

Functions of multiple variables

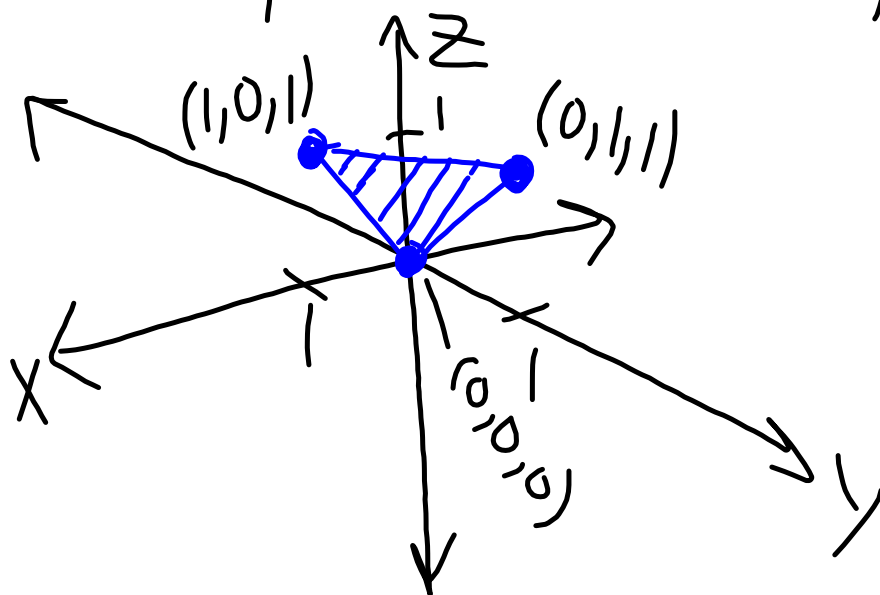
$$f = f(x_1, x_2, x_3, \dots, x_n)$$

with  $f(\cdot)$  a real number

We say that  $f$  is a real, scalar-valued function of  $n$  variables;

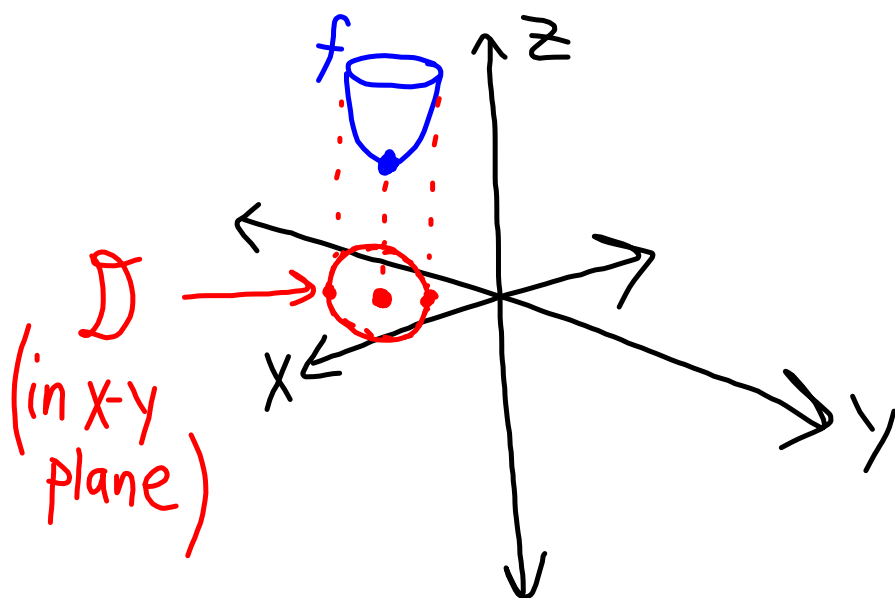
$$f : \mathbb{R}^n \rightarrow \mathbb{R}.$$

Consider  $f(x, y) = x + y$ ;  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ .  
We visualize sometimes via a  
"surface plot" ... set  $z = f(x, y)$ .



$$f : \mathcal{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}$$

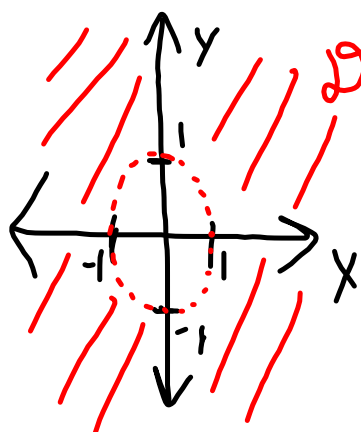
we often need to restrict the domain



Usually, the domain may be determined by inspection.

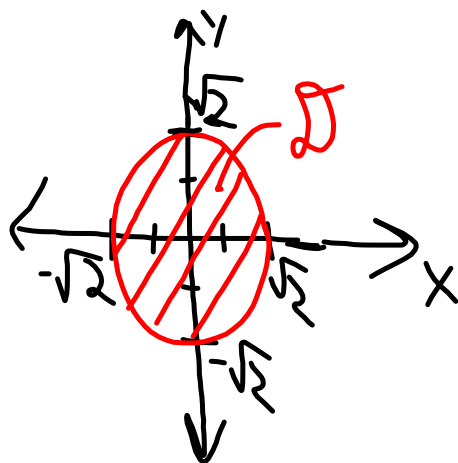
$$f(x, y) = \ln(x^2 + y^2 - 1)$$

$$D = \{(x, y) \mid x^2 + y^2 > 1\}$$

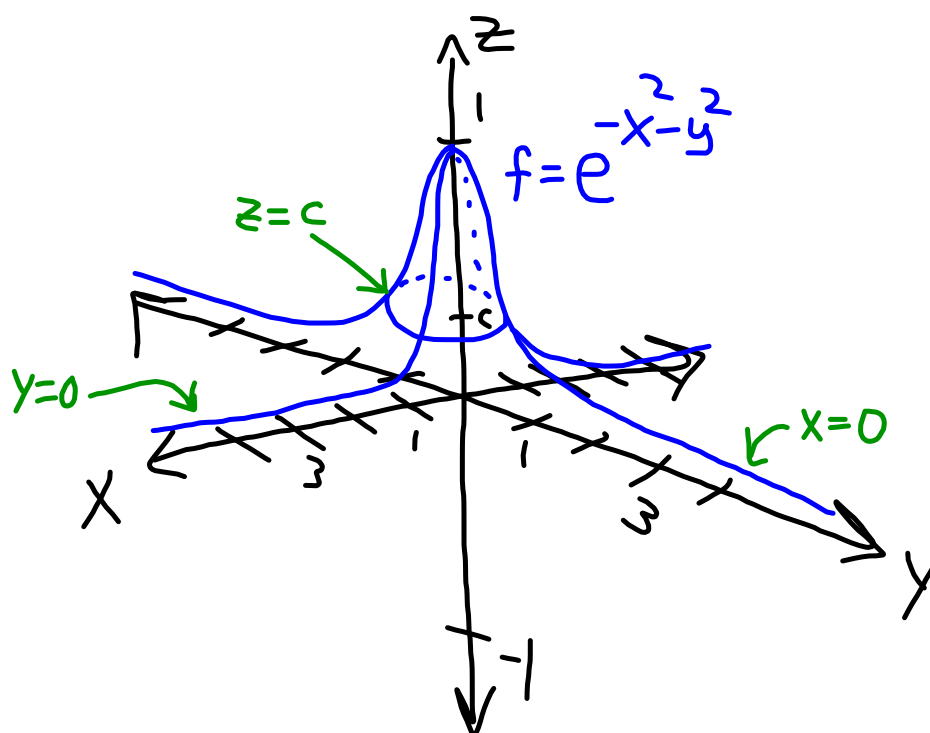


EX: Find the domain of  
 $f(x,y) = \sqrt{2-x^2-y^2}$ .

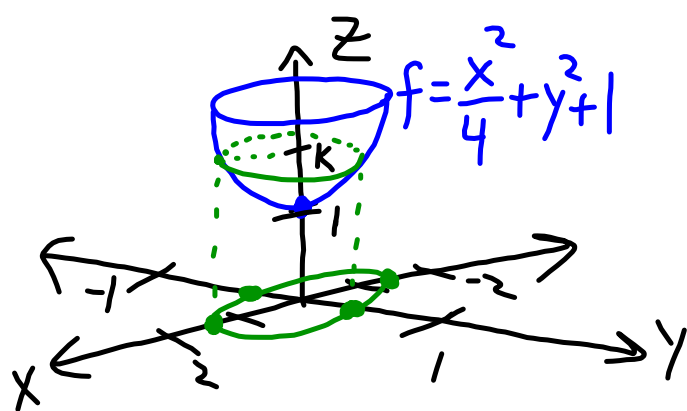
Then  $2-x^2-y^2 \geq 0 \Rightarrow x^2+y^2 \leq 2$



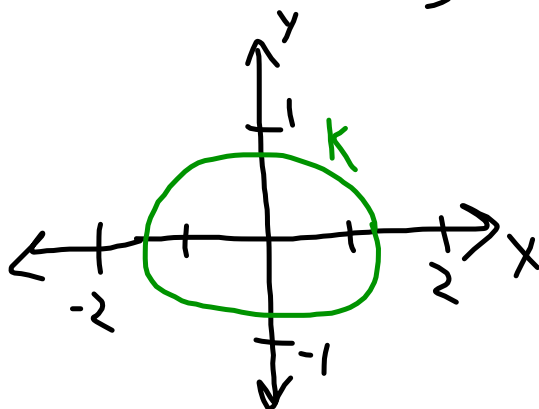
We have seen before that traces may be used to help sketch surface plots.



Level curves are another visualization tool.

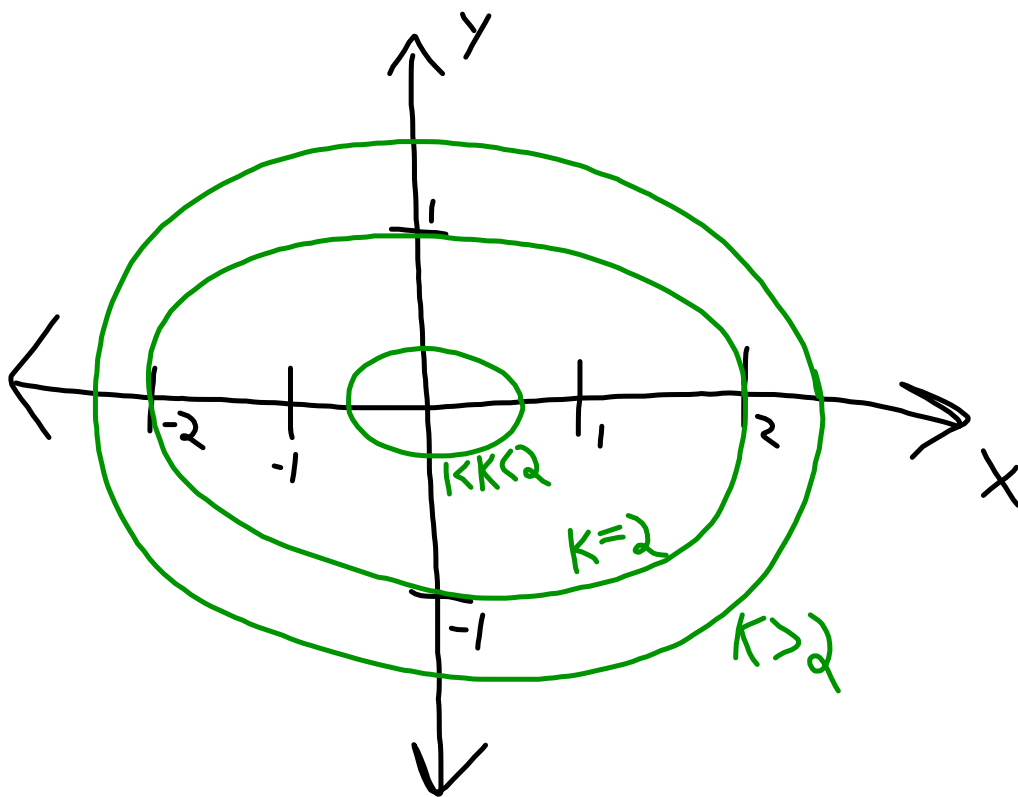


} Project traces  
 $z = k$  down into  
x-y plane.



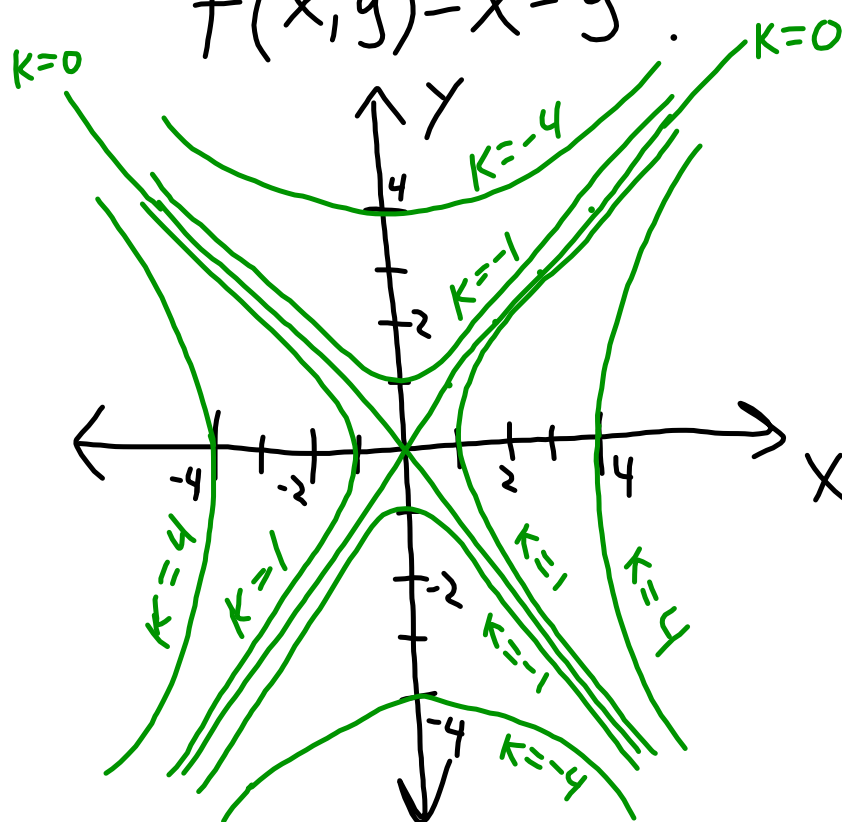
} Visualize  
just in the  
x-y plane

Now add in more k-values...





Ex: Sketch a contour plot for  
 $f(x, y) = x^2 - y^2$ .



Level surfaces

Consider  $f(x, y, z) = k$

e.g.  $f(x, y, z) = x^2 + y^2 + z^2 = k > 0$

  
Spheres of radius  $\sqrt{k}$

One could try to visualize these together,  
but this is difficult (how do you see one sphere  
within the next?) so we will not focus on level  
surfaces.

LIMITS If  $f(x,y) \rightarrow L$  as  $(x,y) \rightarrow (a,b)$   
then we say

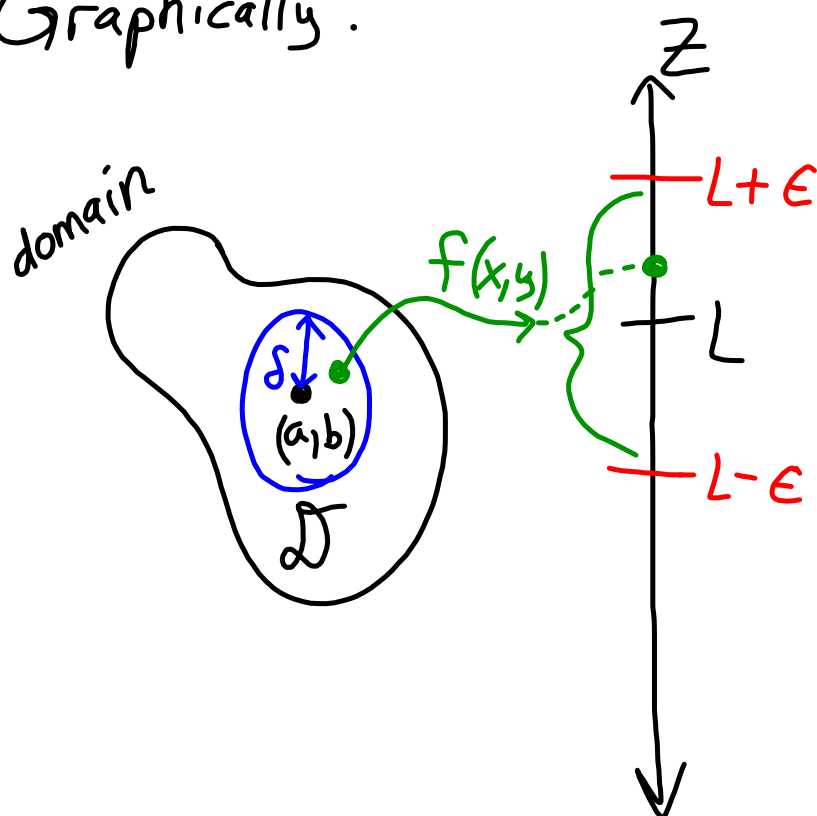
$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L.$$

More precisely, if for ANY  $\epsilon > 0$  there  
is some  $\delta = \delta(\epsilon) > 0$  such that

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \epsilon$$

then the limit exists.

Graphically :

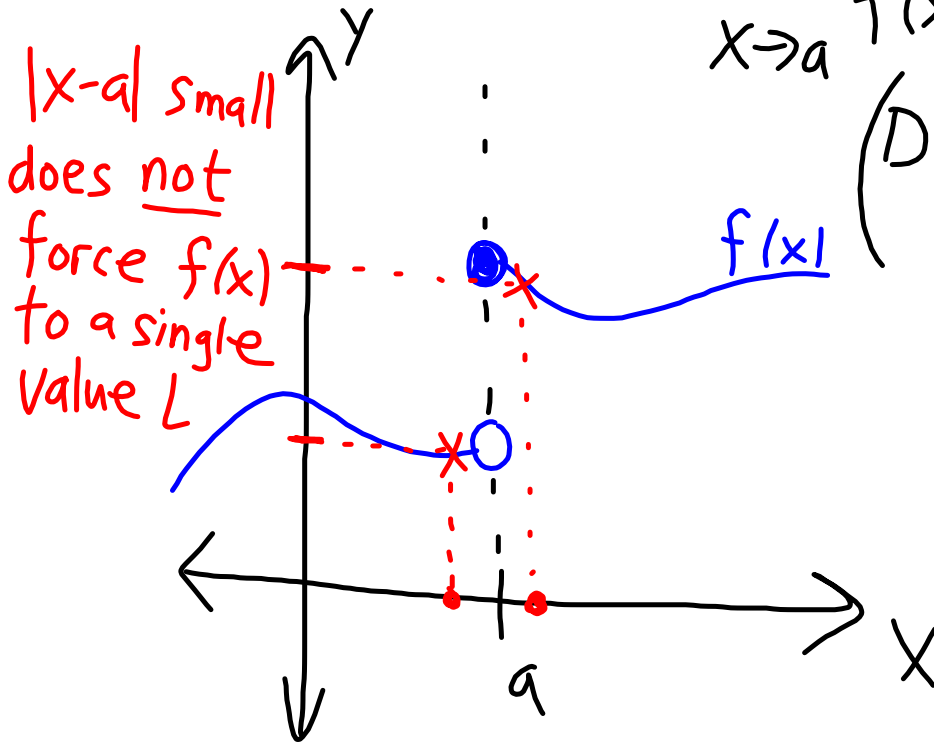


• If we choose  $\epsilon > 0$  very small there must be a neighborhood of  $(a, b)$  that  $f$  maps into  $(L - \epsilon, L + \epsilon)$ .

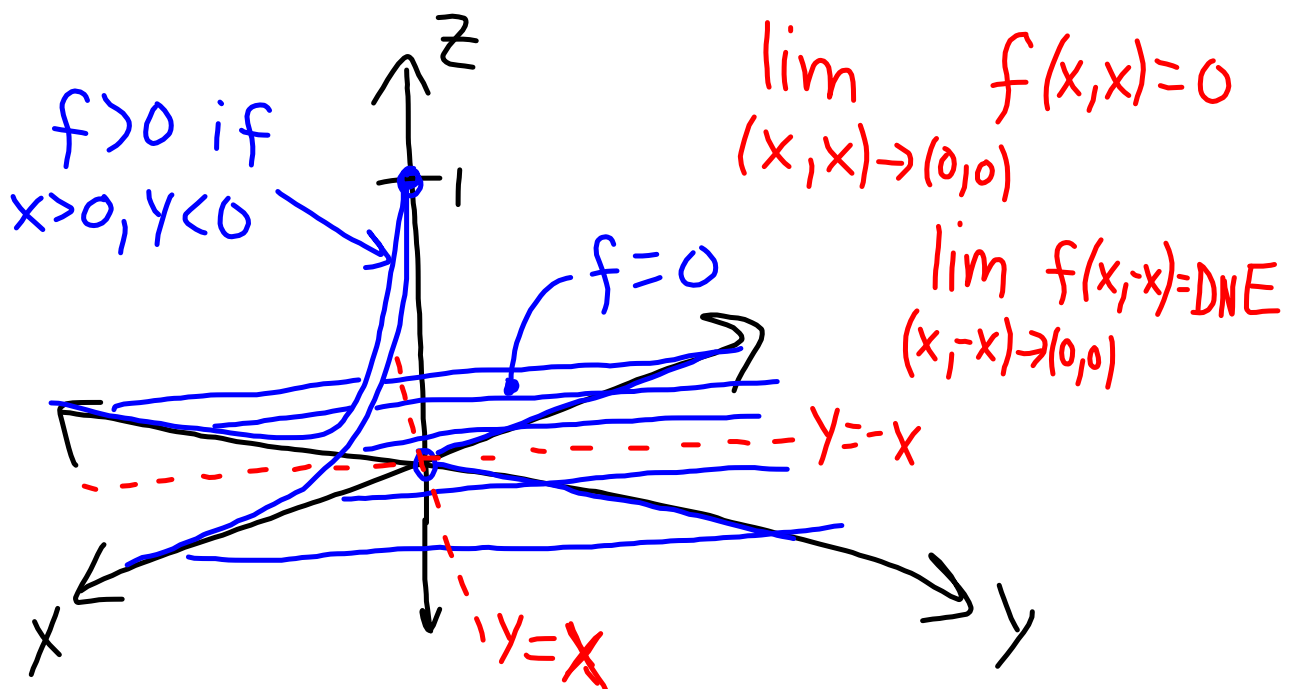
1D reminder:

$$\lim_{x \rightarrow a} f(x) = ?$$

(Does not exist)



In 2D (or higher) the limit may only exist along specific paths.



EX: Consider  $f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$ .

Note  $(0, 0)$  is NOT in the domain, but we can still talk about  $\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = 1$ .

(Same as  $\lim_{r \rightarrow 0^+} \frac{\sin(r)}{r} = 1$ )

EX: Find  $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2}$  or write DNE  
if the limit does not exist.

The way you usually show DNE is to find two paths that yield different limits.

$$1) \text{ Set } y=0 \dots \lim_{\substack{(x,0) \rightarrow (0,0) \\ (x \neq 0)}} \frac{x \cdot 0}{x^2} = 0$$

$$2) \text{ Set } y=-x \dots \lim_{\substack{(x,-x) \rightarrow (0,0) \\ (x \neq 0)}} \frac{x(-x)}{2x^2} = \underline{\underline{-\frac{1}{2} \neq 0}}$$



To show that limits exist we will usually cheat a bit and cite CONTINUITY.

When  $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$  } • limit exists  
• equals  $f(a,b)$

We say  $f$  is continuous at  $(a,b)$ .  
(CTS)

\* If  $f$  is CTS. at every point of a set  $A$

We say " $f$  is CTS on  $A$ ".

Polynomials are CTS everywhere and rational functions are CTS on their domains.

EX: Find  $\lim_{(x,y) \rightarrow (1,1)} \frac{x^2 - y^2}{x^2 + y^2}$  or write

DNE if the limit does not exist.

Since  $(1,1)$  is in the domain, we note  $\frac{x^2 - y^2}{x^2 + y^2}$  is rational, hence  $\lim_{(x,y) \rightarrow (1,1)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{1^2 - 1^2}{1^2 + 1^2} = 0$ .

EX: Define  $g(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ .

Show  $\lim_{(x,y) \rightarrow (0,0)} g(x,y) = DNE$ .

Useful short-cut: try  $y = kx$

$$\lim_{(x,kx) \rightarrow (0,0)} g(x,kx) = \lim_{x \rightarrow 0} \frac{x^2 - k^2 x^2}{x^2 + k^2 x^2} = \lim_{x \rightarrow 0} \frac{1 - k^2}{1 + k^2} = \frac{1 - k^2}{1 + k^2}$$

The limit is  $k$ -dependent  $\Rightarrow DNE$ .

Ex:  $\lim_{(x,y) \rightarrow (0,0)} \frac{1}{x^2+y^2} = \infty$ ; we can

conceivably discuss such limits like in Calc I.

Ex: Is  $f(x,y) = \begin{cases} 1, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$  CTS at  $(0,0)$ ?

No;  $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = 1 \neq f(0,0) = 0$ .

Guidelines to identify continuous functions so you know when certain limits automatically exist

- (1) Sums, differences and products of CTS functions are CTS (check domain)
- (2) Ratios of CTS functions (check domain)
- (3) Compositions of CTS functions are CTS\*\*

$$** \text{ If } g(a) = b = \lim_{x \rightarrow a} g(x) \text{ and } \lim_{x \rightarrow b} f(x) = f(b)$$

$$\text{then } \lim_{x \rightarrow a} f(g(x)) = f(b) = f(g(a)).$$

EX: Find the limit or write DNE...

$$\lim_{(x,y) \rightarrow (1,0)} \ln \left( \frac{1+y^2}{x^2+xy} \right) = ?$$

$\ln(z)$  is CTS at  $z=1$  and  $\frac{1+y^2}{x^2+xy}$  is rational / CTS at  $(1,0)$  so just plug in

$$(x,y) = (1,0) \dots \lim_{(x,y) \rightarrow (1,0)} \ln \left( \frac{1+y^2}{x^2+xy} \right) = \ln(1) = 0.$$

EX:  $\lim_{(x,y) \rightarrow (1,-2)} e^{-x^2y} \sin(x+y^2) = ?$

CTS  
CTS

$= e^{-(1)^2(-2)} \sin(1+(-2)^2) = e^2 \sin(5)$

$$\underline{\text{EX:}} \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2+y^2}} = ?$$

division by zero = red flag...

The limit turns out to be zero, but it takes thought. Note the numerator is effectively "higher-order" than the denominator, so it vanishes faster as  $(x,y) \rightarrow (0,0)$ .

This requires an " $\epsilon$ - $\delta$ " proof... we will skip.



$$\underline{\text{EX}}: \lim_{(x,y) \rightarrow (0,0)} \frac{xy^4}{x^2+y^8} = ?$$

$(0,0)$  is NOT in the domain... need to investigate.

Along paths like  $y=kx$  the limit = 0.

However,  $y^8$  is a red flag... numerator is 5<sup>th</sup> order. Take  $y=x^{1/4}$ ...

$$\lim_{(x, x^{1/4}) \rightarrow (0,0)} \frac{x \cdot x}{x^2 + x^2} = \underline{\frac{1}{2}} \neq 0. \quad \boxed{\text{DNE}}$$

Practice!

#1

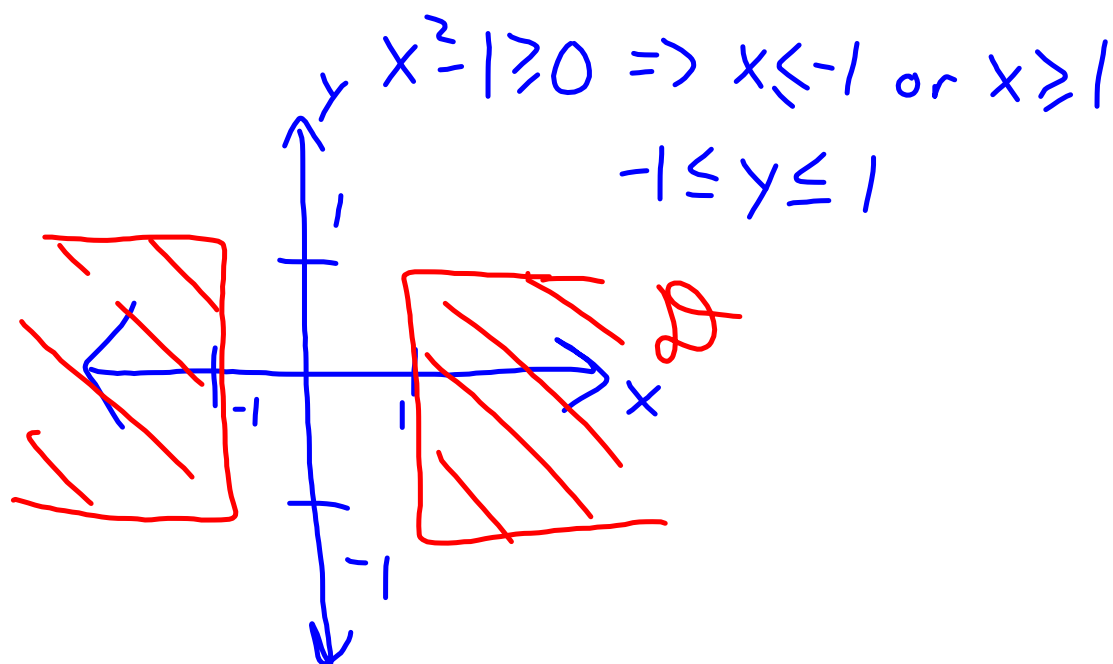
$$(a) \lim_{(x,y) \rightarrow (1,1)} \frac{y^2 \sin^2 x}{x^4 + y^4} = ?$$

$$(b) \lim_{(x,y) \rightarrow (0,0)} \frac{y^2 \sin^2 x}{x^4 + y^4} = ?$$

$$(a) = \frac{(1)^2 \sin^2(1)}{1^4 + 1^4} = \frac{1}{2} \sin^2(1). \quad (\text{CTS})$$

$$(b) = \lim_{(x,y) \rightarrow (0,0)} \frac{y^2 x^2}{x^4 + y^4} \quad \left. \begin{array}{l} y = kx? \quad \frac{k^2 x^4}{x^4(1+k^4)} \\ \underline{\underline{DNE}} \end{array} \right\}$$

#2 Sketch the domain of  
 $f(x,y) = \sqrt{x^2-1} - \sqrt{1-y^2}$

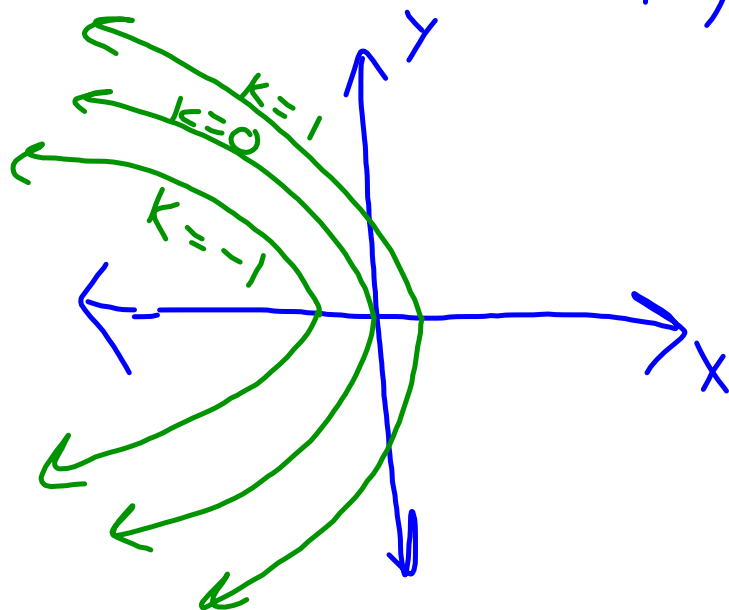


#3 Sketch level sets of  $f(x,y)=x+y^2$ .

$$k=0 \Rightarrow x=-y^2$$

$$k=1 \Rightarrow x=1-y^2$$

$$k=-1 \Rightarrow x=-1-y^2$$



$$\textcircled{\#4} \lim_{(x,y) \rightarrow (0,0)} \frac{x + \sin(y)}{x + y + 1} = \frac{0 + 0}{0 + 0 + 1} = \frac{0}{1} = 0 \text{ (CTS)}$$

$$\textcircled{\#5} \lim_{(x,y) \rightarrow (1,0)} \ln\left(\frac{2x^2 + y^2}{x^2 - 2y^2}\right) = \ln\left(\frac{2}{1}\right) = \ln(2) \text{ (CTS)}$$

$$\textcircled{\#6} \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{2x^2 + y^2} = DNE$$

$$\frac{x^2 + k^2 x^2}{2x^2 + k^2 x^2} = \frac{1 + k^2}{2 + k^2}$$