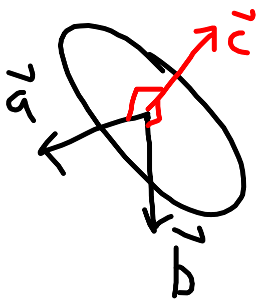
Math 2110: Multivariable Calculus

Lecture 3: Cross-products, lines, planes



Given $\vec{a}$, $\vec{b}$ find $\vec{c}$ such that

$\vec{a} \perp \vec{c}$ & $\vec{b} \perp \vec{c}$?

Could try solving

$$\left.\begin{aligned} \vec{a} \cdot \vec{c} = 0 \\ \vec{b} \cdot \vec{c} = 0 \end{aligned}\right\}$$

2 equations

3 unknowns

$\langle c_1, c_2, c_3 \rangle$

$\vec{c}$ is not unique, but one important example is the cross-product

$$\vec{c} = \vec{a} \times \vec{b}$$
$$= \langle a_2 b_3 - b_2 a_3,\ b_1 a_3 - a_1 b_3,\ a_1 b_2 - b_1 a_2 \rangle.$$

You can check using this definition that

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = (\vec{a} \times \vec{b}) \cdot \vec{b} = 0.$$

Determinants are used to remember the formula...

2x2 determinant: $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2$

3x3: $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

Now we can write

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\vec{k}$$

$$= \langle a_2b_3 - b_2a_3 \,,\, b_1a_3 - a_1b_3 \,,\, a_1b_2 - b_1a_2 \rangle$$

Example: Find $\vec{a}\times\vec{b}$; $\vec{a}=\langle 0,1,-1\rangle$

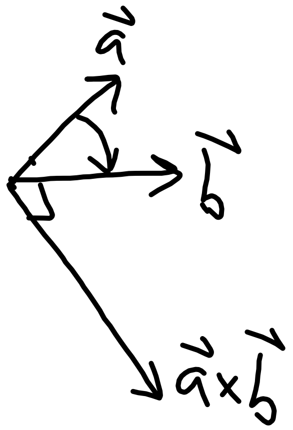
$\vec{b}=\langle 2,4,6\rangle$.

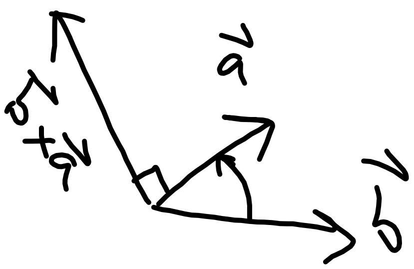
$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 1 & -1\\ 2 & 4 & 6\end{vmatrix}=(6-(-4))\vec{i}-(0\cdot 6-(-2))\vec{j}+(0\cdot 4-2)\vec{k}$$

$$=\langle 10,-2,-2\rangle.$$

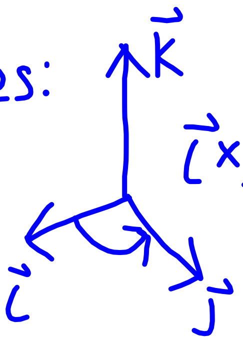
What direction is $\vec{a} \times \vec{b}$?

"Right-hand rule"

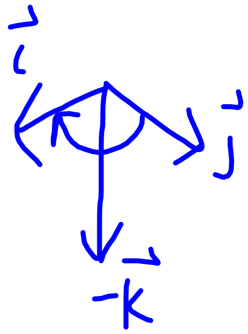




Examples:



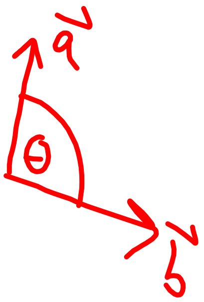
$\vec{i} \times \vec{j} = \vec{k}$

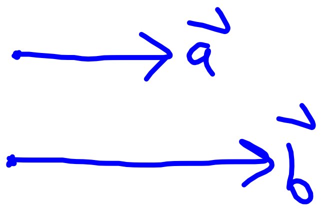


$\vec{j} \times \vec{i} = -\vec{k}$

There is a relationship with the acute angle between vectors:

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta .$$





parallel $\iff \theta = 0$

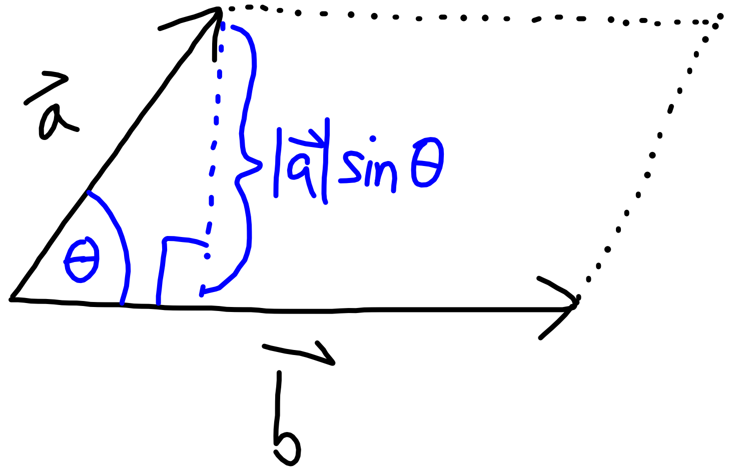
or

$\theta = \pi$

$\iff \sin\theta = 0$

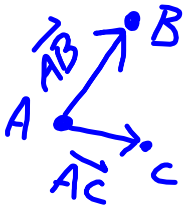
$\iff \vec{a}\times\vec{b} = 0.$

A geometric relationship:



$$\text{Area} = |\vec{a}||\vec{b}|\sin\theta = |\vec{a}\times\vec{b}|.$$

EX: Find $\vec{v}$ orthogonal to the plane containing points $\underbrace{(2,1,1)}_{A}$, $\underbrace{(0,3,-1)}_{B}$ & $\underbrace{(2,-2,3)}_{C}$.



Find $\vec{AB} \times \vec{AC}$... $\vec{AB} = \langle -2, 2, -2 \rangle$

$\vec{AC} = \langle 0, -3, 2 \rangle$

$$\Rightarrow \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & -2 \\ 0 & -3 & 2 \end{vmatrix} = (4-6)\vec{i} - (-4)\vec{j} + (6)\vec{k}$$

$$= \langle -2, 4, 6 \rangle.$$

EX: What is the area of the triangle with the points in the previous example?

Take $\frac{1}{2}$ area of parallelogram...

$$A = \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right| = \frac{1}{2}\left|\langle -2, 4, 6\rangle\right|$$

$$= \frac{1}{2}\sqrt{4+16+36} = \frac{1}{2}\sqrt{56}.$$

PROPERTIES

(1) $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$ (not associative)

(2) $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ (not commutative)

(3) $(c\vec{a})\times\vec{b} = c(\vec{a}\times\vec{b}) = \vec{a}\times(c\vec{b})$.

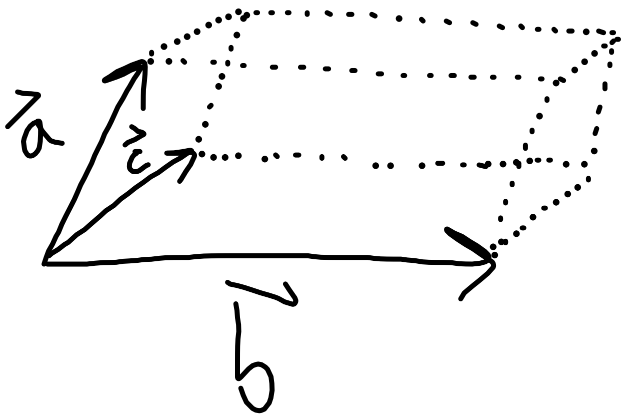
(4) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$.

(5) $(\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c}+\vec{b}\times\vec{c}$.

(6) $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$ $\}$ $\vec{a}\cdot(\vec{b}\times\vec{c})$ "Triple product"

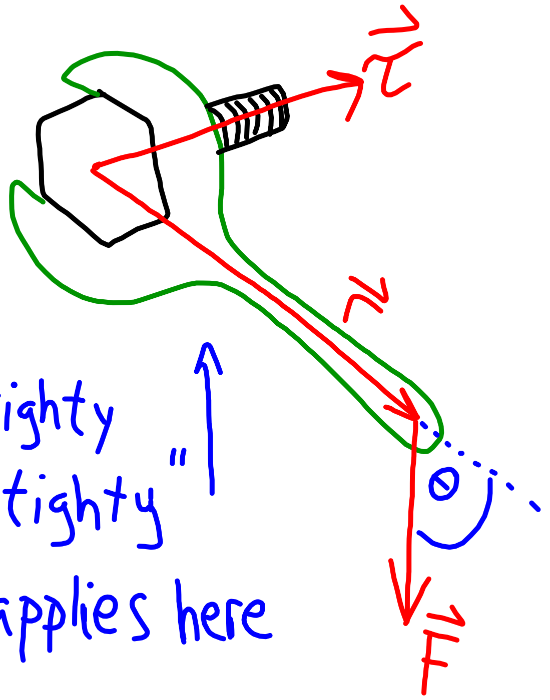
(7) $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$.

Triple product gives the volume of a parallelipiped



$$V = \left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|.$$

Application: Torque $\vec{\tau}$.



"righty-tighty" applies here

$$\vec{\tau} = \vec{r} \times \vec{F}$$

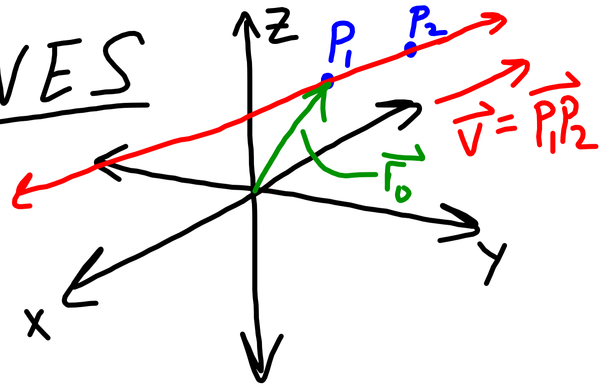
$\vec{F}$: force

$\vec{r}$: lever

EX: How much torque is generated if a force of 50N is applied at an angle of 85° to the end of a 0.1m-long wrench?

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}| \sin\theta$$
$$= (0.1)(50) \sin 85^\circ$$
$$= 5 \sin 85^\circ \quad (N\text{-}m).$$

LINES



Red line can be created by adding multiples of $\vec{v}$ to $\vec{r}_0$ ...

$$\langle x, y, z \rangle = \vec{r}_0 + t\vec{v}$$ vector equation for the line

$$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$$

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

Then

$$\langle x, y, z \rangle = \langle x_0 + t v_1, y_0 + t v_2, z_0 + t v_3 \rangle$$

"Parametric equations"

$$\boxed{\begin{aligned} x &= x_0 + t v_1 \\ y &= y_0 + t v_2 \\ z &= z_0 + t v_3 . \end{aligned}}$$

"Symmetric" equations...

Solve for t: $x = x_0 + t v_1 \Rightarrow t = \dfrac{x - x_0}{v_1}$

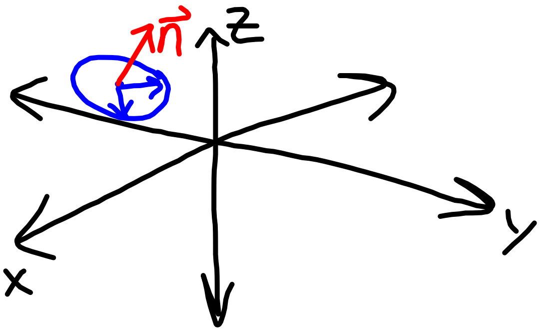
Also, $t = \boxed{\dfrac{y - y_0}{v_2} = \dfrac{z - z_0}{v_3} = \dfrac{x - x_0}{v_1}}$

symmetric equations

If, e.g. $v_3 = 0$, then $z = z_0 + \cancel{t v_3}^{\,0} = z_0$

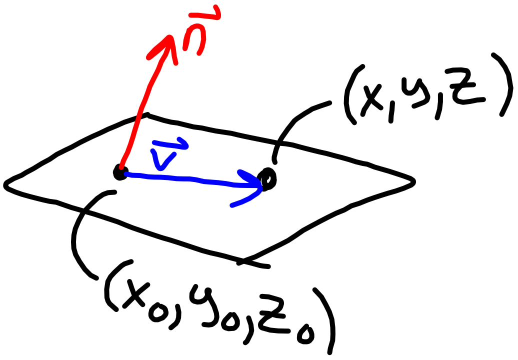
$\Rightarrow \boxed{\dfrac{x - x_0}{v_1} = \dfrac{y - y_0}{v_2} \text{ \& } z = z_0.}$

PLANES



Cross two non-parallel vectors in a plane and a normal vector $\vec{n}$ results.

(*) If $\vec{v}$ is in the desired plane then $\boxed{\vec{n}\cdot\vec{v}=0.}$



$\vec{n} = \langle n_1, n_2, n_3 \rangle$

$\vec{v} = \langle x-x_0, y-y_0, z-z_0 \rangle$

$$\vec{n}\cdot\vec{v} = \underbrace{n_1(x-x_0)+n_2(y-y_0)+n_3(z-z_0)=0}_{\text{Scalar equation for plane}}$$

Note how we can algebraically manipulate the equation of the plane...

$$n_1 x + n_2 y + n_3 z - n_1 x_0 - n_2 y_0 - n_3 z_0 = 0.$$

looks like $\underbrace{ax+by+cz+d=0}$.

So if you see this, then you know $\vec{n} = \langle a, b, c \rangle$. ☺

EX: Find the equation of the plane passing through the points

$(1,0,0) \leftarrow A$
$(0,2,0) \leftarrow B$
$(0,0,-3) \leftarrow C$

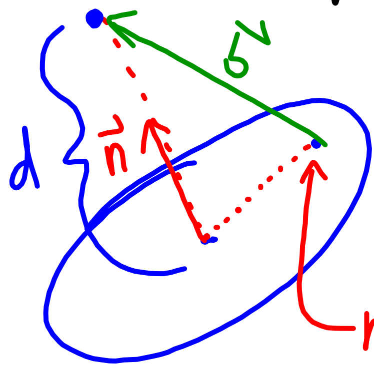
$\overrightarrow{AB} = \langle -1,2,0\rangle$; $\overrightarrow{AC} = \langle -1,0,-3\rangle$

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ -1 & 0 & -3 \end{vmatrix} = \langle -6,-3,2\rangle.$$

choose $(x_0,y_0,z_0) = (1,0,0)$ ... $\vec{n}\cdot\langle x-1, y-0, z-0\rangle = 0$

$$\Rightarrow -6(x-1) - 3y + 2z = 0.$$

Ex: Find the distance from $(3,1,1)$ to the plane $\underbrace{2x-2y+3z}=1$.



So take $\vec{n}=\langle 2,-2,3\rangle$.

$$d=\left|\text{comp}_{\vec{n}}\,\vec{b}\right|=\frac{|\vec{b}\cdot\vec{n}|}{|\vec{n}|} \quad (*)$$

need a point in the plane... $(\frac{1}{2},0,0)$

$$\Rightarrow \vec{b}=\langle 3-\tfrac{1}{2},1-0,1-0\rangle=\langle \tfrac{5}{2},1,1\rangle$$

$$\Rightarrow d=\frac{|\frac{5}{2}\cdot 2-2+3|}{\sqrt{4+4+9}}=\frac{6}{\sqrt{17}}.$$

Practice...

#1 Calculate the triple product $\vec{u}\cdot(\vec{v}\times\vec{w})$, if $\vec{u}=\langle 1,-1,2\rangle$

$\vec{v}=\langle -1,-1,0\rangle$

$\vec{w}=\langle 0,3,-5\rangle$.

$\vec{v}\times\vec{w}=\langle 5,-5,-3\rangle$

$\vec{u}\cdot(\vec{v}\times\vec{w})=5(1)+5-6=4$

#2 Consider two lines:

$$x = 1+2t \qquad x = -1-4t$$
$$y = 1+3t \quad \& \quad y = 5-6t$$
$$z = 1+4t \qquad z = 7-8t.$$

Are the lines parallel? Derive the symmetric equations for the line on the left.

(Yes, parallel.)

$$t = \frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{4}.$$

#3 Consider 3 planes:

(1) $x-2y+5z-9=0$

(2) $2x-3y+10z+7=0$

(3) $-3x+6y-15z+1=0$.

Which of these planes (if any) are parallel?

Yes, (1) & (3) ... multiply (1) by -3 to see this.

#4 Derive equations for the line passing through $(10, 4, -3)$ in the direction $\langle -5, 0, 6 \rangle$.

$\langle x, y, z \rangle = \langle 10 - 5t, 4, -3 + 6t \rangle$

$x = 10 - 5t$

$y = 4$

$z = -3 + 6t$

#5 Derive an equation for the plane passing through $(1,-1,4)$ with normal $\langle -3,-3,1\rangle$.

$$\vec{n}\cdot\vec{v}=0=\langle -3,-3,1\rangle\cdot\langle x-1,y+1,z-4\rangle=0$$

$$-3(x-1)-3(y+1)+z-4=0.$$

#6 Find the distance between the point $(-4, 5, 10)$ and the plane $\underbrace{x+y+z=1}$.

$(0,0,1)$ in the plane

$\vec{n} = \langle 1, 1, 1 \rangle$ "$\vec{b}$" $= \langle -4, 5, 9 \rangle$

$$d = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{n}|} = \frac{|-4+5+9|}{\sqrt{3}} = \frac{10}{\sqrt{3}}.$$