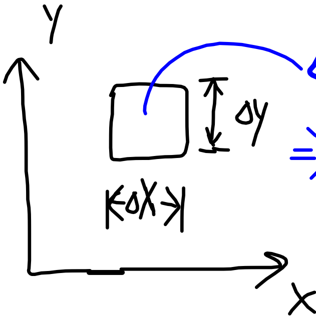
This should be handled using polar coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$



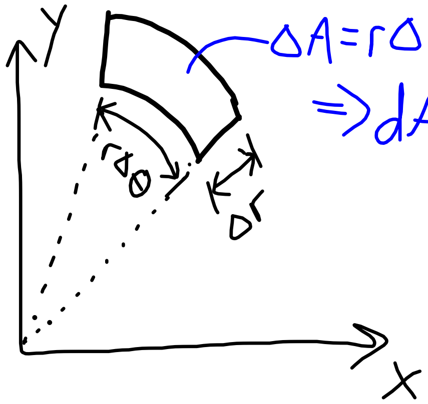
$$\int_{r=1}^{r=2}\int_{\theta=0}^{\theta=\pi/2} f\underbrace{(x,y)}_{\substack{x=r\cos\theta\\ y=r\sin\theta}}\,\underbrace{dA}_{\text{need to discuss this further}} = ?$$

Differential area in polar coordinates



$\Delta A = \Delta x \Delta y$

$\Rightarrow dA = dx\,dy$



$\Delta A = r \Delta r \Delta \theta$

$\Rightarrow dA = r\,dr\,d\theta$

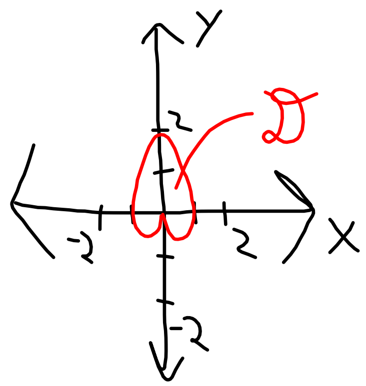
Back to the integral...

$$\int_{r=1}^{r=2} \int_{\theta=0}^{\theta=\pi/2} f(r\cos\theta, r\sin\theta)\, r \, dr\, d\theta = \iint_{\mathcal{D}} f(x,y)\, dA.$$

don't forget!

Ex: Let D be the region bounded by the cardiod $r = 1+\sin\theta$. Find

$$\iint_D y\,dA.$$



$$= \int_0^{2\pi} \int_0^{1+\sin\theta} r\sin\theta \, r\,dr\,d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \sin\theta(1+\sin\theta)^3 d\theta = \frac{1}{3}\int_0^{2\pi} \sin\theta\left(1+3\sin\theta+3\sin^2\theta+\sin^3\theta\right)d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \sin\theta + 3\sin^2\theta + 3\sin^3\theta + \sin^4\theta \quad d\theta$$

$$= \int_0^{2\pi} \sin^2\theta\, d\theta + \frac{1}{3}\int_0^{2\pi} \sin^4\theta\, d\theta \quad \Big\} \int_0^{2\pi} \sin^n\theta\, d\theta = 0 \text{ if } n \geq 1 \text{ is odd}$$

$$= \pi + \frac{1}{3}\,\frac{1}{4}\int_0^{2\pi} \big(1 - \cos(2\theta)\big)^2\, d\theta \quad \Big\} \sin^2\theta = \frac{1}{2}\big(1 - \cos(2\theta)\big)$$

$$= \pi + \frac{1}{12}\int_0^{2\pi} 1 - 2\cos(2\theta) + \cos^2(2\theta)\; d\theta \qquad \cos^2(2\theta) = \frac{1}{2}\big(1 + \cos(4\theta)\big)$$

$$= \pi + \frac{1}{12}\int_0^{2\pi} 1 - 2\cos(2\theta) + \frac{1}{2} + \frac{1}{2}\cos(4\theta)\; d\theta$$

$$= \pi + \frac{1}{12}\left[\frac{3}{2}\theta - \sin(2\theta) + \frac{1}{8}\sin(4\theta)\right]_0^{2\pi}$$

$$= \pi + \frac{3\pi}{12} \qquad = \boxed{\frac{5\pi}{4}}$$

Surface area is a double integral

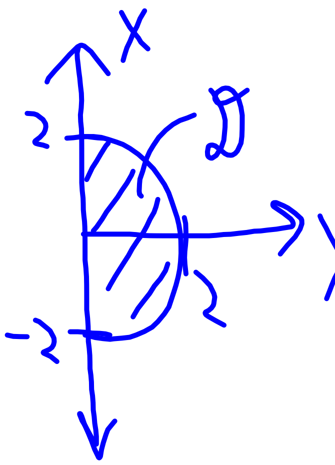
$$S = \iint_{D} \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA$$

for surface $z = f(x,y)$ defined for domain D.

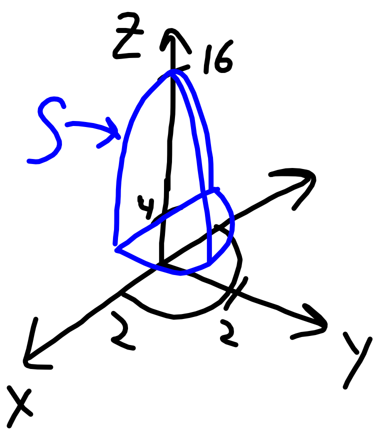
EX: Find the surface area of the portion of the surface $z=16-3x^2-3y^2$ above $z=4$ such that $y \geqslant 0$.

$$z=4=16-3x^2-3y^2 \Rightarrow 3(x^2+y^2)=12$$

$$\Rightarrow x^2+y^2=4$$



Surface lies above this region



$z = 16 - 3x^2 - 3y^2$

$z_x = -6x \quad z_y = -6y$

$1 + z_x^2 + z_y^2 = 1 + 36(x^2+y^2)$

<u>POLAR</u> : $= 1 + 36r^2$

$$\Rightarrow S = \int_0^{\pi}\int_0^2 \left(\sqrt{1+36r^2}\right) r\,dr\,d\theta = \left(\int_0^{\pi} d\theta\right)\left(\int_0^2 r\sqrt{1+36r^2}\,dr\right)$$

$u = 1 + 36r^2$
$du = 72r\,dr$

$$= \frac{\pi}{72}\int_1^{145} u^{1/2}\,du = \frac{\pi}{72}\,\frac{2}{3}\,u^{3/2}\Big|_1^{145} = \underline{\frac{\pi}{108}\left(145^{3/2} - 1\right)}.$$

Moments

Given density $\rho(x,y)$,

$$M_x = \iint_D y\rho \, dA$$

Moment about x-axis

$$M_y = \iint_D x\rho \, dA$$

Moment about y-axis

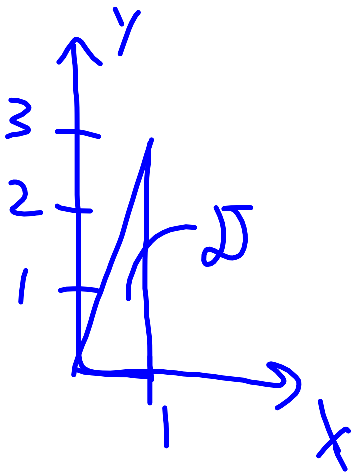
Center of mass

$$\bar{x} = \frac{M_y}{m}.$$

$$\bar{y} = \frac{M_x}{m}.$$

$$m = \iint_D \rho \, dA.$$

EX: Find the center of mass of a solid on the region $\mathcal{D}$ bounded by $y=0$, $y=3x$, $x=1$ if $\rho(x,y)=x+2y$.



$$m=\int_0^1\int_0^{3x} x+2y\,dy\,dx=\int_0^1\left[xy+y^2\right]_0^{3x}dx$$

$$=\int_0^1 3x^2+9x^2\,dx=12\int_0^1 x^2\,dx=\frac{12}{3}=4.$$

m

Need $M_x, M_y \ldots$

$$M_x = \int_0^1 \int_0^{3x} y(x+2y)\,dy\,dx = \int_0^1 \int_0^{3x} yx + 2y^2\,dy\,dx$$

$$= \int_0^1 \left[\frac{x}{2}y^2 + \frac{2}{3}y^3\right]_{y=0}^{y=3x} dx = \int_0^1 \frac{9}{2}x^3 + 18x^3\,dx$$

$$= \frac{45}{2}\int_0^1 x^3\,dx = \frac{45}{8}.$$

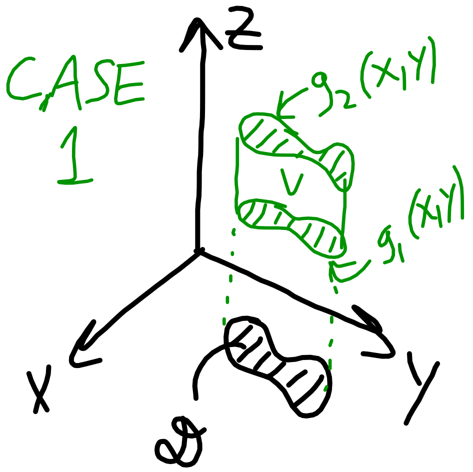
$$M_y = \int_0^1 \int_0^{3x} (x^2 + 2xy)\,dy\,dx = \int_0^1 \left[x^2 y + xy^2\right]_0^{3x} dx$$

$$= \int_0^1 3x^3 + 9x^3 \, dx = 12 \int_0^1 x^3 \, dx = \frac{12}{4} = 3.$$

Thus, $m = 4$, $M_x = \frac{45}{8}$, $M_y = 3$

$$\Rightarrow \boxed{\begin{aligned} \bar{x} &= \frac{M_y}{m} = \frac{3}{4} \\ \bar{y} &= \frac{M_x}{m} = \frac{45}{32}. \end{aligned}}$$

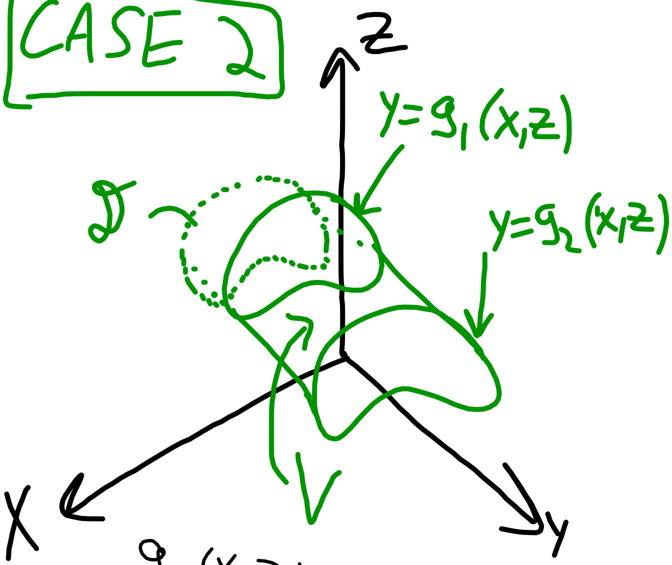
Integration over domains bounded between surfaces; there are three cases.



- Domain of f is $V \subset \mathbb{R}^3$
- Domain of g_1, g_2 is $\mathcal{D} \subset \mathbb{R}^2$

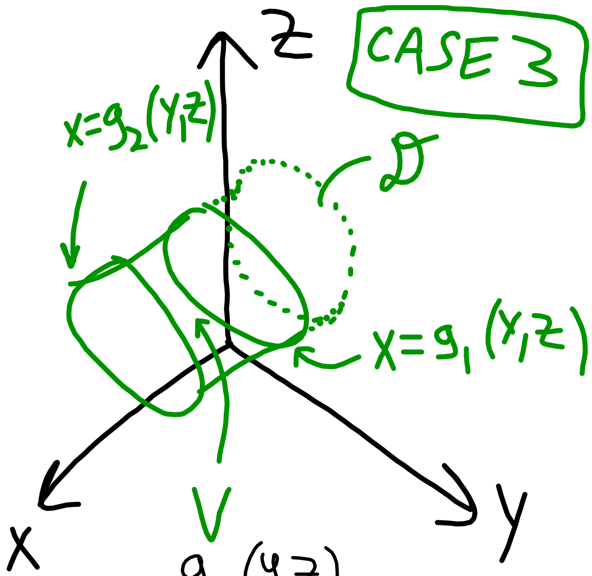
$$\iiint_V f \, dV = \iint_{\mathcal{D}} \left[\int_{g_1(x,y)}^{g_2(x,y)} f \, dz \right] dx\,dy.$$

CASE 2



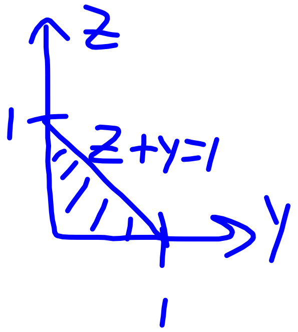
$$\iint_{\mathcal{D}} \int_{g_1(x,z)}^{g_2(x,z)} f(x,y,z)\,dy\,dx\,dz$$

CASE 3

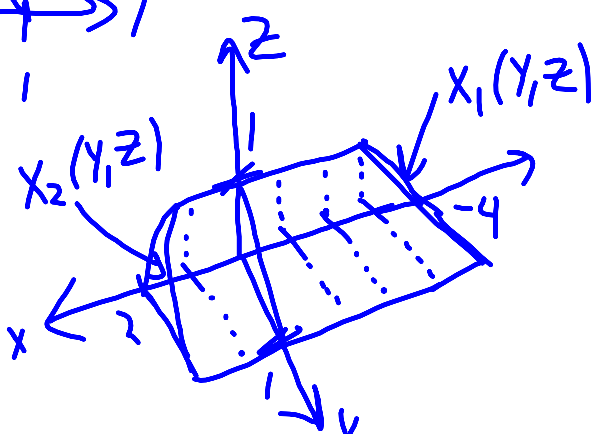


$$\iint_{\mathcal{D}} \int_{g_1(y,z)}^{g_2(y,z)} f(x,y,z)\,dx\,dy\,dz$$

EX: Find the volume of the region bounded by $\underbrace{y=0,\ z=0,\ z+y=1}$, $\underbrace{x=2-z^2,\ x+4=z^2}$.



Think $x_1(y,z)$ & $x_2(y,z)$ bounding surfaces.



$$V = \int_0^1 \int_0^{1-y} \int_{z^2-4}^{2-z^2} dx\,dz\,dy = \int_0^1 \int_0^{1-y} 2 - z^2 - z^2 + 4 \, dz\,dy$$

$$= \int_0^1 \int_0^{1-y} 6 - 2z^2 \, dz\,dy = \int_0^1 6(1-y) - \frac{2}{3}(1-y)^3 \, dy$$

$$= \int_0^1 6 - 6y - \frac{2}{3}(1-y)(1-2y+y^2) \, dy$$

$$= \int_0^1 6 - 6y - \frac{2}{3}\left(1 - 3y + 3y^2 - y^3\right) dy$$

$$= \int_0^1 \frac{16}{3} - 4y - 2y^2 + \frac{2}{3}y^3 \, dy$$

$$= \frac{16}{3} - 2 - \frac{2}{3} + \frac{1}{6} = \frac{32 - 12 - 4 + 1}{6}$$

$$= \boxed{\frac{17}{6}}$$

$$\iint_S f(x,y)\,dx\,dy = \iint_A f\big(x(u,v),y(u,v)\big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv.$$

$$S \xleftarrow{\;T:A\to S\;} A$$

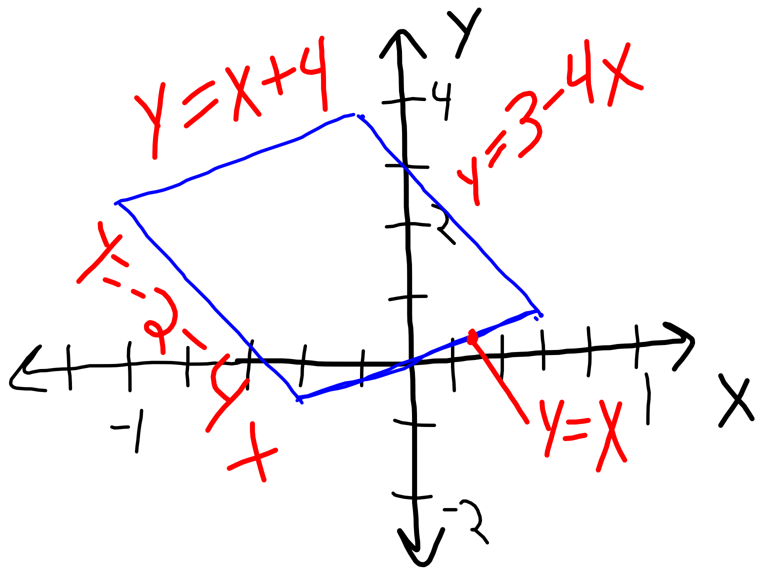
"Change of variables formula"

*Notation: $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = x_u y_v - x_v y_u$

is called the "Jacobian of T"

Transforming domains is important.

EX: A tranformation $(u,v) = (4x+y,\ x-y-3)$ maps a region R in xy-space, shown below, to a region S in uv-space. Find S.



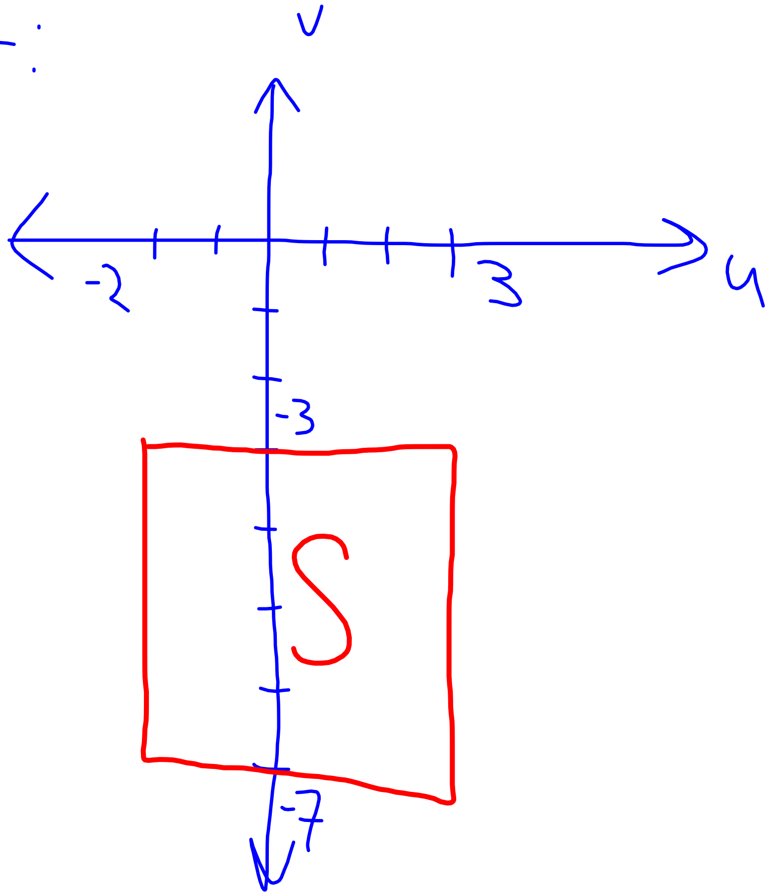
"Intuitive" method : just manipulate the equations.

$$y=3-4x \Rightarrow \overbrace{4x+y}^{u}=3 \quad =u$$

$$y=-2-4x \Rightarrow 4x+y=-2 \quad =u$$

$$\left.\begin{array}{l} y=x \Rightarrow x-y=0 \\ y=x+4 \Rightarrow x-y=-4 \end{array}\right\} x-y=v+3 \left\{\begin{array}{l} v+3=0 \rightarrow v=-3 \\ v+3=-4 \rightarrow v=-7 \end{array}\right.$$

Answer:



The second approach is based on a more general procedure:

> Given $u=u(x,y)$, $v=v(x,y)$ and a curve $g(x,y)=0$ in xy-space.

> We can <u>convert</u> to a curve in uv-space in two steps:

(1) invert ... $x=x(u,v)$, $y=y(u,v)$

(2) Insert into g: $g(x(u,v),y(u,v))$.

Apply to previous example:

Step 1: $u = 4x + y \Rightarrow y = u - 4x$

$v = x - y - 3 \Rightarrow \underbrace{x - (u - 4x) - 3}_{v = 5x - u - 3} = v$

$\Rightarrow x = \frac{1}{5}(v + u + 3)$

$y = u - \frac{4}{5}(v + u + 3)$

$\Rightarrow y = \frac{1}{5}(u - 4v - 12)$

Step 2: Convert the boundary of R.

$$\boxed{y=3-4x} \Rightarrow \frac{1}{5}(u-4v-12)=3-\frac{4}{5}(v+u+3)$$

$$\Rightarrow u-4v-12=15-4v-4u-12$$

$$\Rightarrow 5u=15 \Rightarrow \boxed{u=3}$$

Now repeat for the other 3 edges of R.
The same answer is achieved as before.

Sometimes it makes sense to use the second, more general method.

<u>EX</u>: Transform the ellipsoid

$$x^2+4y^2+2z^2=15+2x$$

into a sphere using $u=\frac{1}{4}(x-1)$, $v=\frac{1}{2}y$,

$$w=\frac{1}{\sqrt{8}}z.$$

Step 1 : invert...

$$x = 4u+1$$
$$y = 2v$$
$$z = w\sqrt{8}$$

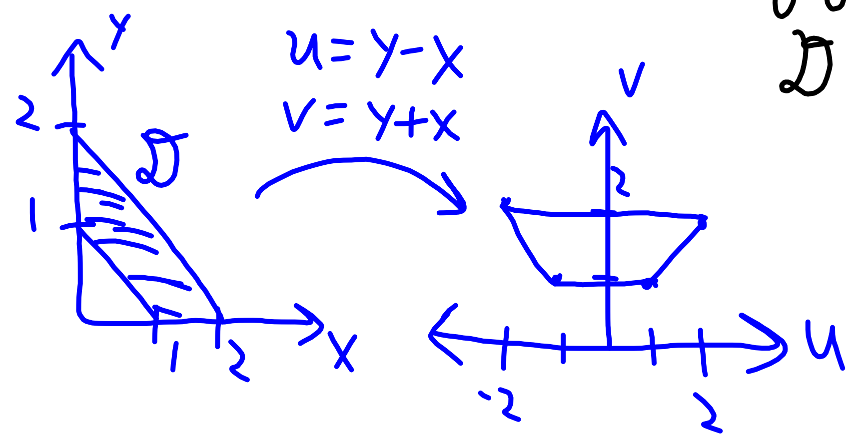
Step 2 : insert into the equation.

$$(4u+1)^2 + 4(2v)^2 + 2(w\sqrt{8})^2 = 15 + 2(4u+1)$$

$$\Rightarrow 16u^2 + 8u + 1 + 16v^2 + 16w^2 = 17 + 8u$$

$$\Rightarrow 16(u^2+v^2+w^2) = 16 \Rightarrow u^2+v^2+w^2 = 1.$$

EX: Let D be the trapezoidal region with corners $(1,0)$, $(2,0)$, $(0,1)$ & $(0,2)$. Use a transformation to find $\displaystyle\iint_D \cos\left(\frac{y-x}{y+x}\right)dA$.



When equations are linear, just plug in the 4 corners & connect the dots.

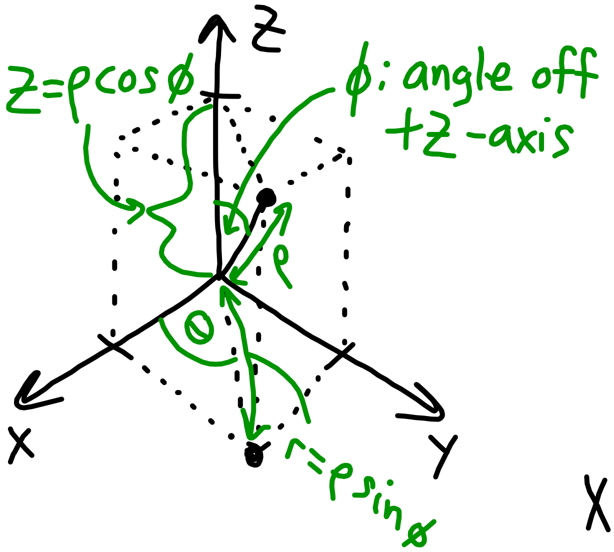
$$\left.\begin{aligned} x &= \tfrac{1}{2}(v-u) \\ y &= \tfrac{1}{2}(u+v) \end{aligned}\right\} \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

Thus $\displaystyle \iint_D \cos\left(\frac{y-x}{y+x}\right) dA = \int_1^2 \int_{-v}^{v} \cos(u/v) \left|-\frac{1}{2}\right| du\,dv$

$$= \frac{1}{2}\int_1^2 \int_{-v}^{v} \frac{\partial}{\partial u}\left(v\sin(u/v)\right) du\,dv = \frac{1}{2}\int_1^2 \left[v\sin\left(\frac{u}{v}\right)\right]_{-v}^{v} dv$$

$$= \frac{1}{2}\int_1^2 v\left(\sin(1) - \sin(-1)\right) dv = \sin(1)\int_1^2 v\,dv = \underline{\frac{3\sin(1)}{2}}.$$

Spherical coordinates



ρ: distance to origin

θ: angle off +x-axis in the xy-plane

ϕ: angle off +z-axis

$0 \leq \rho$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$

$$x = r\cos\theta = \rho\sin\phi\cos\theta$$
$$y = r\sin\theta = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi.$$

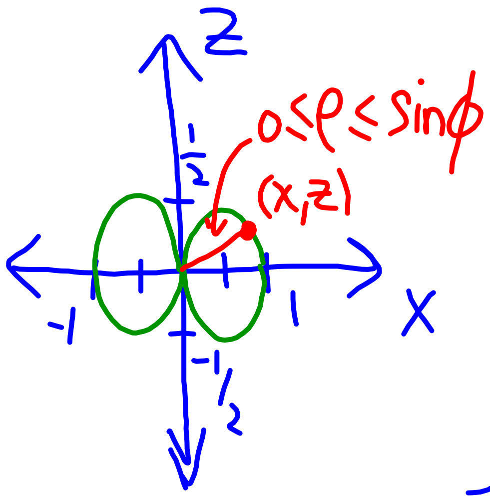
Integration formula:

$$\iiint_V f(x,y,z)\,dV = \iiint_V f(\rho,\theta,\phi)\,\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi .$$

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi .$$

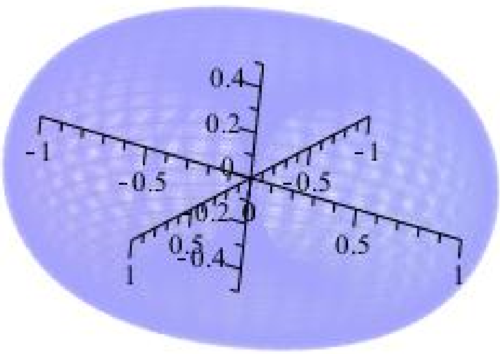
EX: Find the volume of the torus $\rho = \sin\phi$ (spherical).

Look at a cross-section...



θ-independent so the green circles may be rotated around the z-axis...

This is a little easier to see on a computer screen; check out the slides online.



$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right) \int_0^{\pi} \frac{1}{3}\rho^3 \sin\phi \Big|_0^{\sin\phi} d\phi$$

$$= \frac{2\pi}{3}\int_0^{\pi} \sin^4\phi \, d\phi$$

$$(\sin^2\phi)^2 = \left(\frac{1-\cos 2\phi}{2}\right)^2$$

$$= \frac{1}{4}\left(1 - 2\cos 2\phi + \cos^2 2\phi\right)$$

$$\Rightarrow V = \frac{2\pi}{3}\cdot\frac{1}{4}\cdot\int_0^{\pi} 1-2\cos 2\phi+\cos^2 2\phi\, d\phi$$

$$= \frac{\pi}{6}\left[\pi - \cancel{\sin 2\phi\Big|_0^{\pi}}_{\,0} + \int_0^{\pi}\cos^2 2\phi\, d\phi\right]$$

$$\cos^2 2\phi = \frac{1+\cos 4\phi}{4}$$

$$= \frac{\pi}{6}\left(\pi + \frac{1}{4}\int_0^{\pi} 1+\cos 4\phi\, d\phi\right)$$

$$= \frac{\pi}{6}\left(\pi + \frac{\pi}{4} + \frac{1}{16}\cancel{\sin 4\phi\Big|_0^{\pi}}_{\,0}\right) = \frac{\pi^2}{6}\cdot\frac{5}{4} = \boxed{\frac{5\pi^2}{24}}.$$