Curl: an important differential operator on vector fields.

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}, \quad \vec{F} = \vec{F}(x, y, z).$$

$$\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

Think $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

$$\Rightarrow \nabla \times \vec{F} = (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}.$$

$\underline{EX}$: Find $\nabla \times \vec{F}$ if $\vec{F} = \cos(xz)\vec{i} + xy\vec{j} + yz\vec{k}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \cos(xz) & xy & yz \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(yz) - \cancelto{0}{\frac{\partial}{\partial z}(xy)}\right)\vec{i} + \left(\frac{\partial}{\partial z}\cos(xz) - \cancelto{0}{\frac{\partial}{\partial x}(yz)}\right)\vec{j}$$

$$+ \left(\frac{\partial}{\partial x}(xy) - \cancelto{0}{\frac{\partial}{\partial y}(\cos(xz))}\right)\vec{k}$$

$$= \boxed{z\vec{i} - x\sin(xz)\vec{j} + y\vec{k}.}$$

Theorem: If $f(x,y,z)$ is a $C^2$-function (has continuous $2^{nd}$-order derivatives) then

$$\boxed{\nabla \times (\nabla f) = 0.}$$

Proof: The $C^2$-requirement allows application of Clairaut's Theorem (symmetry of mixed partials). So, direct calculation of

$\nabla \times \nabla f$ gives...

$$\nabla \times \nabla f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\vec{j}$$

$$+ \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k}$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}.$$

$\square$

Recall if $\vec{F} = \nabla f$ then $\vec{F}$ is called CONSERVATIVE. It follows that

$$\vec{F} \text{ conservative} \Rightarrow \boxed{\nabla \times \vec{F} = 0},$$

(so long as $f$ is $C^2$). Conversely, if 1) $\vec{F}$ is $C^1$ and 2) $\nabla \times \vec{F} = 0$, then

$\vec{F}$ is conservative.

EX: Determine whether or not

$\vec{F} = \langle x\cos(x+y) + \sin(x+y),\ x\cos(x+y) \rangle$

is conservative.

Extend to 3D: $P = x\cos(x+y) + \sin(x+y)$

$Q = x\cos(x+y)$

$R = 0$

$\nabla \times \vec{F} = \langle 0, 0, Q_x - P_y \rangle$; check if $Q_x = P_y$...

$$Q_x = \cos(x+y) - x\sin(x+y)$$

$$P_y = -x\sin(x+y) + \cos(x+y)$$

(conservative).

* $Q_x = P_y$ was already derived as a condition for 2D vector fields to be conservative.

<u>Ex</u>: Is $\vec{F} = x\vec{i} + y\vec{j} - z^2\vec{k}$ conservative?

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & -z^2 \end{vmatrix} = \left(\frac{\partial}{\partial y}(-z^2) - \frac{\partial}{\partial z}(y)\right)\vec{i}$$

$$+\left(\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(-z^2)\right)\vec{j}$$

$$+\left(\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right)\vec{k}$$

$$= \vec{0}$$

$\Rightarrow$ <u>conservative</u>.

Divergence operator*

*Produces a scalar, unlike $\nabla \times \vec{F}$.

$$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F}$$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$

$$= P_x + Q_y + R_z .$$

EX: Find $\nabla \cdot \vec{F}$, if $\vec{F} = \langle x^2yz, y^2z, z \rangle$.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(y^2z) + \frac{\partial}{\partial z}(z)$$

$$= \boxed{2xyz + 2y + 1}.$$

Theorem: If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, where $P, Q, \& R$ are $C^2$, then

$$\boxed{\nabla \cdot (\nabla \times \vec{F}) = 0}.$$

Proof: $\nabla \times \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

$\Rightarrow \nabla \cdot (\nabla \times \vec{F}) = R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$

$$\underset{\text{Clairaut}}{=} 0. \qquad \square$$

Laplace operator: Let $f(x,y,z)$ be scalar-valued. Then

$$\nabla\cdot(\nabla f) = \nabla^2 f = \nabla\cdot\langle f_x, f_y, f_z\rangle$$
$$= f_{xx} + f_{yy} + f_{zz}.$$

$\nabla^2$: Laplace operator, also known as $\Delta$... as in $\boxed{\Delta f = f_{xx} + f_{yy} + f_{zz}.}$

Apply to vector fields $\vec{F} = \langle P, Q, R \rangle$:

$$\boxed{\Delta \vec{F} = (\Delta P)\vec{i} + (\Delta Q)\vec{j} + (\Delta R)\vec{k}}.$$

<u>EXTRA NOTATION</u>: $\vec{u} = \langle u_1, u_2, u_3 \rangle$

Each $u_j = u_j(x_1, x_2, x_3)$, $j = 1, 2, 3$.

$$\Rightarrow (D\vec{u})_{ij} = \frac{\partial u_j}{\partial x_i}, \quad i, j = 1, 2, 3$$

So $D\vec{u}$ is a $3\times 3$ array:

$$D\vec{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

We extend the dot product:

$$(\vec{u}\cdot D\vec{u})_j = \sum_{i=1}^{3} u_i (Du)_{ij} = \sum_{i=1}^{3} u_i \frac{\partial u_j}{\partial x_i}, \quad j=1,2,3.$$

Application : Navier-Stokes equations of incompressible fluid flow.

$\vec{u}$ : fluid velocity at point $(x_1, x_2, x_3)$

$p$ : pressure

$\nu$ : kinematic viscocity

The following equations are derived from conservation laws.

Conservation of momentum:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} + \nabla p = \nu \Delta \vec{u} .$$

Conservation of mass:

$$\underbrace{\nabla \cdot \vec{u}}_{} = 0.$$

divergence measures rate at which fluid flows "out" from a point.

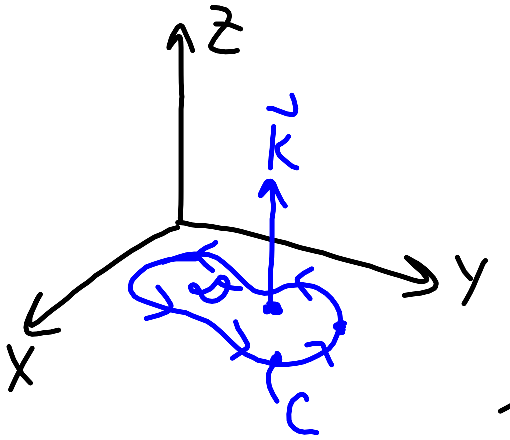
Connections...

- $\nabla \times \vec{u} = \omega$ is the "vorticity", which measures the tendency of the fluid to "rotate" near a point.

- $\nu \Delta u$ measures dissipation of momentum (converts to heat) via molecular friction.

# Green's Theorem in vector form

"Green's in 3D"




Recall

$$\int_C \vec{F}\cdot d\vec{r} = \iint_{\mathcal{D}} (Q_x - P_y)\, dA.$$

But
$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix}$$
$$= \langle 0, 0, Q_x - P_y \rangle.$$

Thus
$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_{\mathcal{D}} (\nabla\times\vec{F})\cdot\vec{k}\, dA}.$$

* Green's Theorem relates the integral of the *tangential part* of the field along *C* to the integral of the *vertical part* of the curl over $\mathcal{D}$.

* We will see this come up again with Stoke's Theorem later in the course.

* We now come up with a similar identity involving the *normal component* of the field, integrated along the curve *C*.

• "Normal part of $\vec{F}$" is $\vec{F}\cdot\hat{n}$.

Let $\vec{r}(t)$ be a parameterization of C,

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}.$$

Derived previously...

$$T(t) = \frac{x'(t)}{|\vec{r}\,'(t)|}\vec{i} + \frac{y'(t)}{|\vec{r}\,'(t)|}\vec{j}$$

$$\hat{n}(t) = \frac{y'(t)}{|\vec{r}\,'(t)|}\vec{i} + \frac{-x'(t)}{|\vec{r}\,'(t)|}\vec{j}$$

Note $\vec{T}\cdot\hat{n} = 0$

It follows that

$$\int_C \vec{F}\cdot\hat{n}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot \underbrace{\left\langle \frac{y'}{|\vec{r}'|}, \frac{-x'}{|\vec{r}'|}\right\rangle}_{\hat{n}} \underbrace{|\vec{r}'|dt}_{ds}$$

$$= \int_a^b \left(P\frac{dy}{dt} - Q\frac{dx}{dt}\right)dt$$

$$= \int_a^b P y'(t)dt - \int_a^b Q x'(t)dt = \int_C P dy - Q dx$$

so $\int_C \vec{F}\cdot\hat{n}\, ds = \int_C P\,dy - Q\,dx$

$= \iint_D \frac{\partial}{\partial x}(P) - \frac{\partial}{\partial y}(-Q)\, dA$

(Green's) $= \iint_D (P_x + Q_y)\,dA = \iint_D (\nabla\cdot\vec{F})\,dA.$

$$\boxed{\int_C (\vec{F}\cdot\hat{n})\,ds = \iint_D (\nabla\cdot\vec{F})\,dA.}$$ USED VERY OFTEN.

EX: Let $\vec{F} = \frac{y}{1+x^2}\vec{i} - \frac{x}{1+x^2}\vec{j}$ on $D = \{(x,y) \mid x^2+y^2 \le 1\}$. Show $\iint_D \nabla\cdot\vec{F}\,dA = 0$.

Note $\nabla\cdot\vec{F} = \frac{\partial}{\partial x}\left(\frac{y}{1+x^2}\right) + \frac{\partial}{\partial y}\left(\frac{-x}{1+x^2}\right)$

$$= \frac{-2xy}{(1+x^2)^2}.$$

You can vaguely see the integral should be zero, but this is easily shown by:

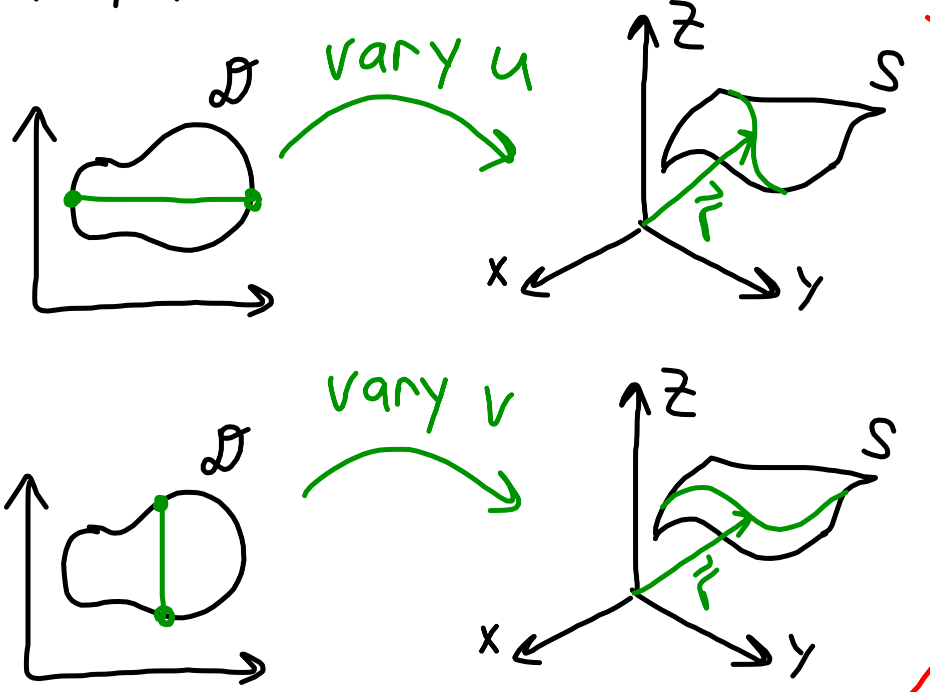
$$\iint_{\mathcal{D}} \nabla\cdot\vec{F}\,dA = \int_C \vec{F}\cdot\hat{n}\,ds, \text{ since}$$

$$\hat{n} = \langle x, y\rangle \text{ on } C \text{ (unit circle)}$$

$$\Rightarrow \vec{F}\cdot\hat{n} = \frac{y}{1+x^2}x - \frac{x}{1+x^2}y = 0.$$

Parametric Surfaces

$$\vec{r}(u,v) = P(u,v)\vec{i} + Q(u,v)\vec{j} + R(u,v)\vec{k}.$$




Varying both u, v creates a surface, S.

<u>EX</u>: The cylinder $x^2+y^2=9$, $-1 \leq z \leq 1$

has parameterization

$$\left.\begin{aligned} x &= 3\cos\theta \\ y &= 3\sin\theta \\ z &= z \end{aligned}\right\} \begin{aligned} 0 \leq \theta \leq 2\pi \\ -1 \leq z \leq 1 \end{aligned}$$

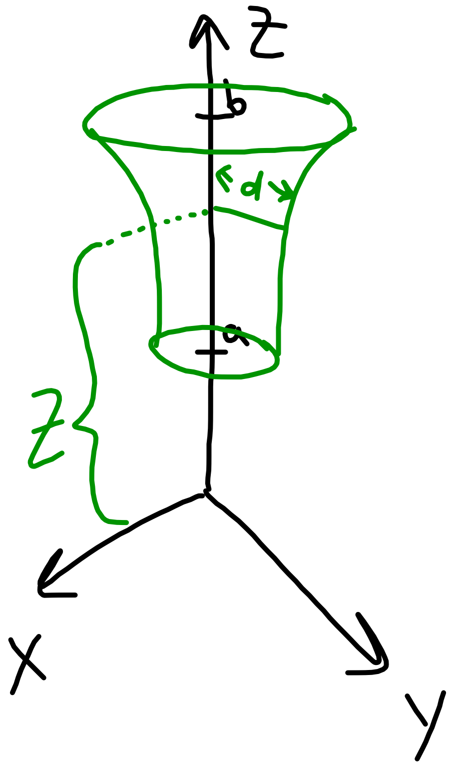
We say $\vec{r}(\theta,z) = \langle 3\cos\theta, 3\sin\theta, z\rangle$.

EX: The sphere $x^2+y^2+z^2=4$ has parameterization

$$x = 2\cos\theta\sin\phi$$
$$y = 2\sin\theta\sin\phi$$
$$z = 2\cos\phi$$

with $0 \le \theta \le 2\pi$ & $0 \le \phi \le \pi$.

Rotational surfaces




$d = d(z)$: distance from z-axis, rotate curve around z-axis.

$$\begin{cases} x = d(z)\cos\theta \\ y = d(z)\sin\theta \\ z = z \end{cases}$$

$a \le z \le b,\ 0 \le \theta \le 2\pi.$

"Natural" parameterizations

When you have a surface $z = f(x,y)$

then just use $\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$.

Ex: Elliptic paraboloid $z = 4x^2 + y^2$

$\Rightarrow \vec{r}(x,y) = \langle x, y, 4x^2 + y^2 \rangle$.

Surface area: We showed previously when discussing surface area that $\Delta S \approx |\vec{r}_x \times \vec{r}_y| \Delta x \Delta y$.

Now we generalize... $\Delta S \approx |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$ (same derivation).

$$\boxed{\text{AREA of } S = \iint_S dS = \iint_{\mathcal{D}} |\vec{r}_u \times \vec{r}_v| \, du\, dv}.$$

For the natural parameterization,

$u = x,\ v = y$

$$\Rightarrow \left| \vec{r}_x \times \vec{r}_y \right| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}.$$

Then $\iint\limits_S dS = \iint\limits_D \sqrt{1 + f_x^2 + f_y^2}\, dx\,dy$

is a special case.

EX: Find the surface area of a sphere of radius $a$.

$$x = a\cos\theta\sin\phi$$
$$y = a\sin\theta\sin\phi$$
$$z = a\cos\phi$$

$$\vec{r}_\theta = \langle -a\sin\theta\sin\phi,\ a\cos\theta\sin\phi,\ 0\rangle$$
$$\vec{r}_\phi = \langle a\cos\theta\cos\phi,\ a\sin\theta\cos\phi,\ -a\sin\phi\rangle$$
$$\vec{r}_\theta\times\vec{r}_\phi = -a^2\cos\theta\sin^2\phi\,\vec{i} - a^2\sin\theta\sin^2\phi\,\vec{j} - a^2\underbrace{(\sin^2\theta+\cos^2\theta)}_{1}\sin\phi\cos\phi\,\vec{k}.$$

$$|\vec{r}_\theta \times \vec{r}_\phi| = \left(a^4\cos^2\theta\sin^4\phi + a^4\sin^2\theta\sin^4\phi + a^4\sin^2\phi\cos^2\phi\right)^{1/2}$$

$$= a^2\left(\sin^4\phi + \sin^2\phi\cos^2\phi\right)^{1/2}$$

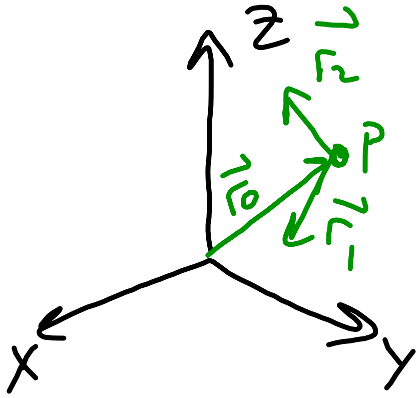
$$= a^2\left(\sin^2\phi\left(\sin^2\phi + \cos^2\phi\right)\right)^{1/2}$$

$$= a^2\sin\phi.$$

$$\text{Area} = \int_0^\pi\int_0^{2\pi} a^2\sin\phi\, d\theta\, d\phi = 2\pi a^2(-\cos\phi)\Big|_0^\pi = 4\pi a^2.$$

Practice!

#1 Find a parameterization of the plane passing through $(0,-1,5)$ and containing $\langle 2,1,4\rangle$ & $\langle -3,2,5\rangle$.




$\vec{r} = \vec{r}_0 + u\vec{r}_1 + v\vec{r}_2$

$\vec{r}_0 = \vec{P} = \langle 0, -1, 5 \rangle$

$\vec{r}_1 = \langle 2, 1, 4 \rangle$

$\vec{r}_2 = \langle -3, 2, 5 \rangle$

$\Rightarrow \vec{r}(u,v) = \langle 2u-3v, -1+u+2v, 5+4u+5v \rangle.$

#2 Is $\vec{F} = \langle xy^2z, y+z, 1 \rangle$ conservative?

$\nabla \times \vec{F} = 0$?

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2z & y+z & 1 \end{vmatrix}$$

$$= \langle -1,\ xy^2,\ -2xyz \rangle \neq \langle 0,0,0 \rangle$$

$\Rightarrow$ NOT conservative.

#3 Find the area of the surface with parameterization

$$x = u^2$$
$$y = uv$$
$$z = \frac{1}{2}v^2$$

$0 \leq u \leq 1$ & $0 \leq v \leq 2$.

$$\vec{r}_u = \langle x_u, y_u, z_u \rangle = \langle 2u, v, 0 \rangle$$

$$\vec{r}_v = \langle x_v, y_v, z_v \rangle = \langle 0, u, v \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle v^2, -2uv, 2u^2 \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{v^4 + 4u^2v^2 + 4u^4} = \sqrt{(2u^2+v^2)^2}$$

$$= 2u^2 + v^2.$$

$$\text{Area} = \int_0^2 \int_0^1 2u^2 + v^2 \, du\, dv = \int_0^2 \left[ \frac{2}{3}u^3 \Big|_0^1 + uv^2 \Big|_0^1 \right] dv$$

$$= \int_0^2 \frac{2}{3} + v^2 \, dv$$

$$= \frac{2}{3} \cdot 2 + \frac{1}{3} v^3 \Big|_0^2 = \frac{4}{3} + \frac{8}{3} = \frac{12}{3}$$

$$= \boxed{4.}$$

#4 Calculate $\nabla \cdot \vec{F}$,

$$\vec{F} = \langle 1-zx, -2y^2x^3, ze^{xz}+y \rangle.$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(1 - zx) + \frac{\partial}{\partial y}(-2y^2x^3) + \frac{\partial}{\partial z}(ze^{xz} + y)$$

$$= \boxed{-z - 4yx^3 + e^{xz} + xze^{xz}}.$$