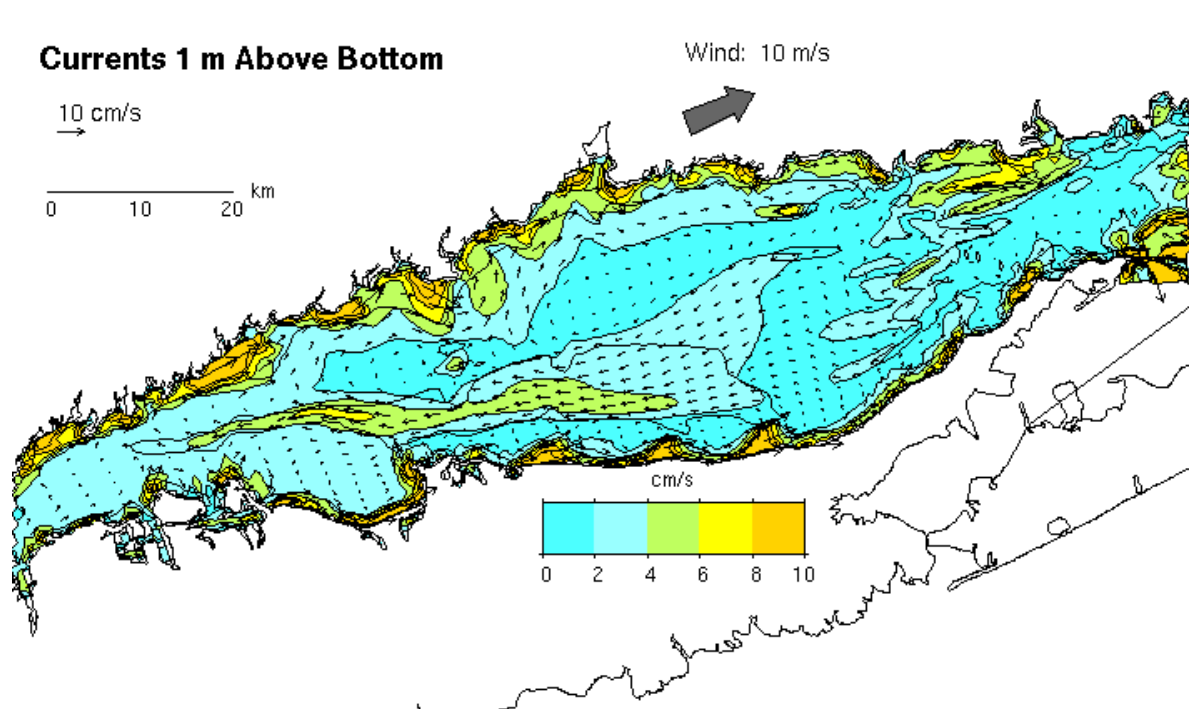


Vector fields - we describe these as vector-valued functions that (1) depend on n variables and have n components.

$$2D : \vec{F}(x, y) = \langle f_1(x, y), f_2(x, y) \rangle$$

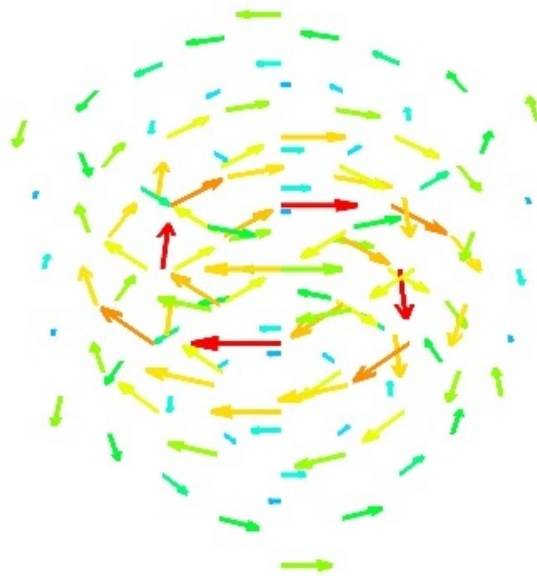
$$3D : \vec{F}(x, y, z) = \langle f_1(x, y, z), f_2(x, y, z), f_3(x, y, z) \rangle$$

At each point (x,y) there is a vector.



Near-bottom currents in Long Island Sound

Hurricane-type wind pattern in 3D



Representations:

$$1) \vec{F} = \langle f_1, f_2, f_3 \rangle .$$

$$2) \vec{F} = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k} .$$

Sketching:

1) Pick some points in the domain.

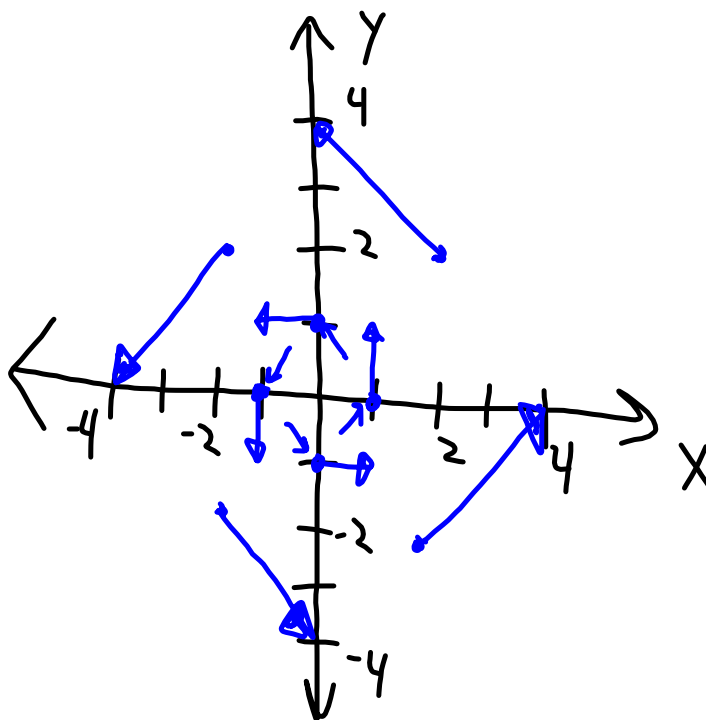
2) Calculate the corresponding vector at each point.

3) Plot each vector with it's tail at that point.

EX: Sketch the vector field

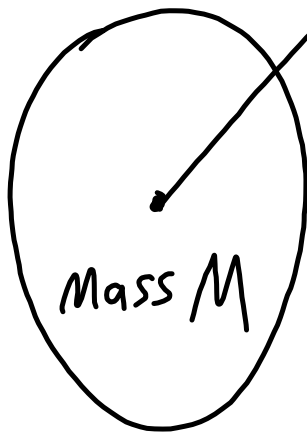
$$\vec{F} = -y\vec{i} + x\vec{j}.$$

(x, y)	\vec{F}
$(1, 0)$	$\langle 0, 1 \rangle$
$(0, 1)$	$\langle -1, 0 \rangle$
$(-1, 0)$	$\langle 0, -1 \rangle$
$(0, -1)$	$\langle 1, 0 \rangle$
\vdots	\vdots
$(\frac{1}{2}, \frac{1}{2})$	$\langle -\frac{1}{2}, \frac{1}{2} \rangle$
\vdots	\vdots
$(2, 2)$	$\langle -2, 2 \rangle$



Force fields

Gravity:



mass m
 at (x, y, z) ; $\vec{r} = \langle x, y, z \rangle$

• Direction of \vec{F} is $-\hat{r}$

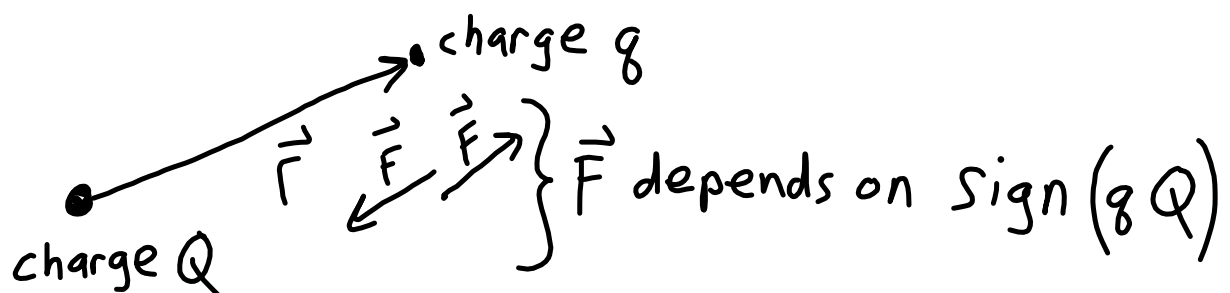
• $|\vec{F}| = \frac{mM G}{|\vec{r}|^2}$

• $\vec{F} = |\vec{F}|(-\hat{r}) = \frac{-mM G}{|\vec{r}|^2} \frac{\vec{r}}{|\vec{r}|}$

$$\vec{F} = \left(\frac{-mM G}{|\vec{r}|^3} \right) \vec{r}$$

$G \approx 6.6738 \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2}$

Electric force field (Coulomb's Law)



$$\vec{F} = \frac{\epsilon q Q}{|\vec{r}|^3} \vec{r}$$

ϵ is a constant;
depends on situation

• $qQ > 0 \Rightarrow$ repulsion, \vec{F} in \vec{r} -direction

• $qQ < 0 \Rightarrow$ attraction, \vec{F} in $(-\vec{r})$ -direction.

Gradient fields

Recall for $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ we have

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

So $\vec{F} = \nabla f$ defines a "gradient vector field".

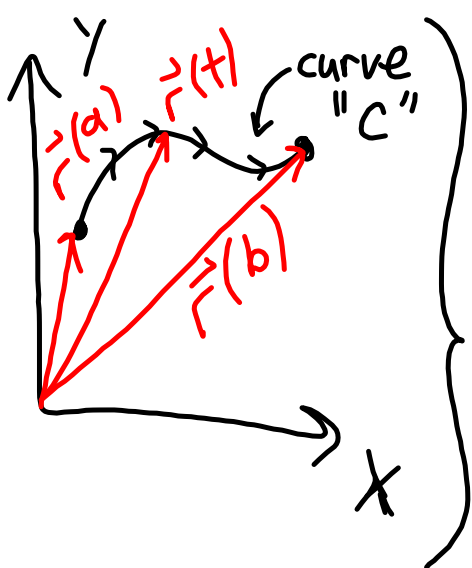
In this case:

1) f is called a "potential function" for \vec{F} .

2) \vec{F} is a "conservative" vector field.

Line integrals

$\vec{r}(t)$: parameterization of a curve
 s : arc length $(a \leq t \leq b)$



Let $f(x,y) = f(x(t), y(t))$
 $= f(\vec{r}(t))$ on the curve C .

LINE INTEGRAL w.r.t. arc-length

$$\int_C f(x,y) ds = ?$$

Recall $S = \int_a^t |\vec{r}'(\tau)| d\tau$

$$\Rightarrow \frac{ds}{dt} = |\vec{r}'(t)|$$

In terms of differentials: $ds = |\vec{r}'(t)| dt$

$$\Rightarrow \int_c^b f(x,y) ds = \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| dt .$$

So if $\vec{r} = \langle x(t), y(t) \rangle$
then $\vec{r}' = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle$

$$\Rightarrow \int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

3D: $\int_C f(x, y, z) ds =$

$$\int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt.$$

Application: find the center of gravity for a "wire"

$\rho(x, y)$: density of a wire

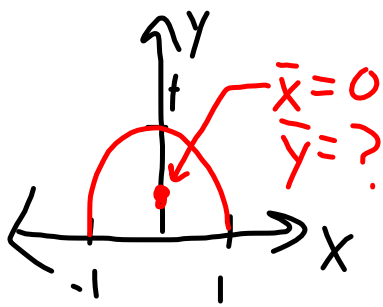
$\vec{r}(t)$: parameterization of wire, $a \leq t \leq b$

Then mass = $m = \int_a^b \rho(x(t), y(t)) \sqrt{(x')^2 + (y')^2} dt$

$$\bar{x} = \frac{1}{m} \int_a^b x(t) \rho(x(t), y(t)) \sqrt{(x')^2 + (y')^2} dt$$

$$\bar{y} = \frac{1}{m} \int_a^b y(t) \rho(x(t), y(t)) \sqrt{(x')^2 + (y')^2} dt.$$

EX: Let a wire have mass density $\rho = 1 - \frac{1}{2}y^2$ and position $x^2 + y^2 = 1$ for $y \geq 0$.
Find the center of mass.



Parameterize the curve!

$$x = \cos(t) \quad y = \sin(t)$$

$$x' = -\sin(t) \quad y' = \cos(t)$$

$$(x')^2 + (y')^2 = \sin^2(t) + \cos^2(t) = 1$$

$$m = \int_0^{\pi} \left(1 - \frac{1}{2}\sin^2(t)\right) \sqrt{1} dt = \pi - \frac{1}{2} \int_0^{\pi} \sin^2(t) dt$$

$$\Rightarrow m = \pi - \frac{1}{2} \int_0^{\pi} \frac{1}{2} (1 - \cos(2t)) dt \quad (\text{Trig. identity})$$

$$= \pi - \frac{1}{4} \left[\pi - \int_0^{\pi} \cos(2t) dt \right] = \frac{3\pi}{4} + \frac{1}{4} \cdot \frac{1}{2} \sin(2t) \Big|_0^{\pi}$$

So $m = 3\pi/4$.

$$\bar{y} = \frac{1}{m} \int_0^{\pi} \sin(t) \left(1 - \frac{1}{2} \sin^2(t) \right) dt = \frac{4}{3\pi} \left[-\cos(t) \Big|_0^{\pi} - \frac{1}{2} \int_0^{\pi} \sin^3(t) dt \right]$$

$$= \frac{4}{3\pi} \left[2 - \frac{1}{2} \int_0^{\pi} \sin(t) (1 - \cos^2(t)) dt \right]$$

$$\begin{aligned}\bar{y} &= \frac{4}{3\pi} \left[2 - \frac{1}{2} \int_0^{\pi} \sin(t) dt + \frac{1}{2} \int_0^{\pi} \sin(t) \cos^2(t) dt \right] \\ &= \frac{4}{3\pi} \left[2 + \frac{1}{2} \cos(t) \Big|_0^{\pi} + \frac{1}{2} \int_0^{\pi} \frac{-1}{3} \frac{d}{dt} \cos^3(t) dt \right] \\ &= \frac{4}{3\pi} \left[2 - \frac{2}{2} - \frac{1}{6} (\cos^3(t)) \Big|_0^{\pi} \right] \\ &\Rightarrow \bar{y} = \frac{4}{3\pi} \left[1 + \frac{1}{3} \right] = \frac{4}{3\pi} \cdot \frac{4}{3} = \boxed{\frac{16}{9\pi}}.\end{aligned}$$

(and $\bar{x} = 0$).

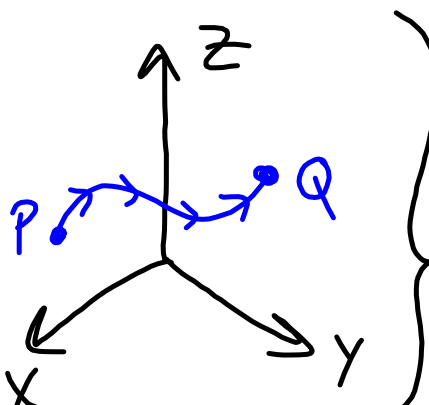
We will sometimes see integrals over a line with respect to x or y ...

$$\int_c f(x,y) dx = \int_a^b f(x(t), y(t)) x'(t) dt$$

$$\int_c f(x,y) dy = \int_a^b f(x(t), y(t)) y'(t) dt$$

$$\text{Notation: } \int_c f_1 dx + \int_c f_2 dy = \int_c f_1 dx + f_2 dy .$$

WORK: We will calculate work done in moving a particle through a force field from point P to Q.



Consider moving over a small segment of the curve...

Work is $\vec{F} \cdot \vec{D}$, in this case $\approx \vec{F}(t_i) \cdot \vec{\Delta D}_i$

$$\vec{\Delta D}_i \approx \Delta s \vec{T}(t_i)$$

\uparrow Unit tangent
 \uparrow Arc length

$$\approx \vec{F}(t_i) \cdot \vec{T}(t_i) \Delta s$$

$$\text{Total work} \approx \sum_i \vec{F}(t_i) \cdot \vec{T}(t_i) \Delta s$$

Pass to the limit as $\Delta s \rightarrow 0 \dots$

$$\text{Total work} = W = \int \vec{F} \cdot \vec{T} ds$$

$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} \quad \left\{ \begin{array}{l} ds = |\vec{r}'(t)| dt \end{array} \right.$$

$$\Rightarrow W = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

Now let $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$\vec{F} = \langle f_1, f_2, f_3 \rangle$$

$$\Rightarrow W = \int_a^b \langle f_1, f_2, f_3 \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle dt$$
$$= \int_a^b f_1 x'(t) dt + \int_a^b f_2 y'(t) dt + \int_a^b f_3 z'(t) dt$$

$$= \int_C f_1 dx + \int_C f_2 dy + \int_C f_3 dz = \int_C f_1 dx + f_2 dy + f_3 dz.$$

SUMMARY of main formulas

$$1) \int_C f ds = \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| dt$$

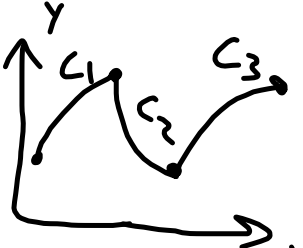
$$2) W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

$$3) W = \int_C \vec{F} \cdot d\vec{r} = \int_C f_1 dx + f_2 dy + f_3 dz.$$

Two notations,
same thing.

Two comments:

1) If a curve has "kinks", break the integral up into multiple integrals over the smooth parts.



$$C = C_1 \cup C_2 \cup C_3$$

$$\int_C f ds = \int_{C_1} f ds + \int_{C_2} f ds + \int_{C_3} f ds .$$

2) Direction of motion/"orientation" of a parameterization matters.





$$\int_C \vec{F} \cdot d\vec{r} = - \int_{-C} \vec{F} \cdot d\vec{r}$$

If your curve is parameterized in the wrong direction you will get a sign error.

EX: Find the work done on a particle when moving from $P(1,0,0)$ to $Q(3,-1,2)$ along a straight line if $\vec{F} = \langle z, x, y \rangle$.

A line segment can be parameterized as follows:

$$\begin{aligned}\vec{r}(t) &= \vec{p} + t\vec{PQ} = \langle 1, 0, 0 \rangle + t\langle 3-1, -1-0, 2-0 \rangle \\ &= \langle 1+2t, -t, 2t \rangle, \text{ for } 0 \leq t \leq 1.\end{aligned}$$

Then $\vec{r}'(t) = \vec{PQ} = \langle 2, -1, 2 \rangle$.

Plug into work formula...

$$\begin{aligned} W &= \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_0^1 \langle 2t, 1+2t, -t \rangle \cdot \langle 2, -1, 2 \rangle dt \\ &= \int_0^1 (4t - 1 - 2t - 2t) dt = \int_0^1 (-1) dt = \boxed{-1}. \end{aligned}$$

EX: Let a particle move along the path $y=2x^2-1$ from $(0,-1)$ to $(2,7)$. Find the work done if it is acted upon by a force $\vec{F} = -y\vec{i} + x\vec{j}$.

Parameterize: $y=y(x)$ so choose $\begin{cases} x=t \\ y=2t^2-1 \end{cases}$
with $0 \leq t \leq 2$.

$$\vec{r}(t) = \langle t, 2t^2-1 \rangle \Rightarrow \vec{r}'(t) = \langle 1, 4t \rangle.$$

$$W = \int_0^2 \langle 1-2t^2, t \rangle \cdot \langle 1, 4t \rangle dt = \int_0^2 1-2t^2+4t^2 dt$$

$$W = \int_0^1 1+2t^2 dt = 2 + \frac{2}{3}t^3 \Big|_0^2 = \frac{6}{3} + \frac{16}{3} = \frac{22}{3}.$$

Practice!

#1 Let a particle move along a straight path from $(-5, -4)$ to $(1, 2)$ subject to a force $\vec{F} = y^2 \vec{i} + x^2 \vec{j}$.

Find the work done.

$$\vec{r}(t) = \langle -5+t(1+5), -4+t(2+4) \rangle = \langle 6t-5, 6t-4 \rangle.$$

$$\vec{r}'(t) = 6\langle 1, 1 \rangle, \quad 0 \leq t \leq 1.$$

$$\begin{aligned} W &= \int_0^1 \langle (6t-4)^2, (6t-5)^2 \rangle \cdot \langle 1, 1 \rangle 6 dt \\ &= 6 \int_0^1 (6t-4)^2 + (6t-5)^2 dt = 6 \int_0^1 6^2 \left(t - \frac{2}{3}\right)^2 + 6^2 \left(t - \frac{5}{6}\right)^2 dt \\ &= 6^3 \int_0^1 \left(t - \frac{2}{3}\right)^2 + \left(t - \frac{5}{6}\right)^2 dt = \frac{6^3}{3} \left[\left(t - \frac{2}{3}\right)^3 + \left(t - \frac{5}{6}\right)^3 \right]_0^1 \end{aligned}$$

$$\begin{aligned} &= \frac{6^3}{3} \left(\frac{1}{3^3} + \frac{1}{6^3} + \frac{8}{3^3} + \frac{125}{6^3} \right) \\ &= \frac{6^3}{3} \left(\frac{3^2}{3^3} + \frac{126}{6^3} \right) = \frac{6^3}{3} \left(\frac{1}{3} + \frac{7}{12} \right) \\ &= \frac{6^3}{3} \cdot \frac{11}{12} = \frac{6 \cdot 36 \cdot 11}{36} = \boxed{66}. \end{aligned}$$

② Let $\vec{F} = -y\vec{i} + x\vec{j}$. Calculate the work done to move a particle from $(1,0)$ around the circle $x^2 + y^2 = 1$ and back to $(1,0)$.

$$x = \cos(t) \quad y = \sin(t), \quad 0 \leq t \leq 2\pi$$
$$x' = -\sin(t) \quad y' = \cos(t)$$

$$W = \int_0^{2\pi} \langle -\sin(t), \cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle dt$$
$$= \int_0^{2\pi} \cos^2(t) + \sin^2(t) dt = \int_0^{2\pi} dt = \boxed{2\pi.}$$