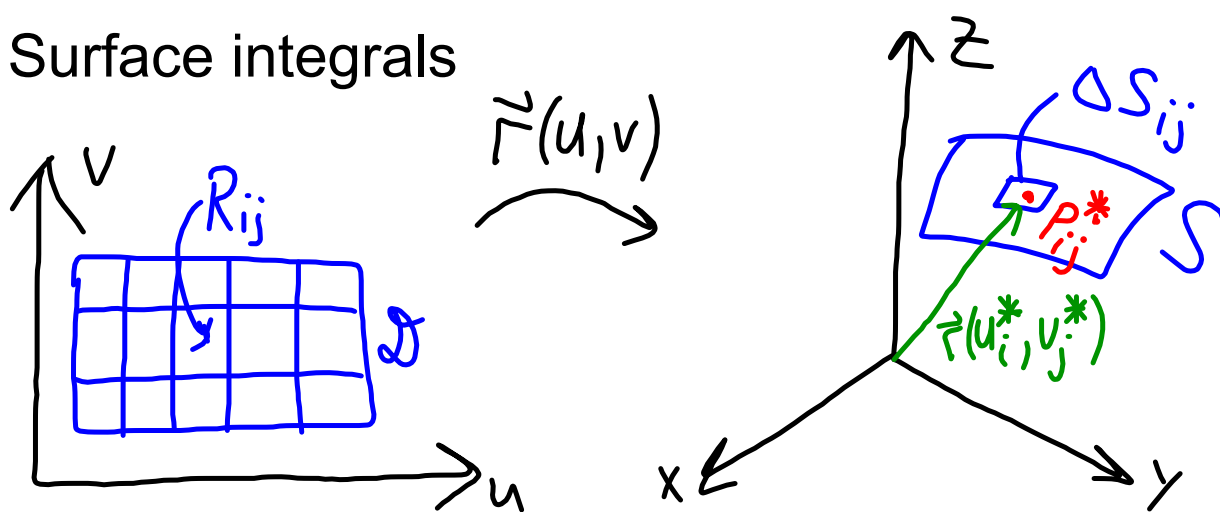


## Surface integrals



- $f(x, y, z)$  is defined on  $S$ .
- $S$  is parameterized by  $\vec{r}(u, v)$ .

Recall  $\Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v| \Delta u \cdot \Delta v$

$$\Rightarrow \iint_S f(x, y, z) dS = \lim_{N \rightarrow \infty} \sum_{i, j=1}^N f(P_{ij}^*) \Delta S_{ij}$$

$$= \lim_{N \rightarrow \infty} \sum_{i, j=1}^N f(\vec{r}(u_i^*, v_j^*)) |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

$$= \iint_D f(\vec{r}(u, v)) |\vec{r}_u \times \vec{r}_v| du dv.$$

Surface integral for scalar-valued  $f$ :

$$\iint_S f dS = \iint_D f(\vec{r}(u,v)) |\vec{r}_u \times \vec{r}_v| du dv.$$

In case  $f \equiv 1$  we recover surface area:

$$|S| = \iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v| du dv.$$

Recall for  $\vec{r}(x,y) = \langle x, y, z(x,y) \rangle$

we get

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}, \text{ so}$$

$$\iint_S f dS = \iint_D f(x,y,z(x,y)) \sqrt{1 + (z_x)^2 + (z_y)^2} dx dy.$$

special case we use often.

EX: Calculate  $\int_S x^2 + y^2 dS$ , where  $S$  is the sphere of radius 3 centered at the origin. Recall

$$x = 3 \cos \theta \sin \phi$$

$$y = 3 \sin \theta \sin \phi$$

$$z = 3 \cos \phi$$

Also,  $|\vec{r}_\theta \times \vec{r}_\phi| = 3^2 \sin \phi = 9 \sin \phi$ , and

$$x^2 + y^2 = r^2 = (3 \sin \phi)^2 = 9 \sin^2 \phi.$$

$$\begin{aligned}\int_S \int x^2 + y^2 dS &= \int_0^{2\pi} \int_0^{\pi} 9\sin^2\phi \cdot 9\sin\phi d\phi d\theta \\ &= 81 \cdot 2\pi \cdot \int_0^{\pi} \sin^3\phi d\phi = 162\pi \int_0^{\pi} \sin\phi (1 - \cos^2\phi) d\phi \\ &= 162\pi \left[ \int_0^{\pi} \sin\phi d\phi - \int_0^{\pi} \sin\phi \cos^2\phi d\phi \right] \\ &= 162\pi \left[ -\cos\phi \Big|_0^{\pi} + \frac{1}{3} \cos^3\phi \Big|_0^{\pi} \right] = \boxed{216\pi}.\end{aligned}$$

Ex: Calculate  $\iint_S z^2 dS$ , where  
 $S$  is given by  $f(x,y) = 1+x+y$ ,  $0 \leq x \leq 1$   
 $0 \leq y \leq 2$ .

$$f_x = 1 = f_y \Rightarrow \sqrt{1+f_x^2+f_y^2} = \sqrt{3}$$

$$\iint_S z^2 dS = \int_0^1 \int_0^2 (1+x+y)^2 \sqrt{3} dy dx = \sqrt{3} \int_0^1 \frac{1}{3} (1+x+y)^3 \Big|_0^2 dx$$

$$= \frac{\sqrt{3}}{3} \int_0^1 (3+x)^3 - (1+x)^3 dx = \frac{\sqrt{3}}{3} \left[ \frac{1}{4} (3+x)^4 - \frac{1}{4} (1+x)^4 \right] \Big|_0^1$$

$$= \frac{\sqrt{3}}{12} [4^4 - 3^4 - 2^4 + 1]$$

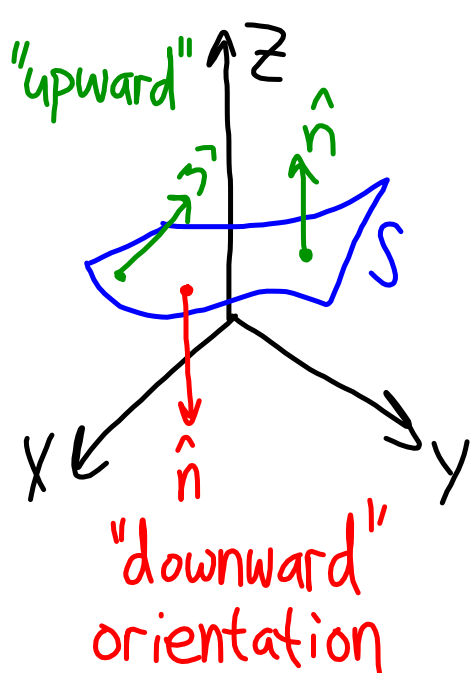
$$= \frac{\sqrt{3}}{12} [256 - 81 - 16 + 1]$$

$$= \frac{\sqrt{3} \cdot 160}{12} = \boxed{\frac{40\sqrt{3}}{3}}$$



## Surface orientation

We will assign a unit normal vector to a surface, depending on where you are on that surface.



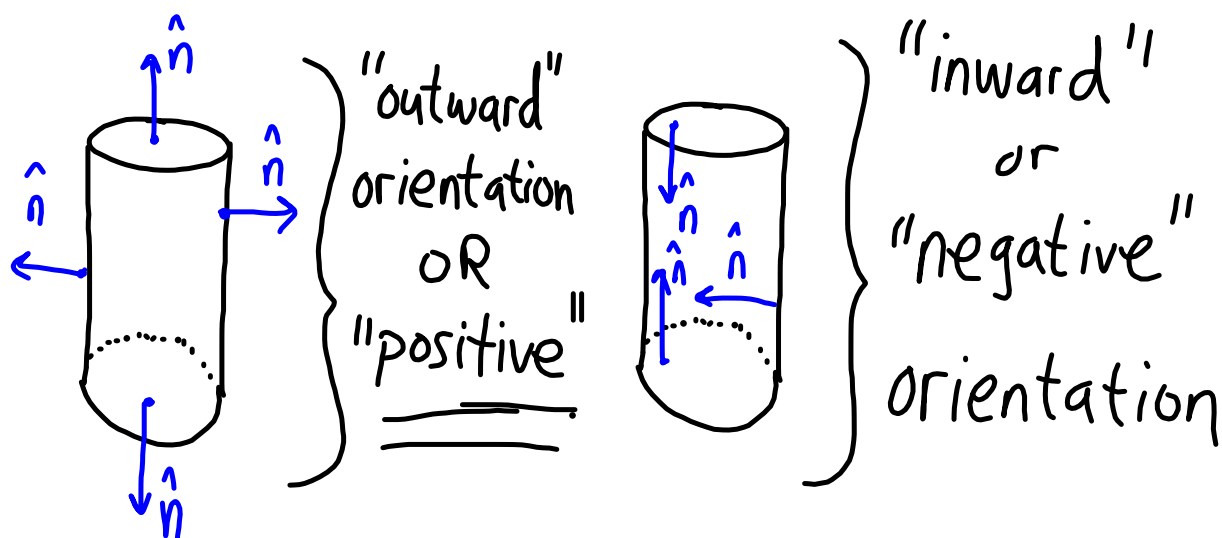
$$\hat{n} = \hat{n}(x, y, z)$$

We only consider surfaces with 2 distinct sides  
(no Möbius strips, etc.)

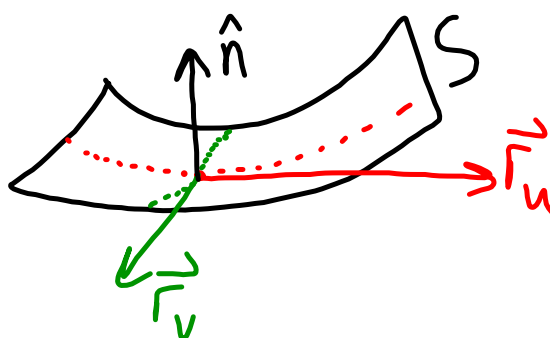
Thus, there are 2 possible "orientations".

More precisely, "upward" means the  $\vec{k}$ -component of  $\hat{n}$  is positive.

Closed surfaces (enclose a volume):



How to get  $\hat{n}$



$$\vec{n} = \vec{r}_u \times \vec{r}_v$$

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

If you calculate  $\hat{n}$  this way and it points the "wrong" way then multiply by  $-1$ !

$$-\hat{n} = \frac{-\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \frac{\vec{r}_v \times \vec{r}_u}{|\vec{r}_v \times \vec{r}_u|}$$

For the natural parameterization,

$$\vec{r}(x,y) = \langle x, y, z(x,y) \rangle \left\{ \begin{array}{l} \vec{r}_x = \langle 1, 0, z_x \rangle \\ \vec{r}_y = \langle 0, 1, z_y \rangle \end{array} \right.$$

$$\vec{r}_x \times \vec{r}_y = \langle -z_x, -z_y, 1 \rangle$$

$$\Rightarrow \hat{n} = \frac{1}{\sqrt{1+(z_x)^2+(z_y)^2}} \langle -z_x, -z_y, 1 \rangle$$

"upward"

## Surface integrals for vector fields

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \hat{n}) dS$$

$\downarrow$  notation scalar, so we know what this integral means

Parameterize  $S$  by  $\vec{r}(u,v)$ , domain  $D$ , then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \underbrace{\left( \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \right)}_{\hat{n}} \underbrace{|\vec{r}_u \times \vec{r}_v|}_{dS} du dv$$

$$\Rightarrow \int_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) du dv.$$

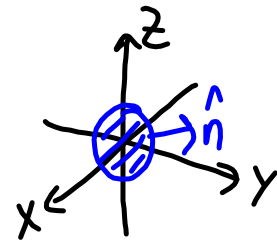
Summary of integrals:

Scalar  $f$   $\left\{ \int_c^b f ds = \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| dt \right.$   $\left. \begin{array}{l} \int_S f dS = \iint_D f(\vec{r}(u,v)) |\vec{r}_u \times \vec{r}_v| du dv \\ \text{LINE} \qquad \qquad \qquad \text{SURFACE} \end{array} \right.$

Vector  $\vec{F}$   $\left\{ \int_c^b \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt \right.$   $\left. \begin{array}{l} \int_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) du dv. \end{array} \right.$

EX: Let  $S$  be the unit sphere with positive orientation. Find  $\iint_S \vec{F} \cdot d\vec{S}$  if  $\vec{F} = \langle 0, 0, 1 \rangle$ .

We want  $\hat{n}$  pointing outward:



$$\left. \begin{array}{l} x = \cos\theta \sin\phi \\ y = \sin\theta \sin\phi \\ z = \cos\phi \end{array} \right\} \text{ We calculated previously}$$

$$\vec{r}_\theta \times \vec{r}_\phi = \begin{array}{l} -\cos\theta \sin^2\phi \vec{i} \\ -\sin\theta \sin^2\phi \vec{j} \\ -\sin\phi \cos\phi \vec{k} \end{array}$$

We don't really need  $\hat{n}$  but will look at it for illustration...  $|\vec{r}_\theta \times \vec{r}_\phi| = \sin\phi$

$$\hat{n} = \frac{\vec{r}_\theta \times \vec{r}_\phi}{|\vec{r}_\theta \times \vec{r}_\phi|} = \langle -\cos\theta \sin\phi, -\sin\theta \sin\phi, -\cos\phi \rangle$$

$$= -\langle x, y, z \rangle \text{ } \left. \vphantom{\frac{\vec{r}_\theta \times \vec{r}_\phi}{|\vec{r}_\theta \times \vec{r}_\phi|}} \right\} \text{ points "inward"}$$

So we use  $-\vec{r}_\theta \times \vec{r}_\phi = \vec{r}_\phi \times \vec{r}_\theta$  in the integral:

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^\pi \underbrace{\langle 0, 0, 1 \rangle \cdot (\vec{r}_\phi \times \vec{r}_\theta)}_{\sin\phi \cos\phi} d\phi d\theta$$



$$= \int_0^{2\pi} \int_0^{\pi} \sin\phi \cos\phi d\phi d\theta = 2\pi \int_0^{\pi} \frac{1}{2} \frac{d}{d\phi} \sin^2\phi d\phi$$

$$= \frac{2\pi}{2} \sin^2\phi \Big|_0^{\pi} = \boxed{0}.$$

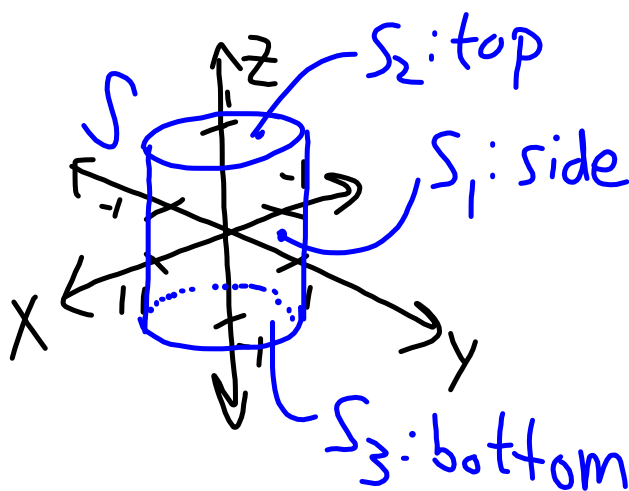
\*Subtlety: note we did not stick in  $\rho^2 \sin\phi$  here, but applied the surface integral formula.

Using a parameterization, not change of variables formula.

EX: Let  $S$  be the surface of the cylinder  $\{(x, y, z) \mid x^2 + y^2 \leq 1, -1 \leq z \leq 1\}$ , with positive orientation. Find

$$\iint_S \vec{F} \cdot d\vec{S} \text{ if } \vec{F} = \langle x, y, z \rangle.$$

$$S = S_1 \cup S_2 \cup S_3$$



$$\underline{\text{On } S_1 : \vec{r}(\theta, z) = \langle \cos \theta, \sin \theta, z \rangle}$$

$$0 \leq \theta \leq 2\pi, -1 \leq z \leq 1$$

$$\begin{aligned} \vec{r}_\theta &= \langle -\sin \theta, \cos \theta, 0 \rangle \\ \vec{r}_z &= \langle 0, 0, 1 \rangle \end{aligned} \left\{ \begin{aligned} \vec{r}_\theta \times \vec{r}_z &= \langle \cos \theta, \sin \theta, 0 \rangle \\ &= \langle x, y, 0 \rangle, \text{ "outward" } \end{aligned} \right.$$

$$F(\vec{r}(\theta, z)) = \langle \cos \theta, \sin \theta, z \rangle$$

$$F \cdot (\vec{r}_\theta \times \vec{r}_z) = \cos^2 \theta + \sin^2 \theta = 1.$$

$$\text{Thus } \iint_{S_1} \vec{F} \cdot d\vec{S}_1 = \int_0^{2\pi} \int_{-1}^1 1 \, dz \, d\theta = \underline{4\pi}$$

On  $S_2$  (top) :  $\vec{r}(x,y) = \langle x, y, 1 \rangle$ ,  $x^2 + y^2 \leq 1$

$$\left. \begin{array}{l} \vec{r}_x = \langle 1, 0, 0 \rangle \\ \vec{r}_y = \langle 0, 1, 0 \rangle \end{array} \right\} \vec{r}_x \times \vec{r}_y = \langle 0, 0, 1 \rangle$$

"outward"

$$\iint_{S_2} \vec{F} \cdot d\vec{S}_2 = \iint_{\mathcal{D}} \langle x, y, 1 \rangle \cdot \langle 0, 0, 1 \rangle \, dx \, dy = \iint_{\mathcal{D}} 1 \, dx \, dy = \pi$$

Area of circle  
=

$$\underline{\underline{\text{On } S_3: \vec{r}(x,y) = \langle x, y, -1 \rangle}}$$

$$\left. \begin{array}{l} \vec{r}_x \times \vec{r}_y = \langle 0, 0, 1 \rangle \end{array} \right\} \begin{array}{l} \text{points} \\ \text{wrong way} \end{array}$$

$$\Rightarrow \iint_{S_3} \vec{F} \cdot d\vec{S}_3 = \iint_D \langle x, y, -1 \rangle \cdot \langle 0, 0, -1 \rangle dx dy$$

$$D = \iint_D 1 dx dy = \underline{\underline{\pi}}$$

$$\text{Sum over 3 surfaces: } \iint_S \vec{F} \cdot d\vec{S} = 4\pi + \pi + \pi = \boxed{6\pi}$$

Stoke's Theorem: relates the integral of the tangential part of a vector field over a curve  $C$ , which bounds a surface  $S$ , to the integral of the CURL of the field over the surface.

\* Very important result.

\* Requires the curve to have positive orientation with respect to the surface.



Theorem :  
(Stoke's)

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

Notation: If  $C$  is a closed curve and is the boundary of  $S$ , a surface, then

$$C = \partial S \leftarrow \text{not a derivative}$$

If a volume  $V$  is bounded by surface  $S$ ,

$$S = \partial V.$$

Stoke's again:

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

or

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

We will pick up here next time.

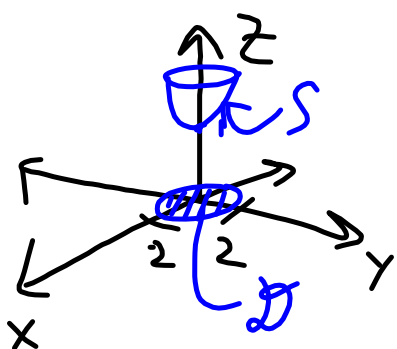


Practice!

$$\textcircled{\#1} \text{ Let } S = \left\{ (x, y, z) \mid \begin{array}{l} x^2 + y^2 \leq 4, \\ z = 1 + x^2 + y^2 \end{array} \right\},$$

with upward orientation.

$$\text{Find } \iint_S \vec{F} \cdot d\vec{S}, \text{ if } \vec{F} = \langle -x, -y, 0 \rangle.$$



$$\vec{r}(x,y) = \langle x, y, 1+x^2+y^2 \rangle$$

$$\vec{r}_x = \langle 1, 0, 2x \rangle$$

$$\vec{r}_y = \langle 0, 1, 2y \rangle$$

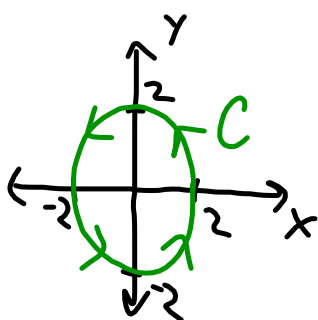
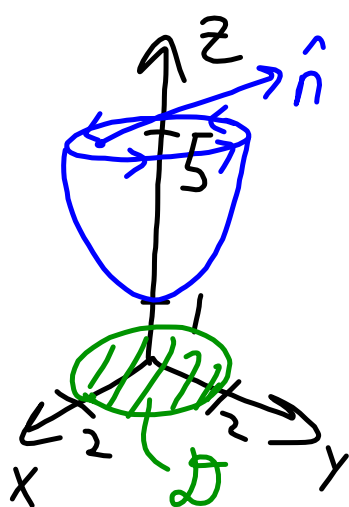
$$\vec{r}_x \times \vec{r}_y = \langle -2x, -2y, 1 \rangle$$

$$\vec{F} \cdot (\vec{r}_x \times \vec{r}_y) = \langle -x, -y, 0 \rangle \cdot \langle -2x, -2y, 1 \rangle$$

$$= 2(x^2 + y^2).$$

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \iint_D 2(x^2 + y^2) dA \\ &= 2 \int_0^{2\pi} \int_0^2 r^2 r dr d\theta = 2 \cdot 2\pi \cdot \frac{1}{4} r^4 \Big|_0^2 \\ &= 2^4 \pi = \boxed{16\pi}\end{aligned}$$

(#2) Parameterize  $\partial S$  in the last problem with positive orientation.



$$\left. \begin{aligned} z &= 1 + x^2 + y^2 \\ &= 1 + r^2 = 1 + 4 = 5 \\ x &= r \cos \theta = 2 \cos \theta \\ y &= 2 \sin \theta \\ z &= 5 \end{aligned} \right\}$$

$$\vec{r}(\theta) = \langle 2 \cos \theta, 2 \sin \theta, 5 \rangle.$$

$$0 \leq \theta \leq 2\pi.$$

