

Review for Exam 2  
Math 2110Q

LIMITS If  $f(x,y) \rightarrow L$  as  $(x,y) \rightarrow (a,b)$   
then we say

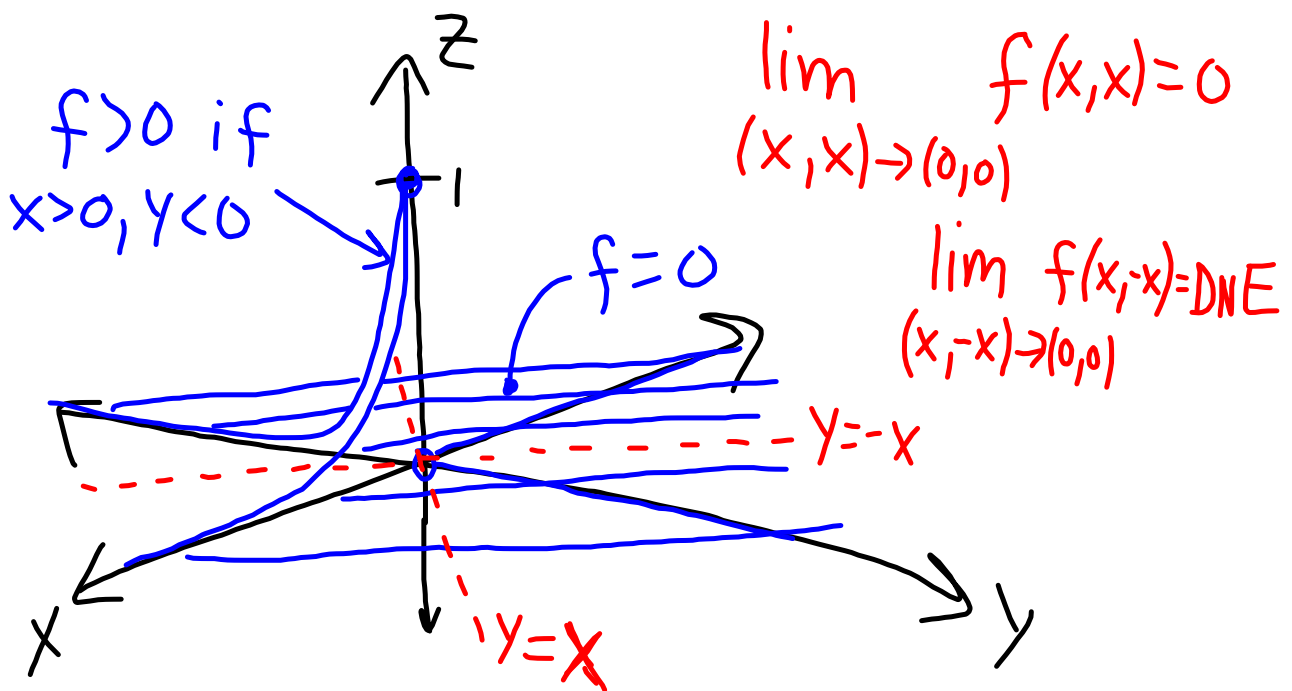
$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L.$$

More precisely, if for ANY  $\epsilon > 0$  there  
is some  $\delta = \delta(\epsilon) > 0$  such that

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \epsilon$$

then the limit exists.

In 2D (or higher) the limit may only exist along specific paths.



Find the limit or else show that it does not exist:

$$(a) \lim_{(x,y) \rightarrow (1,2)} \frac{x+y-2}{x+y-3} \left\{ \begin{array}{l} \leftarrow \text{get "1"} \\ \leftarrow \text{looks like "0"} \end{array} \right.$$

This "blows up" somehow. Take  $x=1$

$$\Rightarrow \frac{1+y-2}{1+y-3} = \frac{y-1}{y-2} \left\{ \begin{array}{l} \text{positive as } y \rightarrow 2^+ \\ \text{negative as } y \rightarrow 2^- \end{array} \right.$$

So this goes to  $\pm \infty$  depending on the way we approach  $\Rightarrow$  D.N.E.

$$(b) \lim_{(x,y) \rightarrow (-1,1)} \frac{x^2 + y^3}{x - y + \sin(x+y)}$$

So  $(-1,1)$  is in the domain here...

- cite continuity
- plug in point to evaluate

$$\lim_{(x,y) \rightarrow (-1,1)} \frac{x^2 + y^3}{x - y + \sin(x+y)} = \frac{1+1}{-1-1+0} = \frac{2}{-2} = -1$$

$$(c) \lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x)}{\sin(y)\cos(y)} \quad \left. \vphantom{\lim} \right\} \text{"0/0"}$$

• Try  $x=0, y \neq 0 \Rightarrow \frac{\sin(0)}{\sin(y)\cos(y)} = 0$

• Look for a way to get a different

result;  $y=x \Rightarrow \frac{\cancel{\sin(x)}}{\cancel{\sin(x)}\cos(x)}$

$$= \frac{1}{\cos(x)} \xrightarrow{x \rightarrow 0} 1 \neq 0 \text{ (DNE)}$$

Implicit differentiation

Let  $z = z(x, y)$  satisfy

$$x^3 + y^2 + z^4 + xyz = 1.$$

Then find  $\frac{\partial z}{\partial x}$  &  $\frac{\partial z}{\partial y}$  by differentiating through the whole equation...

$$\frac{\partial}{\partial x}(x^3 + y^2 + z^4 + xyz) = \frac{\partial}{\partial x}(1) = 0$$

$$\Rightarrow 3x^2 + 4z^3 \frac{\partial z}{\partial x} + yz + xy \frac{\partial z}{\partial x} = 0.$$

Now group  $\frac{\partial z}{\partial x}$  terms and factor it out:

$$3x^2 + yz + \frac{\partial z}{\partial x} \{4z^3 + xy\} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-3x^2 - yz}{4z^3 + xy}$$

$$\frac{\partial}{\partial y} (x^3 + y^2 + z^4 + xyz) = \frac{\partial}{\partial y} (1) = 0$$

$$\Rightarrow 2y + 4z^3 \frac{\partial z}{\partial y} + xz + xy \frac{\partial z}{\partial y} = 0$$



$$\Rightarrow \frac{\partial z}{\partial y}(4z^3 + xy) = -xz - 2y$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{-xz - 2y}{4z^3 + xy}}$$

\* If  $(x, y, z(x, y))$  solve the original equation then you can plug them in to get the partial derivatives.

Given that  $xy+xz+e^z=2$   
find  $\frac{\partial z}{\partial x}$  &  $\frac{\partial z}{\partial y}$  at  $(1,1,0)$ .

$$\frac{\partial}{\partial x} (xy+xz+e^z) = \frac{\partial}{\partial x} (2) = 0$$

$$\Rightarrow y+z+x\frac{\partial z}{\partial x}+e^z\frac{\partial z}{\partial x}=0$$

(plug in  $(1,1,0)$ ...)

$$\Rightarrow 1+0+\frac{\partial z}{\partial x}+e^0\frac{\partial z}{\partial x}=0=1+2\frac{\partial z}{\partial x}$$

$$\Rightarrow \underline{\frac{\partial z}{\partial x} = -\frac{1}{2}}$$

$$\frac{\partial}{\partial y} (xy + xz + e^z) = x + x \frac{\partial z}{\partial y} + e^z \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow 1 + \frac{\partial z}{\partial y} + \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \underline{\frac{\partial z}{\partial y} = -\frac{1}{2}}$$

The Chain Rule revisited

Recall  $y=y(x)$  &  $x=x(t)$

$$\Rightarrow \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad (\text{Chain Rule})$$

But now we can have  $f=f(x,y)$   
with  $x=x(t)$  &  $y=y(t)$

$$\Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} .$$

↑ "note  
partials" ↓

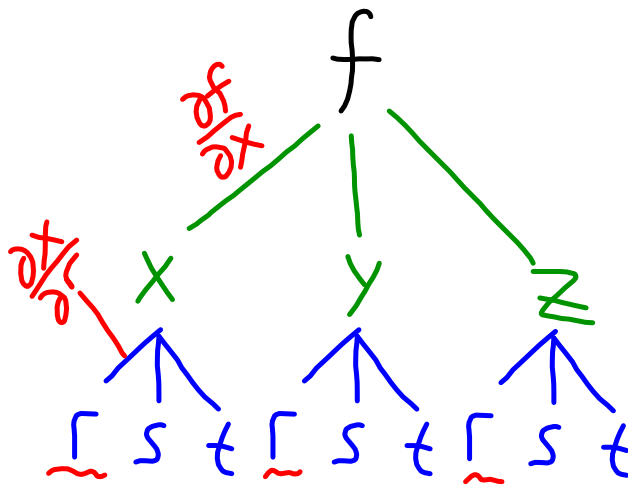
## Tree diagrams

$$f=f(x,y,z)$$

$$x=x(r,s,t)$$

$$y=y(r,s,t)$$

$$z=z(r,s,t)$$

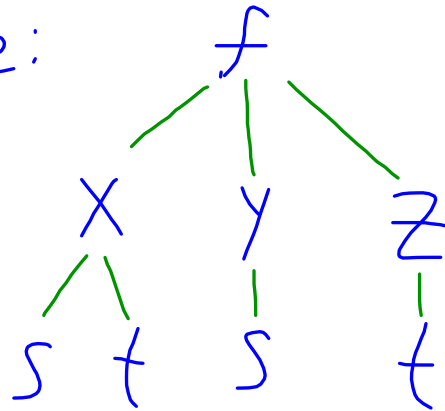


$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r}$$

Ex: If  $f(x, y, z) = xy + z^2$  and we

have  $x = x(s, t)$ ,  $y = y(s)$ ,  $z = z(t)$ , find  $\frac{\partial f}{\partial s}$  and  $\frac{\partial f}{\partial t}$ .

Tree:



$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{dy}{ds}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial z} \frac{dz}{dt}$$

Let  $F(x,y,z) = e^x(2y+4z)$ , with  $x = t+s$ ,  $y = 2r-3t$ ,  $z = r-t+s$ . Use the Chain Rule to find  $\frac{\partial F}{\partial t}$  with  $(r,s,t) = (1,0,1)$ .

$$F_t = F_x x_t + F_y y_t + F_z z_t$$

$$F_x = e^x(2y+4z) \quad x_t = 1$$

$$F_y = 2e^x$$

$$F_z = 4e^x$$

$$y_t = -3$$

$$z_t = -1$$

} still need  
(x, y, z)

$$\begin{aligned}x &= 1+0=1 & \Rightarrow & F_x = e(2(-1)+4(0)) \\y &= 2-3=-1 & & = -2e \\z &= 1-1+0=0 & & F_y = 2e, F_z = 4e\end{aligned}$$

$$\begin{aligned}\text{Thus, } F_t &= -2e + 2e(-3) + 4e(-1) \\&= (-2-6-4)e = \underline{-12e}.\end{aligned}$$



## Directional derivatives

Recall  $f_x$  : rate of change,  $x$ -direction  
 $f_y$  : rate of change,  $y$ -direction

$\hat{u} = \langle a, b \rangle$  : unit vector

Rate of change in direction  $\hat{u}$  is

$$D_{\hat{u}} f = \lim_{h \rightarrow 0} \frac{f(x+ah, y+bh) - f(x, y)}{h}$$

DEFINE :  $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$   
"gradient"

$$\Rightarrow D_{\hat{u}} f = \nabla f \cdot \hat{u}$$

$$\left\langle \overset{\nabla f}{f_x, f_y} \right\rangle \cdot \left\langle \overset{\hat{u}}{a, b} \right\rangle = a f_x + b f_y$$

$$\text{Let } f(x,y) = 4xy + x^2 - 2y^2$$

(a) Find the derivative of  $f$  in the direction  $\langle 4, 3 \rangle$  at  $(-1, 0)$ .

$$|\langle 4, 3 \rangle| = \sqrt{16+9} = \sqrt{25} = 5 \Rightarrow \hat{u} = \frac{1}{5} \langle 4, 3 \rangle.$$

$$\nabla f = \langle 4y + 2x, 4x - 4y \rangle \quad \left. \vphantom{\nabla f} \right\} \text{plug in } \begin{matrix} x=-1 \\ y=0 \end{matrix}$$

$$\nabla f(-1, 0) = \langle -2, -4 \rangle$$

$$\Rightarrow D_{\hat{u}} f = \frac{1}{5} \langle -2, -4 \rangle \cdot \langle 4, 3 \rangle = \frac{-8-12}{5} = \boxed{-4}.$$

(b) Find  $D_{\hat{u}} f$  at  $(2, -1)$  if  $\hat{u}$  points in the direction of fastest increase.

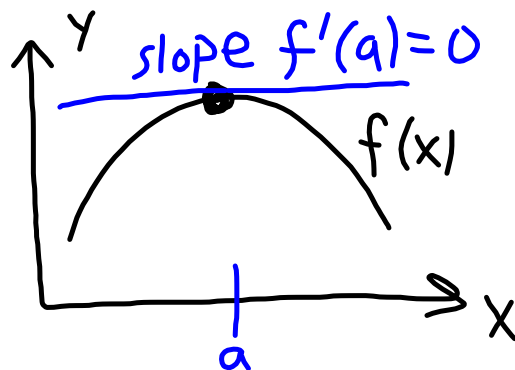
$$\text{Then } \hat{u} = \frac{\nabla f}{|\nabla f|} \text{ and } D_{\hat{u}} f = |\nabla f|$$

$$\begin{aligned} \nabla f|_{(2, -1)} &= \langle 4y + 2x, 4x - 4y \rangle|_{(2, -1)} \\ &= \langle -4 + 4, 8 + 4 \rangle = \langle 0, 12 \rangle \end{aligned}$$

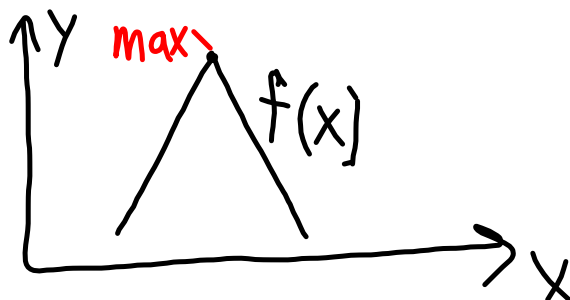
$$\Rightarrow |\nabla f| = D_{\hat{u}} f = \underline{12}.$$

When derivatives exist at a max/min, they must be zero.

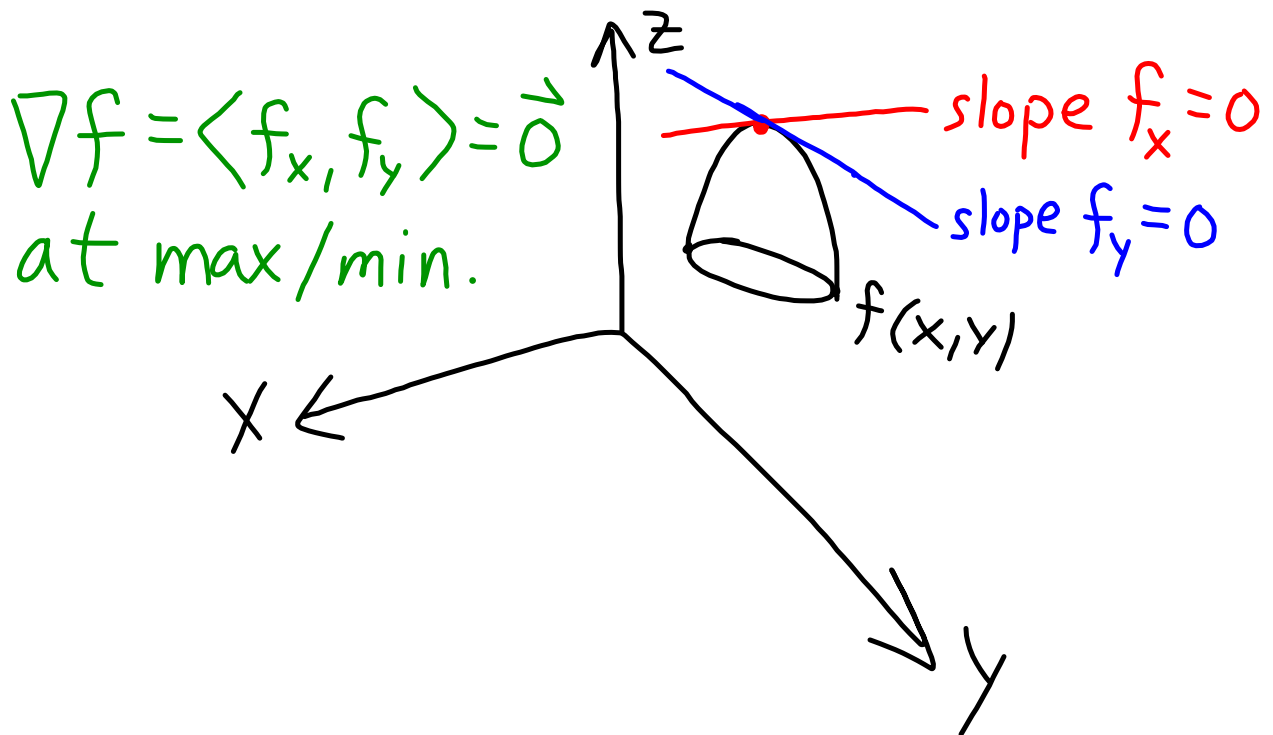
Case 1 : derivative = 0



Case 2 : no derivative



With multiple variables, the partial derivatives will be zero when they exist at max/min points.



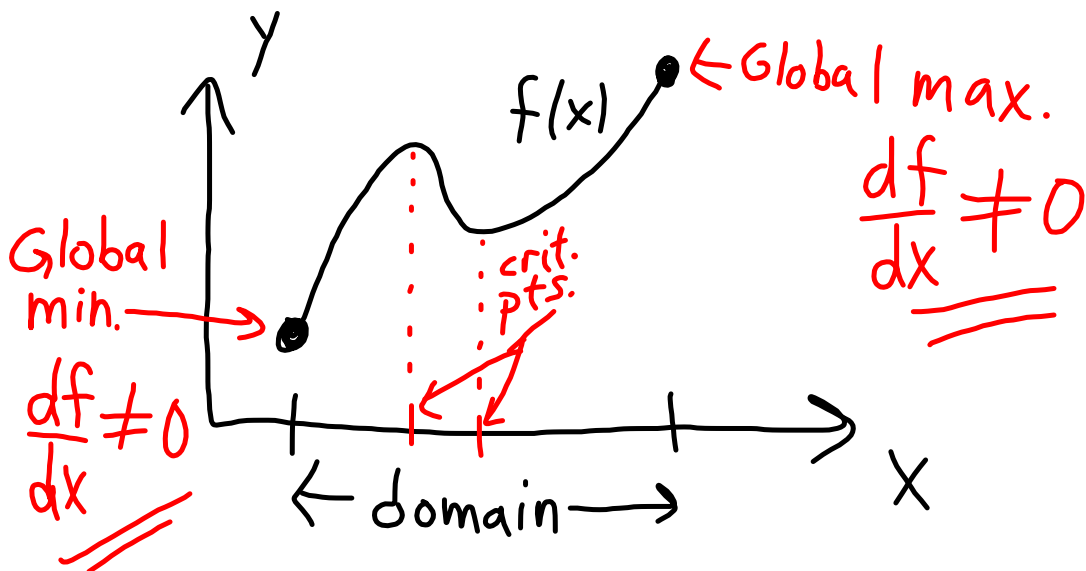
... similarly for more variables

$$f(x, y, z) \text{ has MAX/MIN} \\ \text{at } (a, b, c) \Rightarrow \nabla f = \langle f_x, f_y, f_z \rangle \\ = \langle 0, 0, 0 \rangle \text{ at } (a, b, c)$$

Since  $\nabla f$  may not exist at MAX/MIN points  
we check all **CRITICAL POINTS**, i.e.

$$\text{where } \nabla f = \vec{0} \text{ or } \nabla f = \text{D.N.E.}$$

Looking at critical points gives us local extrema and sometimes global extrema; not generally on a closed and bounded domain.



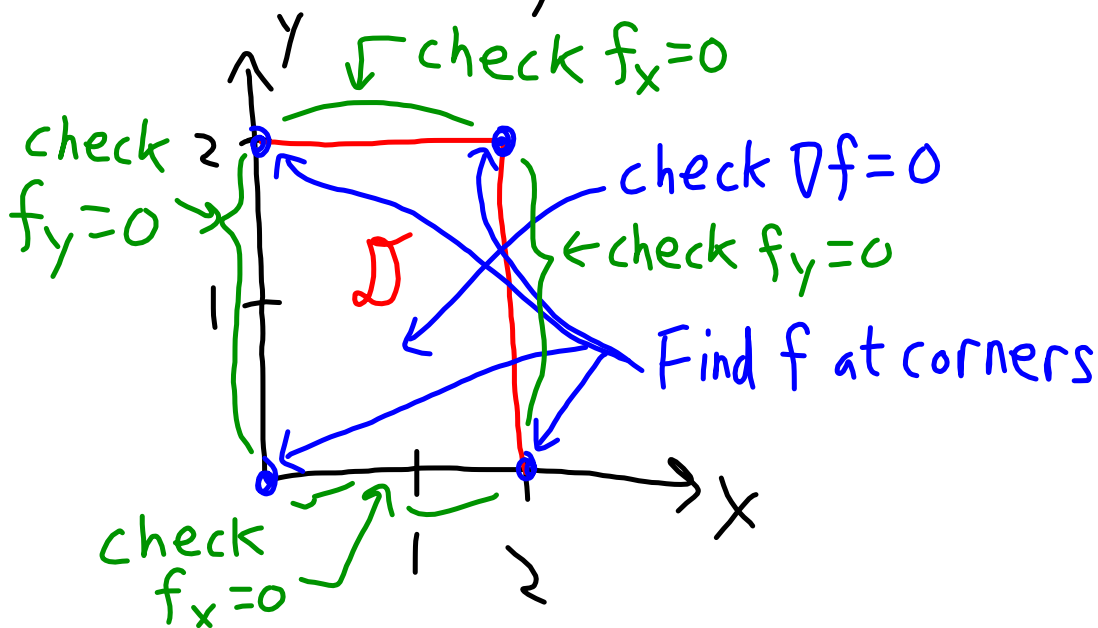


Identifying GLOBAL extrema takes work:

- (1) Find  $f$ -values at all critical points *inside* the domain.
- (2) Find max/min  $f$ -values along the domain boundary.
- (3) Take MAX/MIN of all  $f$ -values from steps (1) & (2).

Given  $f(x,y) = e^{-x^2+2x-2y^2+4y-3}$

find the absolute max/min values for  $f$   
 on  $0 \leq x \leq 2, 0 \leq y \leq 2$ .



$$f_x = 0 = (-2x + 2) e^{-x^2 + 2x - 2y^2 + 4y - 3}$$

$\Rightarrow x = 1$  never = 0

$$f_y = 0 = (-4y + 4) e^{(\dots)} \Rightarrow y = 1$$

Candidate

↓

$$\text{So check } f(1, 1) = e^{-1+2-2+4-3} = e^0 = 1.$$

Check corners of  $D \dots$

$$f(0, 0) = e^{-3} \qquad f(2, 0) = e^{-4+4-3} = e^{-3}$$

$$f(0, 2) = e^{-8+8-3} = e^{-3} \qquad f(2, 2) = e^{-3}$$

So far,  $\max=1$ ,  $\min=e^{-3}$ . Now we check  $f_x=0$  along  $y=0, y=2\dots$

$$f_x=0 \Rightarrow x=1 \text{ (already solved this)}$$

So now check  $f$  at  $(1,0)$  &  $(1,2)$

$$f(1,0) = e^{-1+2-3} = e^{-2}$$

$$f(1,2) = e^{-1+2-8+8-3} = e^{-2}$$

Similarly,  $f_y=0 \Rightarrow y=1$ , so check  $f(0,1), f(2,1)\dots$

$$f(0,1) = e^{-2+4-3} = e^{-1} \quad \& \quad f(2,1) = e^{-1}$$

$$\text{MAX}=1, \text{MIN}=e^{-3}$$

Lagrange Multipliers: used to maximize or minimize a function under constraints.

$f(x, y)$ : A function to minimize or maximize

$g(x, y) = K$  : "Constraint equation" restricts  $(x, y)$  allowed  
(think "design parameters")

\* First step: identify these for your problem.

Let  $f = f(x, y, z) \dots$   $g(x, y, z) = K$  constraint.  
To find the MAX or MIN of  $f$ , solve

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

$$\frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z}$$

$$g(x, y, z) = K$$

solving for  $x, y, z, \lambda$

4 variables

4 equations

The volume of a cylinder in terms of radius  $r$  and height  $h$  is given by  $V = \pi r^2 h$ . The surface area is

$$S = 2\pi r^2 + 2\pi r h.$$

Maximize the volume if  $S=4$ , fixed.

$$\text{Solve } \begin{cases} \nabla V = \lambda \nabla S \\ S = 4 \end{cases}$$

for  $r, h, \lambda$ .

$$1) V_r = 2\pi r h = \lambda S_r = \lambda(4\pi r + 2\pi h)$$

$$2) V_h = \pi r^2 = \lambda S_h = \lambda(2\pi r)$$

$$3) S = 2\pi r^2 + 2\pi r h = 4 \quad (r \neq 0)$$

$$(2) \Rightarrow r^2 = 2\lambda r \Rightarrow r(r - 2\lambda) = 0 \Rightarrow r = 2\lambda$$

$$\text{Insert in (1)... } 4\pi\lambda h = \lambda(8\pi\lambda + 2\pi h)$$

$$\Rightarrow 2\lambda h = 4\lambda^2 + \lambda h$$

$$\Rightarrow 0 = 4\lambda^2 - \lambda h = \lambda(4\lambda - h)$$

$$\Rightarrow \lambda = 0 \text{ or } 4\lambda = h$$



Put  $r=2\lambda$ ,  $h=4\lambda$  into  $S=4$ :

$$2\pi(2\lambda)^2 + 2\pi(2\lambda)(4\lambda) = 4$$

$$8\pi\lambda^2 + 16\pi\lambda^2 = 4$$

positive root

$$24\pi\lambda^2 = 4$$

$$\lambda^2 = \frac{4}{24\pi} = \frac{1}{6\pi} \Rightarrow \lambda = \frac{1}{\sqrt{6\pi}}$$

$$\Rightarrow r = \frac{2}{\sqrt{6\pi}}, \quad h = \frac{4}{\sqrt{6\pi}}$$

$$\Rightarrow V = \pi r^2 h = \pi \frac{4}{6\pi} \cdot \frac{4}{\sqrt{6\pi}} = \frac{8}{3\sqrt{6\pi}}$$

MAX V

8

$\frac{8}{3\sqrt{6\pi}}$