Fundamental Theorem for Line Integrals

$\vec{F} = \nabla f$ (conservative field)

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$.

$$\frac{d}{dt} f(\vec{r}(t)) = f_x x' + f_y y' + f_z z'$$
$$= \nabla f \cdot \vec{r}\,'(t) = \vec{F} \cdot \frac{d\vec{r}}{dt}$$

so $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\, dt.$

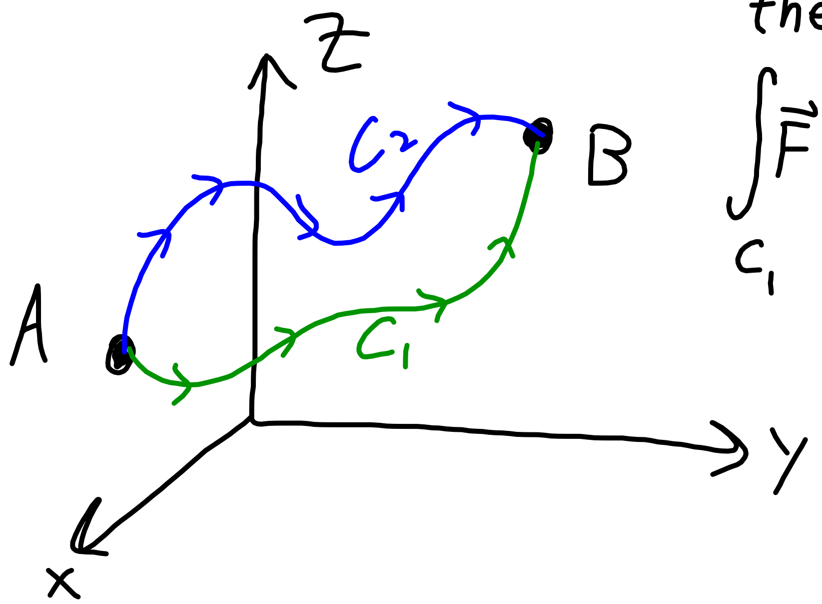
The Fundamental Theorem of Calculus says

$$\int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Summary: $\boxed{\int_C \vec{F}\cdot d\vec{r} = \int_C Df\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$.

Fundamental Theorem for line integrals.

Consequence



If $\vec{F}$ is CONSERVATIVE then

$$\int_{C_1} \vec{F}\cdot d\vec{r}_1 = \int_{C_2} \vec{F}\cdot d\vec{r}_2$$

$$= \underbrace{f(B)-f(A)}_{\text{Path-independent.}}$$

Gravity has a conservative force field

$$\vec{F} = \frac{-mMG}{|\vec{r}|^3}\vec{r} \; ; \; \vec{r} = \langle x, y, z \rangle.$$

Take $f = \dfrac{mMG}{|\vec{r}|}$, then $\vec{F} = \nabla f$.

$$f_x = \frac{\partial}{\partial x}\left(\frac{mMG}{(x^2+y^2+z^2)^{1/2}}\right) = -\frac{1}{2}\frac{mMG(2x)}{(x^2+y^2+z^2)^{3/2}} = \frac{-mMG\,x}{|\vec{r}|^3}.$$

$$f_y = \frac{-mMG}{|\vec{r}|^3}y, \quad f_z = \frac{-mMG}{|\vec{r}|^3}z.$$

Ex: Find the work done by gravity in moving a point mass (m) from $(-1,0,1)$ to $(1,1,2)$.

$$W = \int_C \vec{F}\cdot d\vec{r} = f(1,1,2) - f(-1,0,1)$$

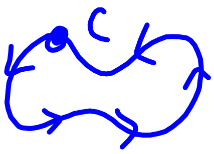
$$= mMG\left(\frac{1}{\sqrt{1+1+4}} - \frac{1}{\sqrt{1+1}}\right)$$

$$= \boxed{mMG\left(\frac{1}{\sqrt{6}} - \frac{1}{\sqrt{2}}\right)}.$$

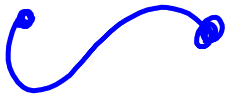
only need endpoints

How can we tell if a field is conservative? To discuss this, we need to define some things.

<u>Closed curve</u>: Initial point = end-point

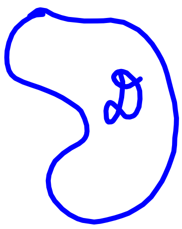


closed

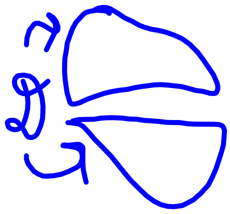


not closed

<u>Connected region</u>: You can get from any point A in $\mathcal{D}$ to any other point B in $\mathcal{D}$ without leaving $\mathcal{D}$.

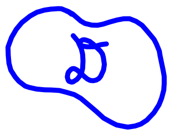


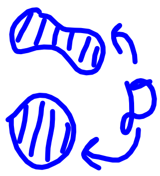
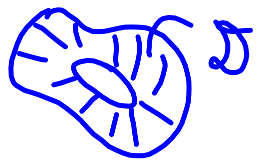
connected



not connected

<u>Simply-connected region</u>: A connected region (say $\mathcal{D}$) such that any simple curve contained in $\mathcal{D}$ only bounds subsets of $\mathcal{D}$.

simply connected 

not  

<u>OPEN region $\mathcal{D}$</u>: $\mathcal{D}$ does not include any points on the boundary.

$\{(x,y) \mid x^2+y^2<1\}$ ← open

$\{(x,y) \mid x^2+y^2\leq 1\}$ ← not open

A nice condition to see if you have a 2D conservative field (3D discussed later on).

If:
1) D is open & simply-connected
2) $\vec{F} = P\vec{i} + Q\vec{j}$; P, Q are C^1-functions
3) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D

then* $\vec{F}$ is CONSERVATIVE on D.

* $\vec{F}$ could still be conservative otherwise, but these conditions guarantee it.

EX: Determine if $\vec{F}$ is conservative.

a) $\vec{F} = \underbrace{xy}_{P}\,\vec{i} \; - \underbrace{xy}_{Q}\,\vec{j}$

$\mathcal{D}$ not specified $\Rightarrow \mathcal{D} = \mathbb{R}^2$.

$\frac{\partial P}{\partial y} = x$

$\frac{\partial Q}{\partial x} = -y$

not always equal $\Rightarrow \vec{F}$ not conservative.

b) $\vec{F} = \underbrace{x^2}_{P}\,\vec{i} + \underbrace{y^2}_{Q}\,\vec{j}$.

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial Q}{\partial x} \Rightarrow \text{CONSERVATIVE.}$$

c) $\vec{F} = \underset{P}{y}\,\vec{i} + \underset{Q}{x}\,\vec{j}$

$$\frac{\partial Q}{\partial x} = 1 = \frac{\partial P}{\partial y} \Rightarrow \text{CONSERVATIVE.}$$

How to find f such that $\vec{F} = \nabla f$.

Best to learn by example here...

Let $\vec{F} = \underbrace{(y^2+2xy-1)}_{\partial f/\partial x}\vec{i} + \underbrace{(x^2+2xy-1)}_{\partial f/\partial y}\vec{j}$

Integrate... $\int \frac{\partial f}{\partial x}\,dx = f(x,y) = xy^2 + x^2y - x + g(y)$

indefinite... we can add in any $g(y)$ since $\frac{\partial}{\partial x} g(y) = 0$

$$\vec{F} = \underbrace{(y^2+2xy-1)}_{\partial f/\partial x}\vec{i} + \underbrace{(x^2+2xy-1)}_{\partial f/\partial y}\vec{j}$$

So $f(x,y) = xy^2 + x^2y - x + g(y) \ldots$

hence $x^2+2xy-1 = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xy^2+x^2y-x+g(y)\right)$

$$= 2xy + x^2 + g'(y)$$

$\Rightarrow -1 = g'(y) \Rightarrow -y = g(y)$

could add $+c$... let $c=0$.

Answer: $f(x,y) = xy^2 + x^2y - x - y$.

Summary: For $\vec{F} = P\vec{i} + Q\vec{j}$ ...

1) Integrate $\int P\,dx$
2) Add on $g(y)$ arbitrary... $f(x,y) = \int P\,dx + g(y)$
3) Get $g(y)$ by setting
$Q = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int P\,dx + g(y)\right)$; solve for $g(y)$.

Method in 3D: $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$
$= f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$.

EX: Let $P = y+z$, $Q = x+z^2$, $R = x+2yz-1$.

$\int (y+z)\,dx = xy + xz + g(y,z) = f(x,y,z)$

depends on y & z now.

$Q = x+z^2 = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xy+xz+g(y,z)\right) = x + \partial g/\partial y$.

$R = x+2yz-1 = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(xy+xz+g(y,z)\right) = x + \frac{\partial g}{\partial z}$.

Solve these for g_y, g_z ...

$$\frac{\partial g}{\partial y} = z^2 \quad \& \quad \frac{\partial g}{\partial z} = 2yz - 1$$

$$\Rightarrow g(y,z) = \int \frac{\partial g}{\partial y}\, dy = \int z^2\, dy = yz^2 + h(z)$$

$$\frac{\partial g}{\partial z} = 2yz - 1 = \frac{\partial}{\partial z}\left(yz^2 + h(z)\right) = 2yz + h'(z)$$

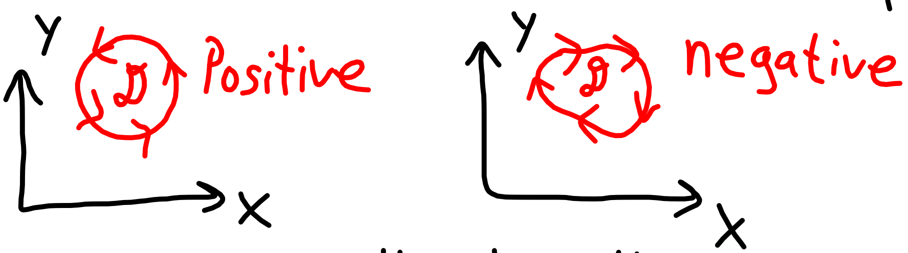
$$\Rightarrow h'(z) = -1 \Rightarrow h(z) = -z$$

$$\Rightarrow g = yz^2 - z$$

$$\Rightarrow f(x,y,z) = xy + xz + g = \boxed{xy + xz + yz^2 - z}.$$

Green's Theorem: allow you to switch between line integrals over a closed curve and double integrals over the region bounded by the curve.

The curve must be POSITIVELY oriented.



If an observer walks along the curve in the indicated direction, positive orientation means $\mathcal{D}$ is to the "left".

<u>Green's Theorem</u>: Let C be a curve that

1) is positively oriented
2) simple, closed
3) piecewise-smooth
4) encloses a region D.

If P, Q are C^1 on D, then

$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.}$$

Implication for conservative fields:

$$\vec{F} = P\vec{i} + Q\vec{j} = \nabla f$$

$$\Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

$$\Rightarrow \int_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = 0.$$

A line integral with a conservative field over a simple closed curve is always zero.

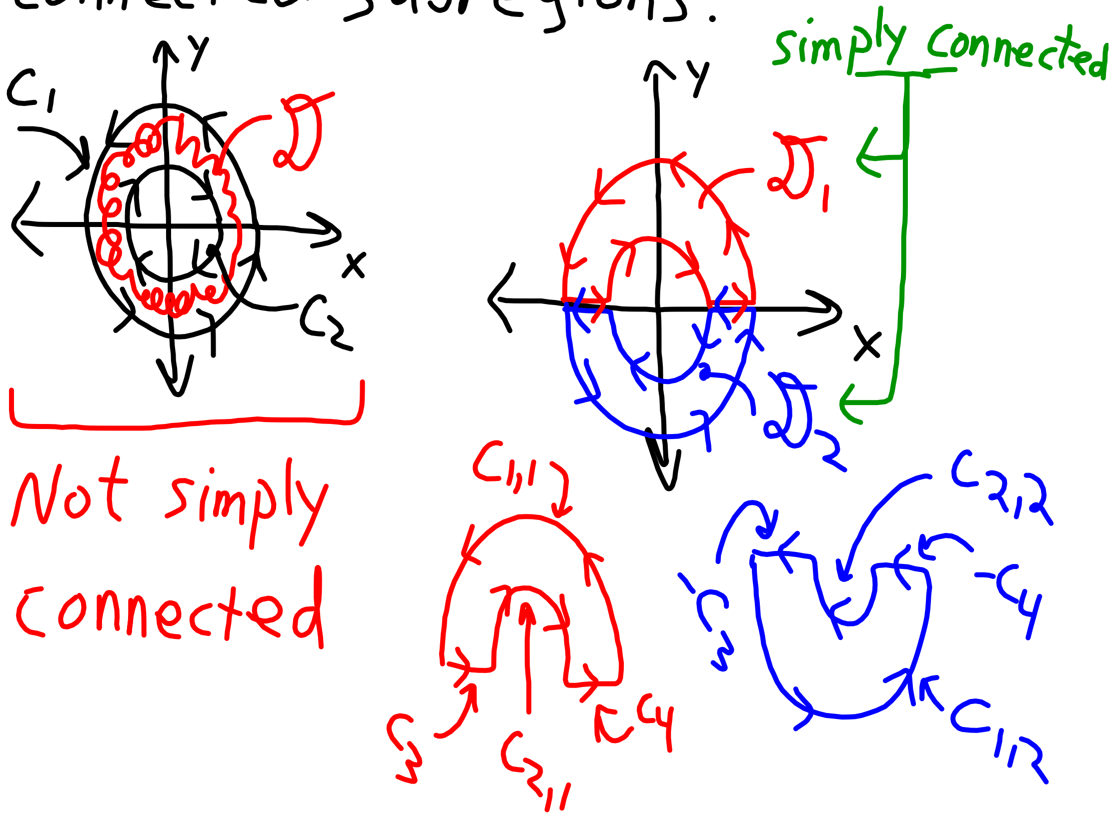
This also follows from the Fundamental Theorem for line integrals...

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A) \Big\}\ B = A \text{ for a closed curve.}$$

Extensions are often possible for Green's to more complicated domains.

Consider when you can break $\mathcal{D}$ into simply-connected subregions.

EX:



$$\boxed{\iint_{D} (Q_x - P_y)\,dA} = \iint_{D_1} (Q_x - P_y)\,dA + \iint_{D_2} (Q_x - P_y)\,dA$$

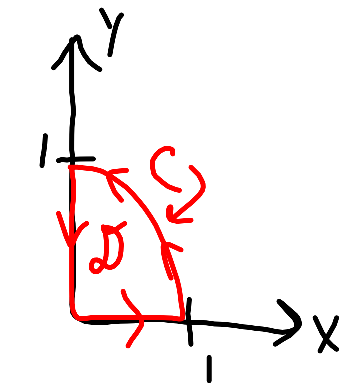
$$= \int_{C_{1,1}} P\,dx + Q\,dy + \int_{C_{2,1}} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy + \int_{C_4} P\,dx + Q\,dy$$

These cancel

$$+ \int_{C_{1,2}} P\,dx + Q\,dy + \int_{C_{2,2}} P\,dx + Q\,dy + \int_{-C_3} P\,dx + Q\,dy + \int_{-C_4} P\,dx + Q\,dy$$

$$= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \boxed{\int_{C} P\,dx + Q\,dy}.$$

EX: Often a line integral may be tedious so we apply Green's... let $\mathcal{D}$ be the region by $x \geq 0$, $y \geq 0$, $x^2+y^2 \leq 1$ and C is the boundary, positively oriented. Find $\int_C xy\,dx + y^2\,dy$.



Green's $\Rightarrow \int_C xy\,dx + y^2\,dy = \iint_{\mathcal{D}} (Q_x - P_y)\,dA$

$= \iint_{\mathcal{D}} (-x)\,dA$ (polar)

$$= \int_0^{\frac{\pi}{2}} \int_0^1 (-r\cos\theta)\, r\, dr\, d\theta$$

$$= -\left(\int_0^{\pi/2} \cos\theta\, d\theta\right)\left(\int_0^1 r^2\, dr\right)$$

$$= -\sin\theta\Big|_0^{\pi/2} \cdot \frac{1}{3} r^3\Big|_0^1 = \boxed{-\frac{1}{3}}.$$

$\underline{EX}$: Let $f(x,y) = xy(1-x^2-y^2)$.
Find $\iint_{\mathcal{D}} \frac{\partial f}{\partial x}\, dA$ if $\mathcal{D}$ is the region bounded by $x^2+y^2=1$.

So think $\iint_{\mathcal{D}} \frac{\partial f}{\partial x}\, dA = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

or $Q = f$, $P = 0$

$$\Rightarrow \iint_{\mathcal{D}} \frac{\partial f}{\partial x}\, dA = \int_C P\,dx + Q\,dy = \underbrace{\cancel{\int_C f\,dy}}_{0}, \text{ where}$$

$x^2+y^2=1$ on $C \Rightarrow f=0$ on C.

Area: Let D be bounded by a simple, closed curve C. Then

$$|D| = \iint_D dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$,

Green's

$$|D| = \int_C P\,dx + Q\,dy$$

$$= \int_C x\,dy = -\int_C y\,dx$$

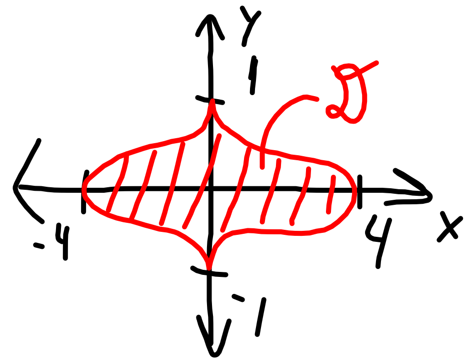
$$= \frac{1}{2}\int_C x\,dy - y\,dx.$$

e.g. $Q = x$ & $P = 0$

$Q = 0$ & $P = -y$

$Q = \frac{1}{2}x$ & $P = -\frac{1}{2}y$

EX: Find the area enclosed by the curve $\vec{r}(t) = \langle 4\cos^3(t), \sin(t)\rangle$, $0 \le t \le 2\pi$.



$|\mathcal{D}| = \iint_{\mathcal{D}} dA$? Use Green's

$$|\mathcal{D}| = \int_C x\,dy \quad \left\{ \begin{aligned} x &= 4\cos^3(t) \\ y &= \sin(t) \\ dy &= y'(t)\,dt \\ &= \cos(t)\,dt \end{aligned} \right.$$

$$\Rightarrow |\mathcal{D}| = \int_0^{2\pi} 4\cos^3(t)\cdot\cos(t)\,dt$$

We've seen this before...

$$\cos^4(t) = \frac{1}{4}\left(\frac{3}{2} + 2\cos(2t) + \frac{1}{2}\cos(4t)\right)$$

$$\Rightarrow |\mathcal{D}| = \int_0^{2\pi}\left(\frac{3}{2} + 2\cos(2t) + \frac{1}{2}\cos(4t)\right)dt$$

$$= \frac{3}{2}\int_0^{2\pi} dt + 2\underbrace{\int_0^{2\pi}\cos(2t)\,dt}_{0} + \frac{1}{2}\underbrace{\int_0^{2\pi}\cos(4t)\,dt}_{0}$$

$$\Rightarrow |\mathcal{D}| = \frac{3}{2}\cdot 2\pi = \boxed{3\pi}.$$

Practice!

#1 Is $\vec{F} = \cos(y)\,\vec{i} - (x\sin(y)+1)\,\vec{j}$ conservative? If so, find f such that $\vec{F} = \nabla f$ and calculate

$\int_C \vec{F}\cdot d\vec{r}$, where C is a simple curve from $(0,0)$ to (π,π).

$P = \cos(y)$, $Q = -x\sin(y) - 1$

$$\frac{\partial Q}{\partial x} = -\sin(y) = \frac{\partial P}{\partial y} \Rightarrow \text{conservative.}$$

$$\int \cos(y)\,dx = x\cos(y) + g(y) = f(x,y)$$

$$\Rightarrow -x\sin(y) - 1 = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x\cos(y) + g(y)\right)$$

$$= -x\sin(y) + g'(y)$$

$$\Rightarrow -1 = g'(y) \Rightarrow -y = g(y)$$

$$\Rightarrow \boxed{f(x,y) = x\cos(y) - y}.$$

$$\int_C \vec{F}\cdot d\vec{r} = f(\pi,\pi) - f(0,0)$$

$$= \pi\cos(\pi) - \pi - 0$$

$$= \boxed{-2\pi}$$

#2 Find $\int_C (x+y)dx + x^2y\,dy$ if C is the boundary of the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$, positively oriented.

Method 1:

Apply Green's...

$P = x+y$

$Q = x^2y$

$$\iint_D 2xy - 1\, dA$$

$$= \int_0^1 \int_0^{1-x} 2xy - 1\, dy\, dx$$

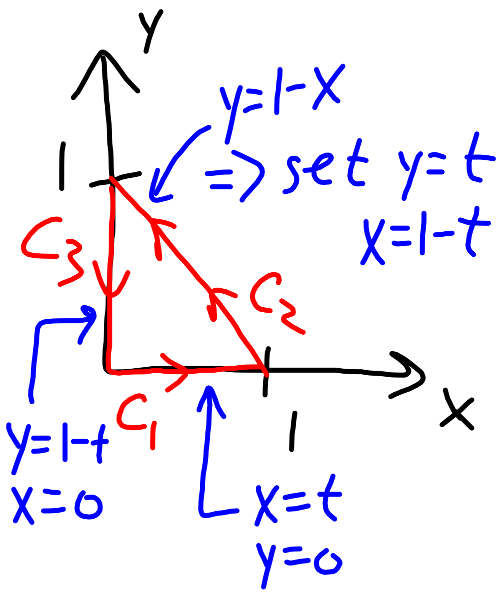
$$\int_0^1 \int_0^{1-x} 2xy - 1 \, dy \, dx$$

$$= \int_0^1 \left[xy^2 - y \right]_0^{1-x} dx = \int_0^1 x(1-x)^2 - (1-x) \, dx$$

$$= \int_0^1 x - 2x^2 + x^3 - 1 + x \, dx$$

$$= \int_0^1 x^3 + 2x - 2x^2 - 1 \, dx = \left[\frac{1}{4}x^4 + x^2 - \frac{2}{3}x^3 - x \right]_0^1$$

$$= \frac{1}{4} + 1 - \frac{2}{3} - 1 = \frac{-5}{12}.$$



Method 2: line integral.

These parameterizations work for $0 \le t \le 1$; they give proper orientation.

$$\int_C (x+y)\,dx + x^2 y\,dy = \int_{C_1}(\dots) + \int_{C_2}(\dots) + \int_{C_3}(\dots)$$

$$\int_{C_1} (x+y)\,dx + \cancel{x^2 y\,dy} = \int_0^1 t\,dt = \frac{1}{2}t^2\Big|_0^1 = \frac{1}{2}.$$

$dx = x'(t)dt = dt$ $\quad y=0$ $\quad 0$

$$\int_{C_3} (x+y)\cancel{dx} + \cancel{x^2 y}\,dy = 0$$

$dx=0$ $\quad 0$ $\quad x=0$ $\quad 0$

$$\int_{C_1}+\int_{C_2}+\int_{C_3} = \frac{1}{2}-\frac{11}{12} = \boxed{\frac{-5}{12}}.$$

$$\int_{C_2} (x+y)\,dx + x^2 y\,dy = -\int_0^1 dt + \int_0^1 t(1-t)^2\,dt$$

$$= -1 + \int_0^1 (t-2t^2+t^3)\,dt = -1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4} = \frac{-11}{12}$$