Lecture 5: Vector derivatives, integrals, arc length

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

Let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$. This is the same as solving 3 "Calc-I" problems:

$$\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

Ex: Helix... $\vec{r}(t) = \langle \underbrace{\cos(t)}_{x(t)}, \underbrace{\sin(t)}_{y(t)}, \underbrace{2t}_{z(t)} \rangle$.

Find $\vec{r}\,'(t)$.

$x'(t) = -\sin(t)$

$y'(t) = \cos(t)$

$z'(t) = 2$

$\Rightarrow \vec{r}\,'(t) = \langle -\sin(t), \cos(t), 2 \rangle$.

A closer look at that limit definition:

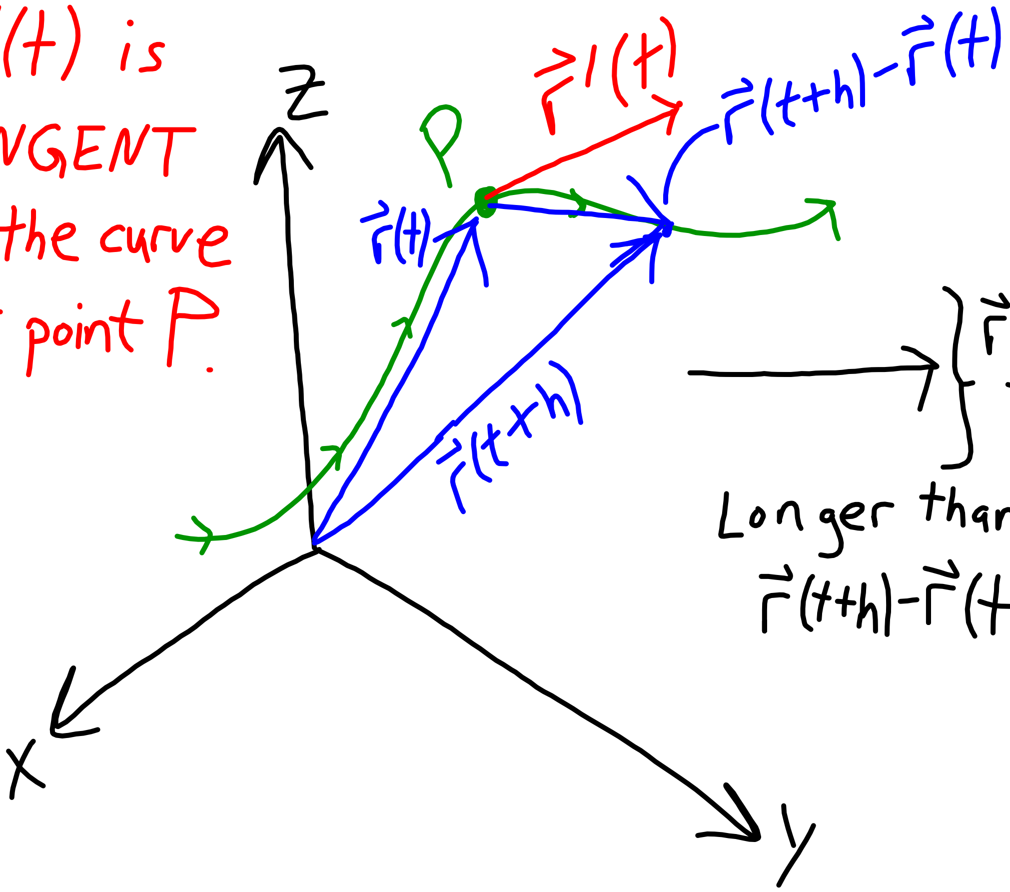
$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$= \lim_{h \to 0} \frac{\langle x(t+h) - x(t),\, y(t+h) - y(t),\, z(t+h) - z(t) \rangle}{h}$$

$$= \left\langle \lim_{h \to 0} \frac{x(t+h) - x(t)}{h},\, \lim_{h \to 0} \frac{y(t+h) - y(t)}{h},\, \lim_{h \to 0} \frac{z(t+h) - z(t)}{h} \right\rangle$$

$$= \langle x'(t), y'(t), z'(t) \rangle.$$

$\vec{r}\,'(t)$ is TANGENT to the curve at point P.



$$\left.\right\}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$

Longer than $\vec{r}(t+h)-\vec{r}(t)$; $h<1$

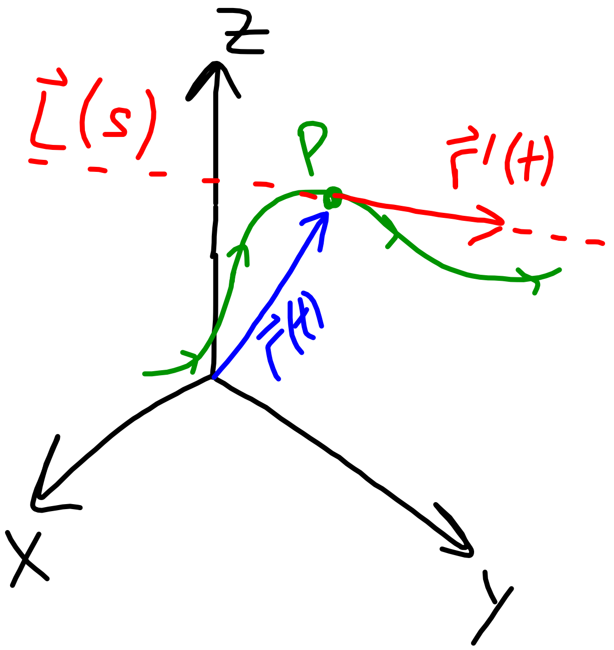
Notation... the unit tangent vector will be $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$.

Ex: Let $\vec{r}(t) = \langle t, t^2, t^2 \rangle$.
Find the unit tangent vector.

$\vec{r}\,'(t) = \langle 1, 2t, 2t \rangle$, $|\vec{r}\,'| = \sqrt{1+4t^2+4t^2}$

$$\vec{T}(t) = \frac{1}{\sqrt{1+8t^2}} \langle 1, 2t, 2t \rangle.$$

Tangent Lines



- $\vec{r}(t)$ points to P
- $\vec{r}'(t)$ gives direction vector for line
- $\vec{L}(s) = \vec{r}(t) + s\,\vec{r}'(t)$.

EX: Find the tangent line for $\vec{r}(t) = \langle 15t, t^2-1, t/(t+1) \rangle$ at the point $(45, 8, 3/4)$.

Note $\vec{r}(3) = \langle 45, 8, 3/4 \rangle$, so $\underline{t=3}$.

$$\vec{r}\,'(t) = \left\langle 15, 2t, \frac{t+1-t}{(t+1)^2} \right\rangle$$

$$\Rightarrow \vec{r}\,'(3) = \langle 15, 6, 1/16 \rangle.$$

$$\vec{L}(s) = \left\langle 45+15s, 8+6s, \frac{3}{4} + s/16 \right\rangle.$$

EX: Find the tangent line to

$\vec{r}(t) = \langle t^2 - t + 2, t^2 + 5t, t \rangle$

when $t = 5$.

$\vec{r}(5) = \langle 25 - 5 + 2, 25 + 25, 5 \rangle$
$= \langle 22, 50, 5 \rangle$.

$\vec{r}\,'(t) = \langle 2t - 1, 2t + 5, 1 \rangle$
$\Rightarrow \vec{r}\,'(5) = \langle 9, 15, 1 \rangle$
$\Rightarrow \vec{L}(s) = \langle 22 + 9s, 50 + 15s, 5 + s \rangle$.

Differentiation Rules

(1) $\frac{d}{dt}(\vec{u}+\vec{v})=\frac{d\vec{u}}{dt}+\frac{d\vec{v}}{dt}$.

(2) $\frac{d}{dt}(c\vec{u})=c\frac{d\vec{u}}{dt}$.

(3) $\frac{d}{dt}(f(t)\vec{u}(t))=\left(\frac{df}{dt}\right)\vec{u}+f\frac{d\vec{u}}{dt}$.

(4) $\frac{d}{dt}(\vec{u}\cdot\vec{v})=\frac{d\vec{u}}{dt}\cdot\vec{v}+\vec{u}\cdot\frac{d\vec{v}}{dt}$.

(5) $\frac{d}{dt}(\vec{u}\times\vec{v})=\left(\frac{d\vec{u}}{dt}\right)\times\vec{v}+\vec{u}\times\left(\frac{d\vec{v}}{dt}\right)$.

(6) $\frac{d}{dt}\vec{u}(f(t))=\left(\frac{d\vec{u}(f(t))}{dt}\right)\frac{df(t)}{dt}$.

EX: (Useful trick.) Find $\frac{d}{dt}|\vec{r}(t)|^2$.

Apply $|\vec{r}(t)|^2 = \vec{r}(t)\cdot\vec{r}(t)$, so

$$\frac{d}{dt}|\vec{r}|^2 = \frac{d}{dt}(\vec{r}\cdot\vec{r}) = \vec{r}\,'\cdot\vec{r} + \vec{r}\cdot\vec{r}\,'$$

$$= 2\vec{r}(t)\cdot\vec{r}\,'(t).$$

Integration of vectors

Definite: $\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b x\,dt, \int_a^b y\,dt, \int_a^b z\,dt \right\rangle$.

Indefinite: Let $X(t) = \int x(t)\,dt$, $Y(t) = \int y(t)\,dt$, ...

$$\Rightarrow \int \vec{r}(t)\,dt = \left\langle X(t)+c_1, Y(t)+c_2, Z(t)+c_3 \right\rangle$$

$$= \left\langle X(t), Y(t), Z(t) \right\rangle + \vec{C}.$$

Ex: Let $\vec{r}(t) = \langle t, \cos(t), 1-t^2 \rangle$.

Find $\int_0^{\pi} \vec{r}(t)\,dt$.

$$\int_0^{\pi} t\,dt = \frac{1}{2}\pi^2$$

$$\int_0^{\pi} \cos(t)\,dt = 0$$

$$\int_0^{\pi} 1-t^2\,dt = \pi - \frac{1}{3}\pi^3$$

$$\int_0^{\pi} \vec{r}(t)\,dt = \left\langle \frac{1}{2}\pi^2, 0, \pi - \frac{1}{3}\pi^3 \right\rangle.$$

Ex: Find $\int \langle \frac{1}{t}, e^t, t^{10} \rangle dt.$

$= \langle \ln|t|, e^t, \frac{t^{11}}{11} \rangle + \vec{C}.$

One comment... what is $\vec{r}''(t)$?

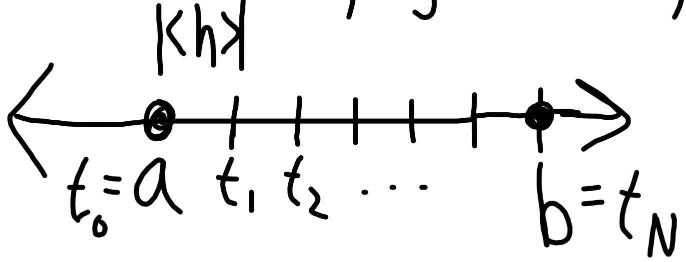
Note that $\vec{r}'(t)$ traces out some curve, so then $\vec{r}''(t)$ is tangent to <u>that</u> curve.

Arc Length - Consider $\vec{r}(t)$ with $a \leq t \leq b$.

- DIVIDE the arc by choosing

$$t_0 = a, \quad t_1 = a+h, \quad t_2 = a+2h,$$

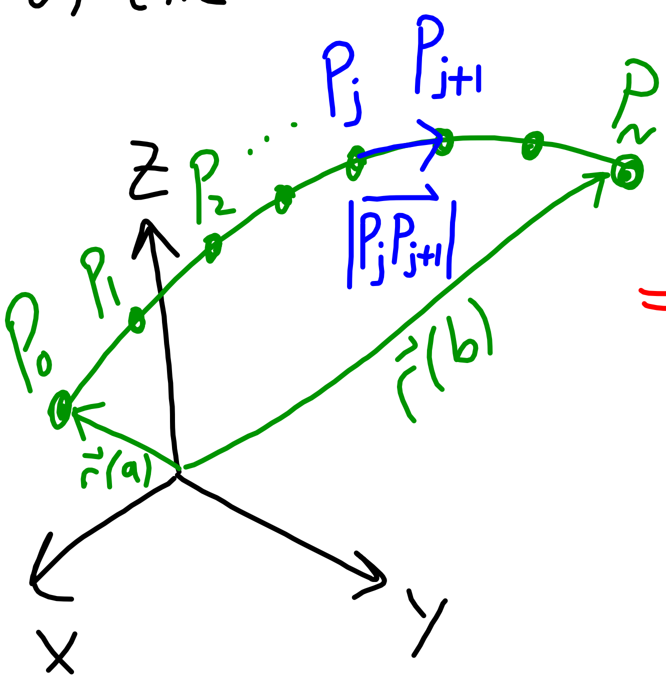
$$\ldots, \quad t_j = a+jh, \ldots, \quad t_N = b.$$



- Let P_j be the point for $\vec{r}(t_j)$.

Let L be the length of the curve.

$$L \approx \sum_{j=0}^{N-1} \left|\overrightarrow{P_j P_{j+1}}\right|.$$



But $\overrightarrow{P_j P_{j+1}} = \vec{r}(t_{j+1}) - \vec{r}(t_j)$

$$\Rightarrow L \approx \sum_{j=0}^{N-1} \left|\vec{r}(t_{j+1}) - \vec{r}(t_j)\right|$$

$$= h \sum_{j=0}^{N-1} \left|\frac{\vec{r}(t_{j+1}) - \vec{r}(t_j)}{h}\right|$$

Riemann sum!

$$L = \lim_{N\to\infty} h \sum_{j=0}^{N-1} \left| \frac{\vec{r}(t_j+h) - \vec{r}(t_j)}{h} \right|$$

$$= \lim_{N\to\infty} h \sum_{j=0}^{N-1} \left| \vec{r}\,'(t_j^*) \right| = \int_a^b |\vec{r}\,'(t)|\,dt.$$

$t_j \leq t_j^* \leq t_{j+1}$

$$\boxed{L = \int_a^b |\vec{r}\,'(t)|\,dt}$$

EX: Find the length of the helical segment of $\langle \cos(t), \sin(t), t \rangle$ for $0 \leq t \leq 2\pi$.

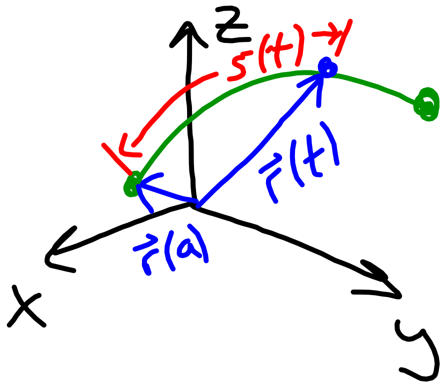
$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\underbrace{\sin^2(t) + \cos^2(t)}_{=1} + 1} = \sqrt{2}$$

$$\Rightarrow L = \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2}.$$

The arc-length function

$$s(t) = \int_a^t |\vec{r}\,'(\tau)|\,d\tau.$$



Note that $s'(t) = |\vec{r}\,'(t)|$
so $|\vec{r}\,'(t)|$ is the rate of increase of the arc length at t.

If we can invert $s = s(t)$ to get $t = t(s)$ then we can parameterize $\vec{r}$ in terms of s (arc length):

$$\vec{r}(t) = \vec{r}(t(s)) = \vec{r}(s).$$

EX: Reparameterize the curve $\vec{r}(t) = \langle t^2, t^2, 2t^2 \rangle$ starting from the origin in terms of arc length

$$\vec{r}\,'(t) = \langle 2t, 2t, 4t \rangle$$

$$\Rightarrow s(t) = \int_0^t \sqrt{4\tau^2 + 4\tau^2 + 16\tau^2}\, d\tau$$

$$= \sqrt{24} \int_0^t \tau\, d\tau = 2\sqrt{6} \int_0^t \tau\, d\tau = t^2\sqrt{6}$$

... so $s = t^2\sqrt{6} \Rightarrow t^2 = \frac{s}{\sqrt{6}}$

$$\vec{r}(t) = \langle t^2, t^2, 2t^2 \rangle$$

$$\Rightarrow \vec{r}(s) = \left\langle \frac{s}{\sqrt{6}}, \frac{s}{\sqrt{6}}, \frac{2s}{\sqrt{6}} \right\rangle.$$

* So, for example, when arc length $= \frac{1}{2} = s$ you now know

$$\vec{r} = \frac{1}{\sqrt{6}} \left\langle \frac{1}{2}, \frac{1}{2}, 1 \right\rangle.$$

#1 Find a vector tangent to
$\vec{r}(t) = \langle t^3, 1+t^2, 4t \rangle$
at the point $(8, 5, 8)$.

$\vec{r}\,' = \langle 3t^2, 2t, 4 \rangle$
$\vec{r}(2) = \langle 12, 4, 4 \rangle$

#2 Find $\vec{T}(t)$ if $\vec{r}(t) = \langle \cos(t), \sin(t), 4t \rangle$.

$$\vec{T}(t) = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \underbrace{\frac{1}{\sqrt{17}} \langle -\sin(t), \cos(t), 4 \rangle}.$$

$$\vec{r}\,' = \langle -\sin(t), \cos(t), 4 \rangle$$

$$|\vec{r}\,'| = \sqrt{\cos(t)^2 + \sin^2(t) + 16} = \sqrt{17}$$

#3 Find $\int_0^1 \vec{r}(t)\,dt$; $\vec{r}(t) = \langle e^{2t}, 3t, 1 \rangle$.

$$\int_0^1 e^{2t}\,dt = \left(\frac{1}{2}e^{2t}\right)\Bigg|_{t=0}^{t=1} = \frac{1}{2}e^2 - \frac{1}{2}$$

$$\int_0^1 3t\,dt = \frac{3}{2}t^2\Big|_0^1 = \frac{3}{2}$$

$$\int_0^1 \vec{r}\,dt = \left\langle \frac{1}{2}(e^2-1), \frac{3}{2}, 1 \right\rangle.$$

#4 Let $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 3t \rangle$.
Find the length of the curve traced out between $(2,0,0)$ & $(0,2,3\pi/2)$.

$t=0$ $t=\pi/2$

$\vec{r}\,' = \langle -2\sin(t), 2\cos(t), 3 \rangle$

$|\vec{r}\,'| = \sqrt{4(\sin^2(t)+\cos^2(t))+9} = \sqrt{13}$

$$\int_0^{\pi/2} |\vec{r}\,'|\,dt = \int_0^{\pi/2} \sqrt{13}\,dt = \boxed{\frac{\pi\sqrt{13}}{2}}.$$

#5 Reparameterize $\vec{r}(t)=\frac{1}{4}\langle\cos(t),\sin(t),t\sqrt{2}\rangle$ in terms of arc length from $(\frac{1}{4},0,0)$.

$$s=\int_{t=0}^{t}|\vec{r}\,'(\tau)|\,d\tau=\frac{t\sqrt{3}}{4}\Rightarrow t=\frac{4s}{\sqrt{3}}$$

$t=0$

$$\vec{r}\,'=\frac{1}{4}\langle-\sin(t),\cos(t),\sqrt{2}\rangle$$

$$|\vec{r}\,'|=\frac{1}{4}\sqrt{1+2}=\frac{\sqrt{3}}{4}$$

$$\vec{r}(s)=\frac{1}{4}\left\langle\cos\left(\frac{4s}{\sqrt{3}}\right),\sin\left(\frac{4s}{\sqrt{3}}\right),4s\sqrt{\frac{2}{3}}\right\rangle.$$