The Chain Rule revisited

Recall $y=y(x)$ & $x=x(t)$

$$\Rightarrow \frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt} \quad \text{(Chain Rule)}$$

But now we can have $f=f(x,y)$
with $x=x(t)$ & $y=y(t)$

$$\Rightarrow \frac{df}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}.$$

note "partials"

Similarly, for $f = f(x, y, z)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}.$$

- Clear
- Precise

"Compact" notation:

$$f'(t) = f_x \cdot x' + f_y \cdot y' + f_z \cdot z'.$$

- Fast, easy
- Need to recall meaning

let $x = x(s,t)$, $y = y(s,t)$

$f = f(x,y)$

Now there are two cases; s and t

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Ex: $f(x,y)=e^{x+y}$, $x=st$, $y=s^2-t^2$.

Find f_s, f_t.

$$\frac{\partial x}{\partial s}=t \quad \frac{\partial y}{\partial s}=2s$$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=e^{x+y}, \quad \frac{\partial x}{\partial t}=s \quad \frac{\partial y}{\partial t}=-2t$$

$$\Rightarrow \frac{\partial f}{\partial s}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}=e^{x+y}(t+2s)$$

$$\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}=e^{x+y}(s-2t).$$

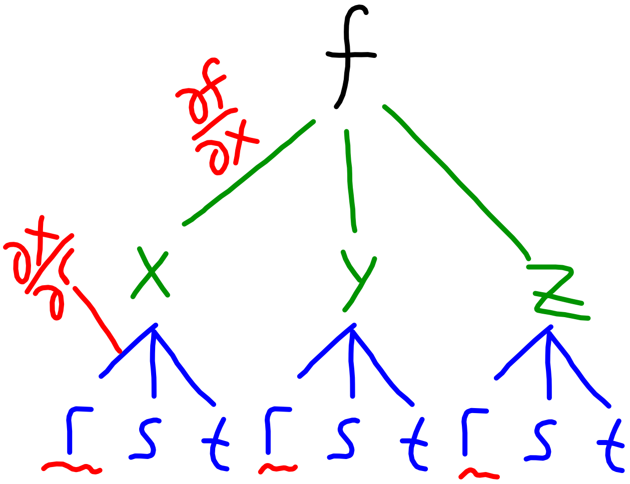
Tree diagrams

$f=f(x,y,z)$

$x=x(r,s,t)$

$y=y(r,s,t)$

$z=z(r,s,t)$

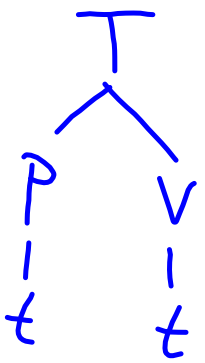


$$\frac{\partial f}{\partial r}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r}$$

Ex: Temperature $T = T(P, v)$

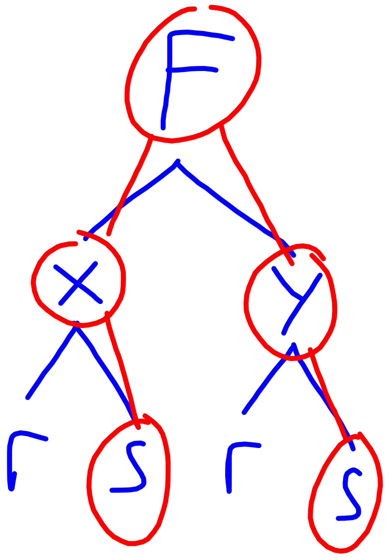
$\left.\begin{array}{l} P: \text{pressure} \\ V: \text{volume} \end{array}\right\}$ depend on time "t".

Find $\dfrac{dT}{dt}$.

Tree: 

$$\frac{dT}{dt} = \frac{\partial T}{\partial P} \cdot \frac{dP}{dt} + \frac{\partial T}{\partial V} \cdot \frac{dV}{dt}.$$

Ex: Find $\frac{\partial F}{\partial s}$; $F = F(x, y)$ and x, y depend on r, s.



$$F_s = F_x \cdot x_s + F_y \cdot y_s .$$

Implicit differentiation revisited

$F = F(x,y) = 0$ is often encountered.

E.g. $x^3 + y^3 = -xy$ } so $F(x,y) = x^3 + xy + y^3 = 0$.

Sometimes* we may think of $y = y(x)$ but lack an EXPLICIT formula... so $y'(x) = ?$

* "Implicit function theorem" says when $y = y(x)$ is valid... depends on x, y choice.

Differentiate F to figure out dy/dx

$$F(x,y)=0$$

$$\Rightarrow \frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\begin{tikzcd}[arrows=-] & F \arrow[dl] \arrow[dr] & \\ x & & y \arrow[d] \\ & & x \end{tikzcd}$$

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

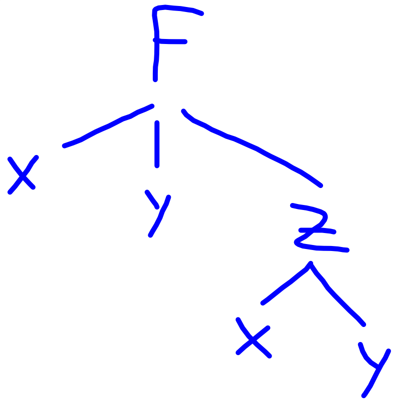
Ex: $F = x^3 + xy + y^3 = 0 \ldots \quad \frac{dy}{dx} = ?$

$F_x = 3x^2 + y \qquad F_y = x + 3y^2$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2+y}{x+3y^2}.$$

Similarly, if $F = F(x, y, z) = 0$,
$z = z(x, y)$,

then $F_x + F_z \cdot z_x = 0$
$F_y + F_z \cdot z_y = 0$



$$\Rightarrow \quad z_x = -\frac{F_x}{F_z}$$

$$z_y = -F_y / F_z \, .$$

Directional derivatives

Recall f_x : rate of change, x-direction
f_y : rate of change, y-direction

$\hat{u} = \langle a, b \rangle$: unit vector

Rate of change in direction $\hat{u}$ is

$$D_{\hat{u}} f = \lim_{h \to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h}.$$

So how do we calculate this?

$$= \lim_{h\to 0}\left[\frac{f(x+ah,y+bh)-f(x,y+bh)}{h}+\frac{f(x,y+bh)-f(x,y)}{h}\right]$$

$$= a\lim_{h\to 0}\frac{f(x+ah,\cdot)-f(x,\cdot)}{ah}+b\lim_{h\to 0}\frac{f(\cdot,y+bh)-f(\cdot,y)}{bh}$$

$$= a\lim_{z\to 0}\frac{f(x+z,\cdot)-f(x,\cdot)}{z}+b\lim_{z\to 0}\frac{f(\cdot,y+z)-f(\cdot,y)}{z}$$

$$= a\frac{\partial f}{\partial x}(x,y)+b\frac{\partial f}{\partial y}(x,y).$$

DEFINE: $\underbrace{\nabla}_{\text{"gradient"}} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$

$$\Rightarrow \boxed{D_{\hat{u}} f = \nabla f \cdot \hat{u}}$$

$$\underbrace{\langle f_x, f_y \rangle}^{\nabla f} \cdot \overbrace{\langle a, b \rangle}^{\hat{u}} = a f_x + b f_y$$

EX: $f(x,y) = x^2 + 4y^2 + 10$. Find the derivative of f in the direction $\langle 1,2 \rangle$ at $(x=3, y=-1)$.

$\Rightarrow \hat{u} = \frac{1}{\sqrt{5}} \overset{\vec{u}}{\langle 1,2 \rangle}$

$\nabla f = \langle 2x, 8y \rangle$

$\nabla f(3,-1) = \langle 6, -8 \rangle$

$D_{\hat{u}} f = \langle 6,-8 \rangle \cdot \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle = \frac{6-16}{\sqrt{5}} = \frac{-10}{\sqrt{5}}$.

Maximum rate of change: It turns out that

(1) the fastest rate of change is $|\nabla f|$

(2) this occurs in the direction of ∇f

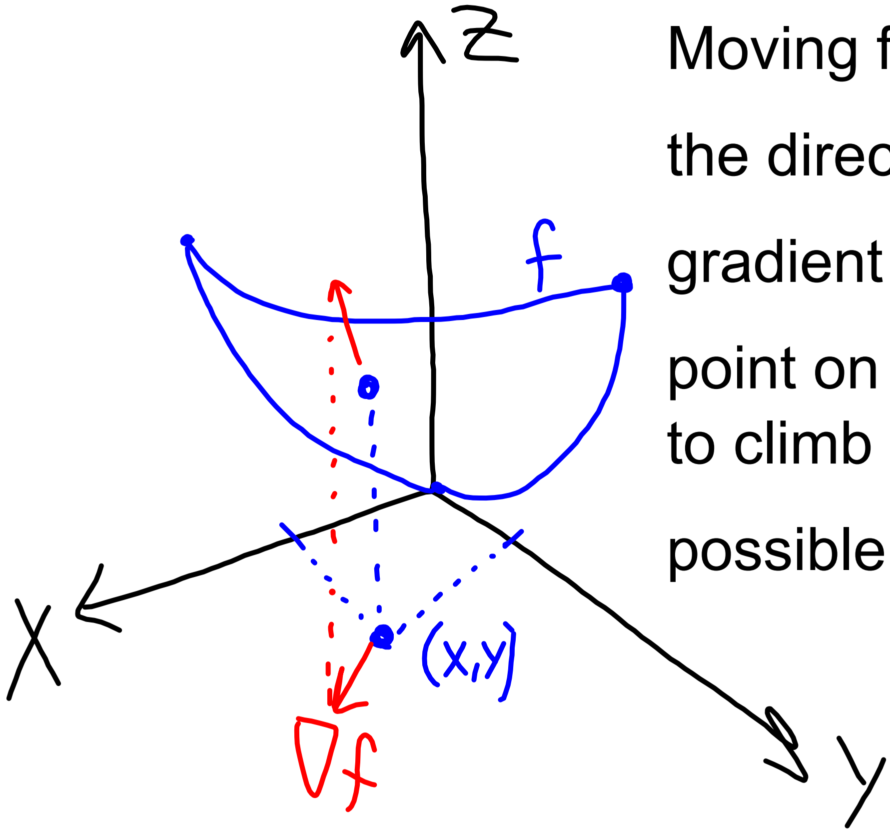
Proof: $\nabla f \cdot \hat{u} = |\nabla f| \underbrace{|\hat{u}|}_{1} \cos\theta$

$$\Rightarrow \underset{\theta=\pi}{-|\nabla f|} \le \nabla f \cdot \hat{u} \le \underset{\theta=0}{|\nabla f|}$$

$\Rightarrow \hat{u}, \nabla f$ opposite direction

$\Rightarrow \hat{u}, \nabla f$ same direction

Picture of the gradient



Moving from (x,y) in the direction of the gradient causes the point on the surface to climb as fast as possible.

<u>EX</u>: Find the max rate of change for $f(x,y) = \cos(x-2y)$ at $(1, \frac{1}{2})$ and at $(\pi/2, 0)$.

$$\nabla f = \langle -\sin(x-2y), 2\sin(x-2y) \rangle$$

$$|\nabla f(1, \tfrac{1}{2})| = |\langle -\sin(0), 2\sin(0) \rangle| = 0$$

$$|\nabla f(\tfrac{\pi}{2}, 0)| = |\langle -1, 2 \rangle| = \sqrt{5}.$$

Tangent plane to a surface

We will think of something like

$$z = x^2 + y^2 \quad \text{as } F(x,y,z) = 0,$$

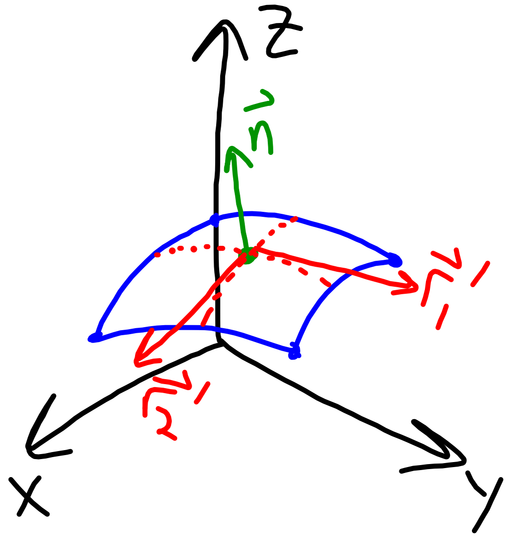
$$F = x^2 + y^2 - z \quad \text{or } F = z - x^2 - y^2.$$

Now consider a point (x_0, y_0, z_0) and ANY curve on the surface, passing through this point; $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Since $(x(t), y(t), z(t))$ is on the surface, $F(x(t), y(t), z(t)) = 0$

$$\Rightarrow \frac{dF}{dt} = F_x \cdot x' + F_y \cdot y' + F_z \cdot z' = 0$$

$$\Rightarrow \underbrace{\langle F_x, F_y, F_z \rangle}_{\nabla F} \cdot \underbrace{\langle x', y', z' \rangle}_{\frac{d\tilde{r}}{dt}} = 0$$



Given two surface curves through $P=(x_0, y_0, z_0)$, say $\vec{r}_1(t)$, $\vec{r}_2(t)$ it follows

$$\nabla F \cdot \vec{r}_1{}' = 0$$
$$\nabla F \cdot \vec{r}_2{}' = 0$$

$\Rightarrow \nabla F \parallel \vec{n}$ ← normal to tangent plane

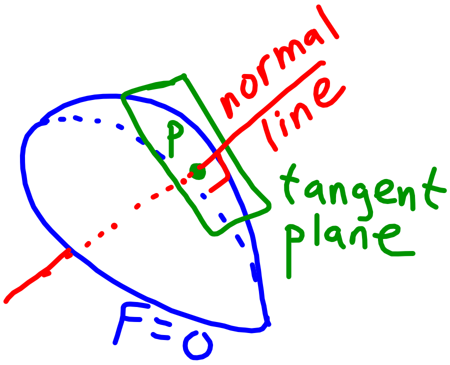
* ∇F is a normal for the tangent plane

* $P=(x_0,y_0,z_0)$ is in the plane

$$\Rightarrow \boxed{\nabla F \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0.}$$

TANGENT PLANE

Normal line: The line through the point P that is perpendicular to the tangent plane at P.



Since ∇F is the direction vector for the normal line,

$$\frac{x-x_0}{F_x} = \frac{y-y_0}{F_y} = \frac{z-z_0}{F_z} = t.$$

EX: Find the tangent plane for

$$\frac{x^2}{9}+y^2+z^2=1 \text{ (ellipsoid) at } \underbrace{\left(\sqrt{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)}_{P}.$$

$$F=\frac{x^2}{9}+y^2+z^2-1=0$$

$$\nabla F(P)=\left\langle \frac{2x}{9}, 2y, 2z \right\rangle\Big|_P=\left\langle \frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3} \right\rangle$$

$$\Rightarrow \left\langle \frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3} \right\rangle \cdot \left\langle x-\sqrt{3}, y-\frac{\sqrt{3}}{3}, z-\frac{\sqrt{3}}{3} \right\rangle = 0.$$

EX: Find the normal line through P in the previous example.

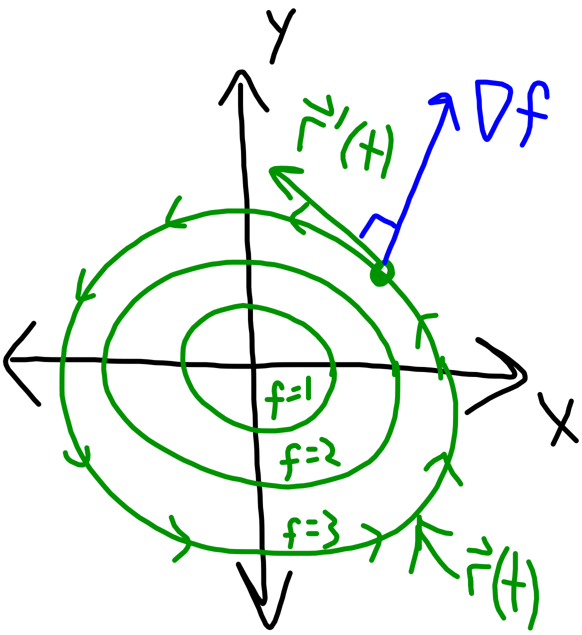
$$\nabla F = \left\langle \frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{3}, \frac{2\sqrt{3}}{3} \right\rangle$$

$$\Rightarrow \frac{9(x-\sqrt{3})}{2\sqrt{3}} = \frac{3(y-\sqrt{3}/3)}{2\sqrt{3}} = \frac{3(z-\sqrt{3}/3)}{2\sqrt{3}}$$

$$\Rightarrow 3(x-\sqrt{3}) = y-\sqrt{3}/3 = z-\sqrt{3}/3 = t$$

$$x = \frac{1}{3}t+\sqrt{3},\ y = t+\frac{\sqrt{3}}{3},\ z = t+\frac{\sqrt{3}}{3}.$$

Similarly, $\nabla f(x,y)$ is perpendicular to level curves $f(x,y)=K$.



$f=K$, constant

$\Rightarrow \frac{df}{dt} = f_x \cdot x' + f_y \cdot y' = 0$

$\Rightarrow \nabla f \cdot \vec{r}\,' = 0$

Practice!

#1 Let $f(x,y) = x^2 + xy + y^2$, $x = \cos(t)$, $y = \sin(t)$.

Find $\frac{df}{dt}$ using the Chain Rule.

$f_x = 2x + y \quad f_y = x + 2y$

$x' = -\sin(t) \quad y' = \cos(t)$

$$\frac{df}{dt} = f_x \cdot x' + f_y \cdot y'$$

$$= (2x+y)(-\sin(t)) + (x+2y)\cos(t).$$

#2 Let $f=f(x,y,z)$ and x,y,z are all functions of r,s,t,u,v. Find $\frac{\partial f}{\partial u}$.

$$\frac{\partial f}{\partial u}=f_x\cdot x_u+f_y\cdot y_u+f_z\cdot z_u$$

#3 Find the derivative of $f(x,y)=2x+\sqrt{y}$ at $(3,4)$ in the direction of $\langle -1,1\rangle$.

$$\hat{u}=\frac{1}{\sqrt{2}}\langle -1,1\rangle$$

$$D_{\hat{u}}f=\nabla f\cdot\hat{u}$$

$$f_x=2$$

$$f_y=\frac{1}{2\sqrt{y}}$$

$$=\left\langle 2,\frac{1}{4}\right\rangle\cdot\left\langle\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right\rangle=\frac{-2}{\sqrt{2}}+\frac{1}{4\sqrt{2}}$$

$$=\frac{-7}{4\sqrt{2}}$$

#4 Find the tangent plane and normal line through $z = \sin(x+y)$ at $\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$.

$$F = \sin(x+y) - z = 0$$

$$\left.\begin{aligned} F_x &= \cos(x+y) \\ F_y &= \cos(x+y) \\ F_z &= -1 \end{aligned}\right\} \begin{aligned} x_0 &= \frac{\pi}{2} \\ y_0 &= \frac{\pi}{2} \\ z_0 &= 0 \end{aligned} \Rightarrow \begin{aligned} F_x &= -1 \\ F_y &= -1 \\ F_z &= -1 \end{aligned}$$

$$-\left(x - \frac{\pi}{2}\right) - \left(y - \frac{\pi}{2}\right) - z = 0. \quad \left(\text{TANGENT PLANE}\right)$$

Normal line:

$$\frac{x-x_0}{F_x} = \frac{y-y_0}{F_y} = \frac{z-z_0}{F_z}$$

$$\Rightarrow \frac{x-\frac{\pi}{2}}{-1} = \frac{y-\frac{\pi}{2}}{-1} = \frac{z-0}{-1}$$