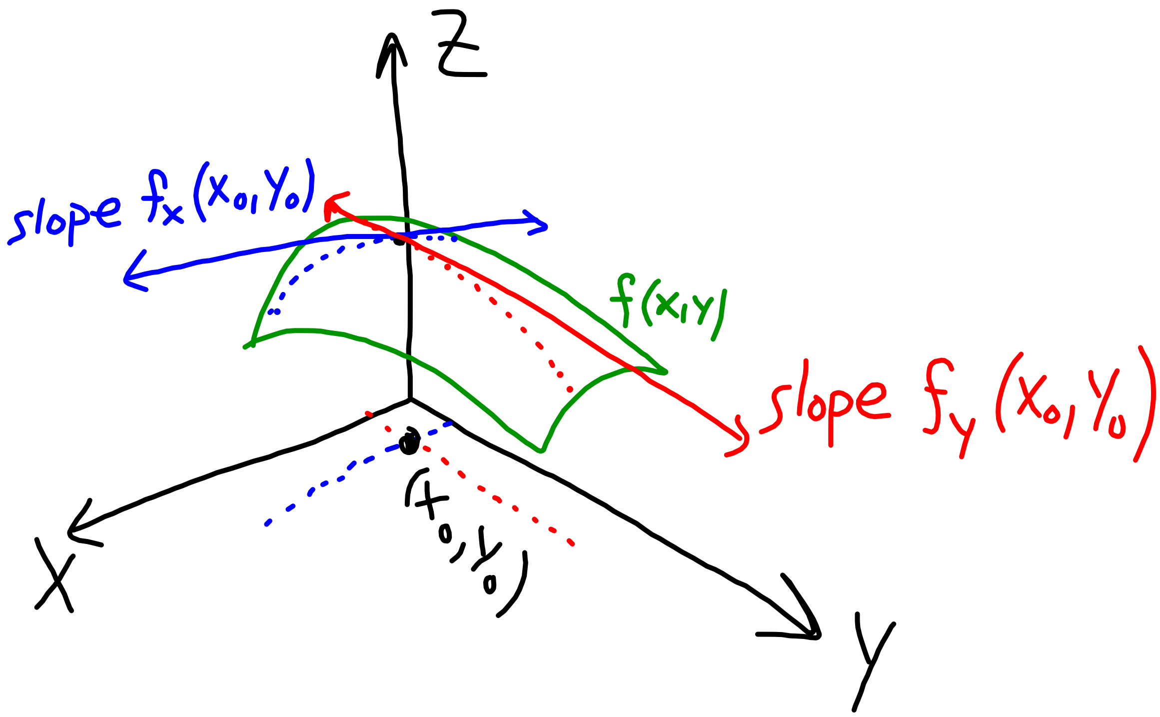
Partial Derivatives

For $f = f(x,y)$ we can discuss derivatives with respect to x, y independently:

$$f_x(x,y) = \frac{\partial}{\partial x} f(x,y) = \lim_{h\to 0} \underbrace{\frac{f(x+h,y) - f(x,y)}{h}}_{y \text{ is constant}}.$$

$$f_y(x,y) = \frac{\partial}{\partial y} f(x,y) = \lim_{h\to 0} \overbrace{\frac{f(x,y+h) - f(x,y)}{h}}^{x \text{ is constant}}.$$

Partial derivatives are the slopes of the tangent lines in the coordinate directions.



EX: Find f_x, f_y if $f(x,y) = x + y + \sin(x)\cos(y)$.

Hold y fixed...

$$f_x = 1 + \cos(x)\cos(y).$$

Hold x fixed...

$$f_y = 1 - \sin(x)\sin(y).$$

Ex: Calculate f_x, f_y for

(a) $f(x,y) = \ln(x^2+xy+y^2)$

$$f_x = \frac{1}{x^2+xy+y^2}\frac{\partial}{\partial x}(x^2+xy+y^2)$$

$$\Rightarrow f_x = \frac{2x+y}{x^2+xy+y^2}$$

$$f_y = \frac{2y+x}{x^2+xy+y^2}$$

(b) $f(x,y) = xy\sin(x+y)$

$f_x = y\sin(x+y) + xy\cos(x+y)$

$f_y = x\sin(x+y) + xy\cos(x+y)$

Implicit differentiation

Let $z = z(x,y)$ satisfy

$$x^3 + y^2 + z^4 + xyz = 1.$$

Then find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ by differentiating through the whole equation...

$$\frac{\partial}{\partial x}\left(x^3 + y^2 + z^4 + xyz\right) = \frac{\partial}{\partial x}(1) = 0$$

$$\Rightarrow 3x^2 + 4z^3\frac{\partial z}{\partial x} + yz + xy\frac{\partial z}{\partial x} = 0.$$

Now group $\frac{\partial z}{\partial x}$ terms and factor it out:

$$3x^2+yz+\frac{\partial z}{\partial x}\left\{4z^3+xy\right\}=0$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x}=\frac{-3x^2-yz}{4z^3+xy}}.$$

$$\frac{\partial}{\partial y}\left(x^3+y^2+z^4+xyz\right)=\frac{\partial}{\partial y}(1)=0$$

$$\Rightarrow 2y+4z^3\frac{\partial z}{\partial y}+xz+xy\frac{\partial z}{\partial y}=0$$

$$\Rightarrow \frac{\partial z}{\partial y}(4z^3+xy) = -xz-2y$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{-xz-2y}{4z^3+xy}}.$$

* If $(x, y, z(x,y))$ solve the original equation then you can plug them in to get the partial derivatives.

Higher-order derivatives

2^{nd}-order of $f(x,y)$: f_{xx} f_{xy}
f_{yx} f_{yy}

THEOREM: If f_{xy} & f_{yx} are both continuous then $f_{xy} = f_{yx}$.

EX: Find all 2^{nd}-order derivatives of

$$f(x,y) = x^3 + y^3 + x^2y^2 + xy + 1.$$

$$f_x = 3x^2 + 2xy^2 + y \qquad f_y = 3y^2 + 2x^2y + x$$

$$f_{xy} = 4xy + 1 \xleftrightarrow{\text{equal}} f_{yx} = 4xy + 1$$

$$f_{xx} = 6x + 2y^2 \qquad f_{yy} = 6y + 2x^2$$

Some other notations

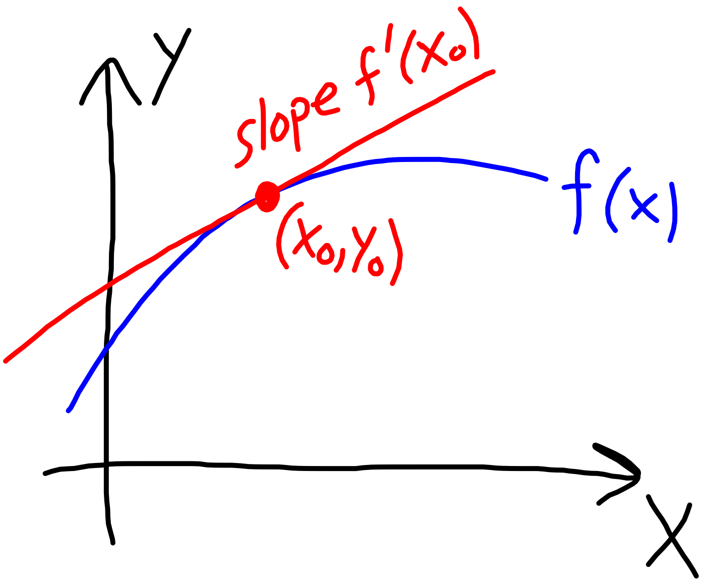
$$\frac{\partial}{\partial x} f = \frac{\partial f}{\partial x} = D_x f = f_x$$

$$\frac{\partial^2}{\partial x \partial y} f = \frac{\partial^2 f}{\partial x \partial y} = D_{xy} f = f_{xy}$$

These have various uses depending on the circumstances and are worth being familiar with for convenience in writing.

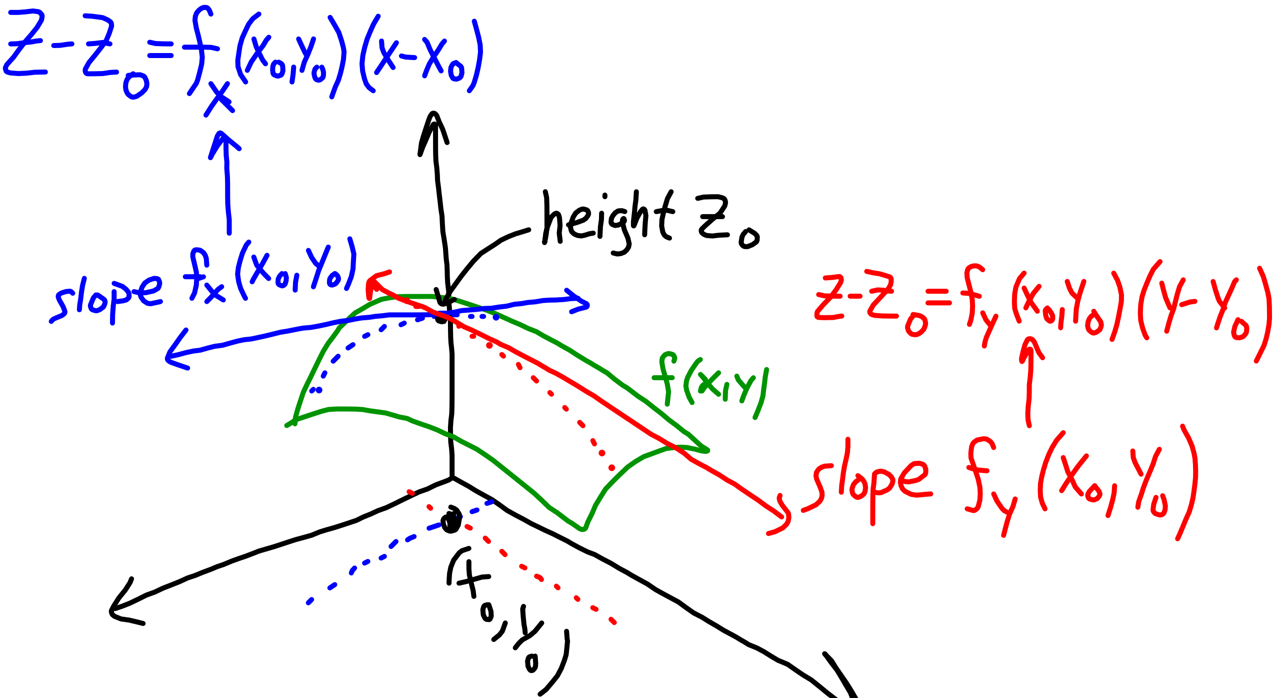
Tangent planes

Let's recall tangent lines in 1D:



$$y - y_0 = m(x - x_0)$$
$$= f'(x_0)(x - x_0).$$

Now recall the tangent lines on a surface. The plane containing these is the tangent plane.



Equation for the tangent plane

Recall the plane equation of the form $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$$\Rightarrow c(z-z_0)=-a(x-x_0)-b(y-y_0)$$

$$\Rightarrow z-z_0=\frac{-a}{c}(x-x_0)-\frac{b}{c}(y-y_0)$$

$$\Rightarrow z-z_0=A(x-x_0)+B(y-y_0).$$

$y = y_0$ fixed $\Rightarrow z - z_0 = A(x - x_0)$

$$= \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0)$$

$x = x_0$ fixed $\Rightarrow z - z_0 = B(y - y_0)$

$$= \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

$$\Rightarrow \boxed{z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).}$$

TANGENT PLANE

Ex: Let $f(x,y) = \ln(x^2+y^2)$... find the tangent plane equation at $(1, 1, \ln(2))$.

$$\left.\frac{\partial f}{\partial x}\right|_{(1,1)} = \left.\left(\frac{2x}{x^2+y^2}\right)\right|_{(1,1)} = \frac{2}{2} = 1.$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} = \left.\left(\frac{2y}{x^2+y^2}\right)\right|_{(1,1)} = 1$$

$$z - \ln(2) = (x-1) + (y-1) = x + y - 2.$$

EX: Let $f(x,y)=\sqrt{x^2-y^2+1}$. Find the equation of the tangent plane at the point $(\sqrt{8},0,3)$.

$$f_x(\sqrt{8},0)=\left.\frac{2x}{2\sqrt{x^2-y^2+1}}\right|_{(\sqrt{8},0)}=\frac{\sqrt{8}}{\sqrt{9}}=\frac{1}{3}\sqrt{8}.$$

$$f_y(\sqrt{8},0)=\left.\frac{-2y}{2\sqrt{x^2-y^2+1}}\right|_{(\sqrt{8},0)}=0.$$

$$\Rightarrow \boxed{z-3=\frac{\sqrt{8}}{3}(x-\sqrt{8})}$$

Differentials (particularly useful in engineering)

$$\underbrace{z-z_0}_{\Delta z}=(f_x)\underbrace{(x-x_0)}_{\Delta x}+(f_y)\underbrace{(y-y_0)}_{\Delta y}$$

$$\Delta z=\frac{\partial f}{\partial x}\cdot\Delta x+\frac{\partial f}{\partial y}\cdot\Delta y \approx \Delta f$$

For small perturbations of design variables we may estimate the effect on the output in this way.

EX: A cylinder has volume $V = \pi r^2 h$
r: radius h: height

Consider machining a cylinder (solid metal) and can control r to $\pm 0.1\%$ and $h \pm 0.01\%$. Put limits on how big/small the cylinder could turn out.

$$\Delta V \cong V_r \Delta r + V_h \Delta h = 2\pi r h \Delta r + \pi r^2 \Delta h.$$

$$\Delta r = 0.001 r \quad \Delta h = 0.0001 \cdot h$$

MAX V :

$$V_{MAX} \approx \pi r^2 h + \left(0.001 \cdot 2\pi r^2 h + 0.0001 \cdot \pi r^2 h\right)$$

$\approx V + \Delta V$

$$V_{MIN} \approx \pi r^2 h - \left(0.001 \cdot 2\pi r^2 h + 0.0001 \cdot \pi r^2 h\right).$$

$\approx V - \Delta V$

Practice!

#1 Find all 2^{nd}-order partial derivatives of $f(x,y) = \sin(x^2y^2)$.

$f_x = 2xy^2\cos(x^2y^2)$ $f_y = 2x^2y\cos(x^2y^2)$

$f_{xx} = 2y^2\cos(x^2y^2) - \sin(x^2y^2)\,4x^2y^4$.

$f_{xy} = 4xy\cos(x^2y^2) - 2x^2y \cdot 2xy^2\sin(x^2y^2)$.

$f_{yy} = 2x^2\cos(x^2y^2) - 2x^2y \cdot 2x^2y\,\sin(x^2y^2)$.

#2 Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ at $(1,1,-1)$ if

$z = z(x,y)$ satisfies

$$xy + xz^2 + y^2 z = 1.$$

$$y + z^2 + 2xz\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial x} = 0$$

$$1 + 1 - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} = 0$$

$$\boxed{2 = \frac{\partial z}{\partial x}}$$

Take $\partial/\partial y$...

$$x + 2xz\frac{\partial z}{\partial y} + 2yz + y^2\frac{\partial z}{\partial y} = 0$$

$$1 - 2\frac{\partial z}{\partial y} - 2 + \frac{\partial z}{\partial y} = 0$$

$$-\frac{\partial z}{\partial y} = 1$$

$$\boxed{\frac{\partial z}{\partial y} = -1}$$

#3 Find the tangent plane to $f(x,y) = x^2 + xy + y^3$ when $x=2,\ y=-1$.

$$\frac{\partial f}{\partial x} = 2x + y \Rightarrow f_x(2,-1) = 4-1 = 3$$

$$\frac{\partial f}{\partial y} = x + 3y^2 \Rightarrow f_y(2,-1) = 2+3 = 5$$

$$z_0 = f(2,-1) = 4-2-1 = 1$$

$$\boxed{z-1 = 3(x-2) + 5(y+1)}.$$

#4 Consider a box of length L, width W that is 4 ft. high. You want to construct the box with L=2 ft and W=1 ft, but are only able to measure these to within 1/8 inch precision. Estimate the largest volume the box could potentially have by applying differentials.

$$V = 4LW$$

$$\Delta V \approx 4W\Delta L + 4L\Delta W = 4(\Delta L + 2\Delta W)$$

$$= 4\left(\frac{1}{8\cdot 12} + \frac{2}{8\cdot 12}\right) = \frac{3}{2\cdot 12} = \frac{3}{24}$$

$$\Delta V \cong \frac{1}{8}(ft.)^3$$

$$V_{MAX} \approx \left(8 + \frac{1}{8}\right) ft^3$$