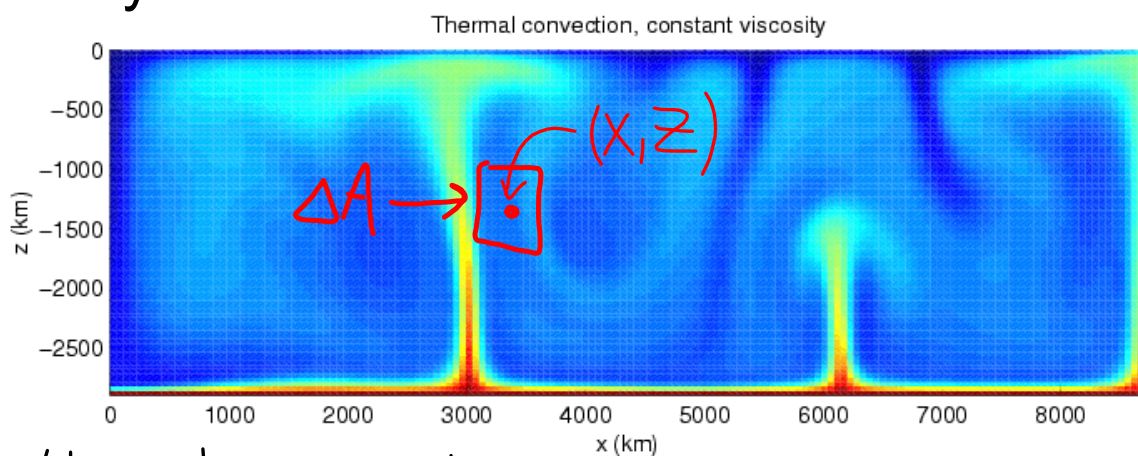


## Density



$E$ : thermal energy in region  $\Delta A$   
 $\Rightarrow$  Thermal energy density is  $E/\Delta A$  "energy per unit area"

$$\lim_{\Delta A \rightarrow 0} \frac{E}{\Delta A} = \rho(x, z) : \text{"local thermal energy density"}$$

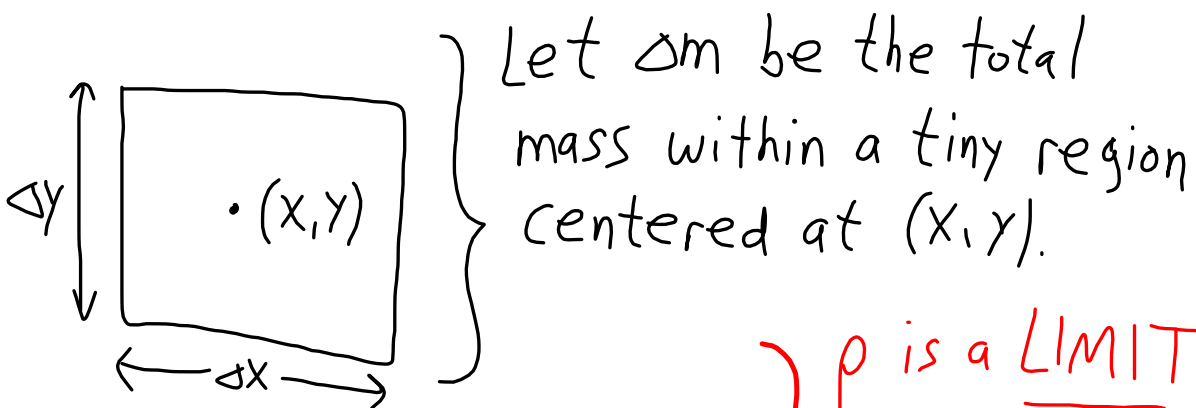
Generally, if  $\rho(x,y)$  is a local density function for some quantity, then the total amount of the quantity over an area  $A$  is

$$\iint_A \rho(x,y) dA.$$

For example, if  $\rho$  is mass density, then the total mass,  $m$ , is  $m = \iint_A \rho dA.$

NOTE: Point-wise densities are a mathematical tool and MODEL, not a "real" thing. For example, what does mass density at a point mean? There is no physical description of mass at a single point; even sub-atomic particles are considered to take up some non-zero volume of space.

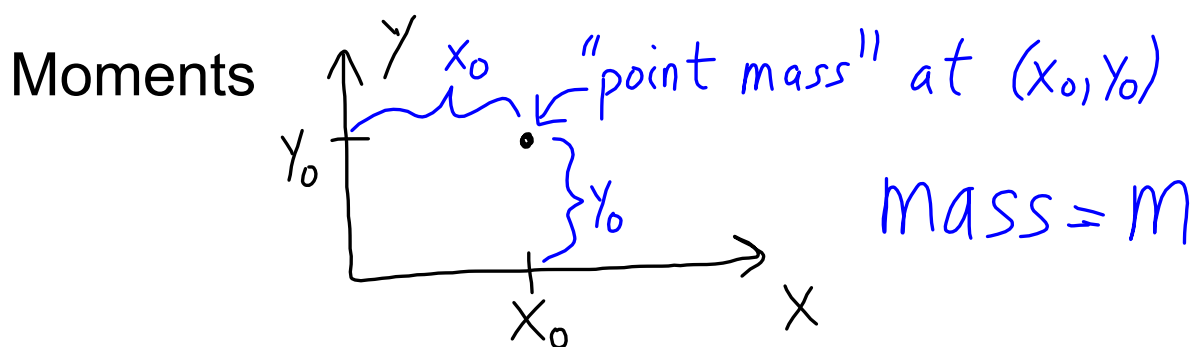
To clarify the point...



Let  $\Delta m$  be the total mass within a tiny region centered at  $(x, y)$ .

$$\rho(x, y) = \lim_{\Delta x, \Delta y \rightarrow 0} \frac{\Delta m}{\Delta x \cdot \Delta y}$$

$\rho$  is a LIMIT at each point. It just models the behavior of mass near a particular point.

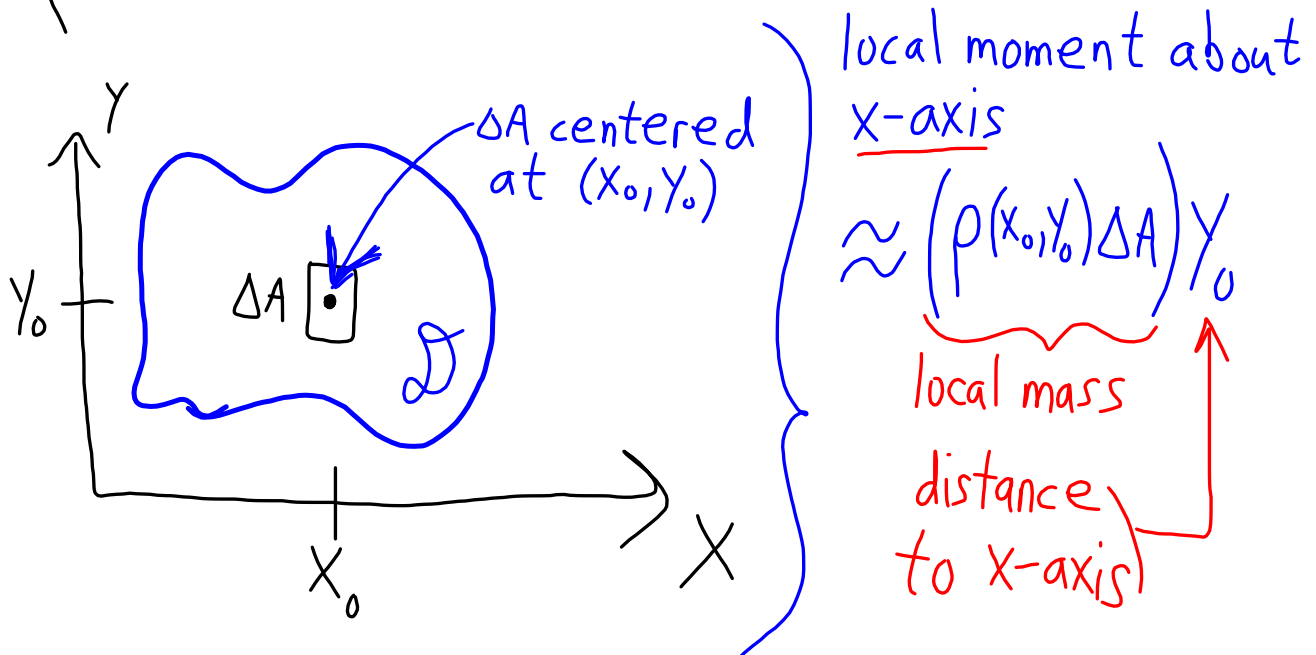


$$X\text{-moment} : my_0$$

$$Y\text{-moment} : mx_0$$

Now consider a solid body as a collection/  
continuum of particles.

$\rho(x, y)$ : local mass density



$M_x^o$  moment of body about x-axis

Riemann sum...  $M_x \approx \sum_{i,j} \rho(x_i^*, y_j^*) \Delta A_{ij} y_j^*$

$$(\Delta A_{ij} = \Delta A, \text{ constant}) = \Delta A \sum_{i,j} \rho(x_i^*, y_j^*) y_j^*$$

$$\Rightarrow M_x = \lim_{\Delta A \rightarrow 0} \Delta A \sum_{i,j} \rho(x_i^*, y_j^*) y_j^* = \iint_D y \rho(x, y) dA.$$

$$M_x = \iint_D y \rho \, dA$$

Moment  
about  
X-axis

$$M_y = \iint_D x \rho \, dA$$

Moment  
about  
y-axis



EX: Let a region  $D = \{(x,y) \mid 0 \leq x \leq 2, 1 \leq y \leq 4\}$  represent a solid object with mass density  $\rho(x,y) = 1+x^2$ . Find the moments about the  $x$ - &  $y$ -axes.

$$\begin{aligned} M_x &= \int_0^2 \int_1^4 y(1+x^2) dy dx = \frac{1}{2} \int_0^2 (16-1)(1+x^2) dx \\ &= \frac{15}{2} \left[ x + \frac{1}{3}x^3 \right]_0^2 = \frac{15}{2} \left( 2 + \frac{8}{3} \right) = 35. \end{aligned}$$

$$\begin{aligned}M_y &= \int_0^2 \int_1^4 x(1+x^2) dy dx \\&= 3 \int_0^2 x+x^3 dx = 3 \left[ \frac{1}{2}x^2 + \frac{1}{4}x^4 \right]_0^2 \\&= 3[2+4] = \boxed{18.}\end{aligned}$$

Center of mass

$$\bar{x} = \frac{M_y}{m} .$$

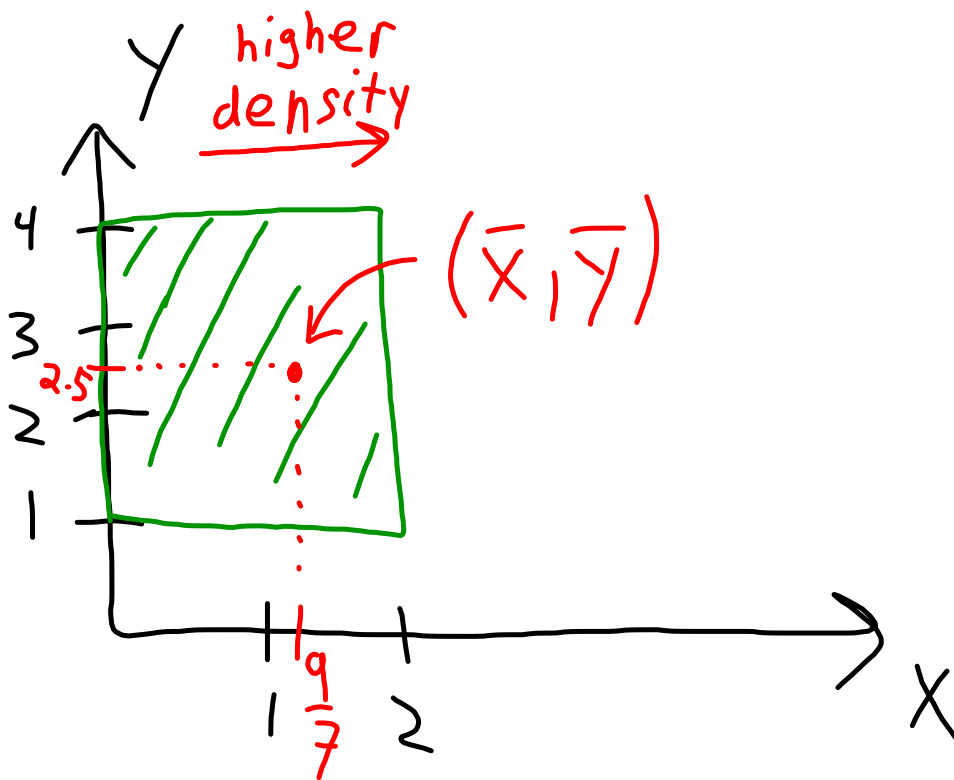
$$\bar{y} = \frac{M_x}{m} .$$

$$m = \iint_D \rho dA .$$

EX: For the previous example, find the center of mass.

$$\begin{aligned}\rho = 1+x^2 &\Rightarrow m = \int_0^2 \int_1^4 (1+x^2) dy dx = 3 \int_0^2 1+x^2 dx \\ &= 3 \left[ x + \frac{1}{3}x^3 \right]_0^2 = 3 \left( 2 + \frac{8}{3} \right) = 14.\end{aligned}$$

$$\text{Thus } \bar{x} = \frac{M_y}{m} = \frac{18}{14} = \frac{9}{7}, \quad \bar{y} = \frac{35}{14} = \frac{5}{2}.$$



Moments of inertia: find the torque needed to generate some angular acceleration.

$$1) I_x = \iint_{\mathcal{D}} y^2 \rho \, dA \quad 2) I_y = \iint_{\mathcal{D}} x^2 \rho \, dA$$

about x-axis                      about y-axis

$$3) I_o = I_x + I_y = \iint_{\mathcal{D}} (x^2 + y^2) \rho \, dA.$$

} rotation  
around  
origin

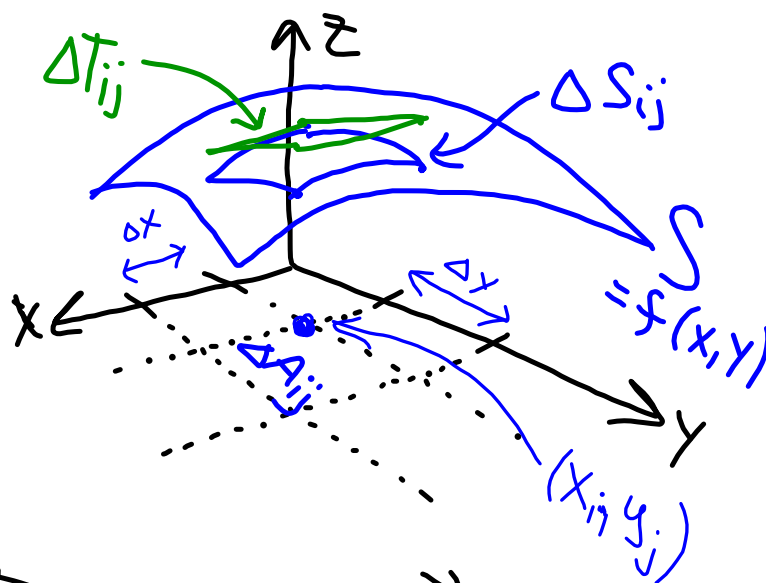
## Comments

- \* The required torque increases with distance.
- \* The required torque increases with density, but not as quickly as with distance.
- \* Think of walking a tight-rope... people hold a long pole rather than something smaller but heavier.
- \* Abstractly, the n-th moment of  $f(x,y)$  with respect to variable  $\omega$  is

$$\iint_D \omega^n f dA.$$

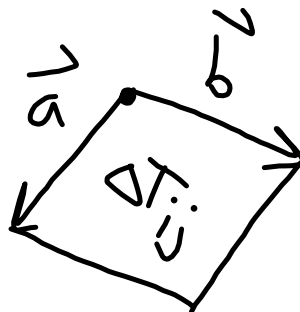
Surface area

$\Delta T_{ij}$ : piece of  
tangent plane to  
 $f$  at  $(x_{ij}, y_{ij})$  lying  
above  $\Delta A_{ij}$ .



Formula for  $\Delta T_{ij}$ :

$$\Delta T_{ij} = |\vec{a} \times \vec{b}|$$





Note  $\vec{a}$  lies along tangent line in the X-direction &  $\vec{b}$  " " in y-direction;

$$\vec{a} = \langle \Delta x, 0, \Delta x \cdot f_x \rangle = \Delta x \langle 1, 0, f_x \rangle$$

$$\vec{b} = \langle 0, \Delta y, \Delta y \cdot f_y \rangle = \Delta y \langle 0, 1, f_y \rangle$$

$$\Rightarrow \vec{a} \times \vec{b} = \Delta x \cdot \Delta y \langle -f_x, -f_y, 1 \rangle$$

$$|\vec{a} \times \vec{b}| = \Delta T_{ij} = \Delta x \Delta y \sqrt{1 + (f_x)^2 + (f_y)^2}$$

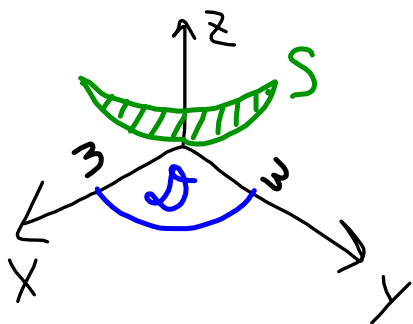
$$\Rightarrow S \approx \sum_{ij} \Delta T_{ij} = \Delta x \Delta y \sum_{ij} \sqrt{1 + f_x(x_{ij}, y_{ij})^2 + f_y(x_{ij}, y_{ij})^2}$$

Let  $\Delta T_{ij} = \Delta x \cdot \Delta y \rightarrow 0 \dots$

Surface area is a double integral

$$S = \lim_{\Delta x \Delta y \rightarrow 0} \sum_{ij} \Delta T_{ij} = \iint_D \sqrt{1 + (f_x)^2 + (f_y)^2} dA.$$

Ex: Calculate the surface area of the paraboloid  $z = x^2 + y^2$  over the domain  $D = \{(x, y) \mid x^2 + y^2 \leq 9, x \geq 0, y \geq 0\}$ .



$$z_x = 2x \dots z_y = 2y$$

$$\text{Polar: } 0 \leq \theta \leq \pi/2$$

$$0 \leq r \leq 3$$

$$z_x^2 + z_y^2 = 4(x^2 + y^2) = 4r^2$$

$$S = \int_0^{\pi/2} \int_0^3 \sqrt{1 + 4r^2} r dr d\theta \quad \left. \begin{array}{l} u = 4r^2 \\ \frac{1}{8} du = r dr \end{array} \right\}$$

$$\begin{aligned} S &= \frac{1}{8} \int_0^{\pi/2} \int_{u=0}^{u=4 \cdot 3^2=36} \sqrt{1+u} \, du \, d\theta \\ &= \frac{1}{8} \left( \frac{\pi}{2} \right) \left( \frac{2}{3} (1+u)^{3/2} \Big|_0^{36} \right) \\ &= \frac{\pi}{24} (37^{3/2} - 1). \end{aligned}$$

Practice!

(#1) Let a solid disc have radius  $r=3$  and density  $\rho(x,y) = e^{-x^2-y^2}$ . Find (a) the total mass and (b) center of mass.

$$\int_0^{2\pi} \int_0^3 e^{-r^2} r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^9 e^{-u} \, du \, d\theta$$

$u=r^2$   
 $\frac{1}{2} du = r \, dr$

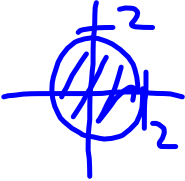
$$\begin{aligned} &= \frac{1}{2} \int_0^{2\pi} \left[ \int_0^9 e^{-u} du \right] d\theta \\ &= \frac{1}{2} \int_0^9 e^{-u} du \cdot \int_0^{2\pi} d\theta \\ &= \pi \left[ -e^{-u} \right]_0^9 = \pi (1 - e^{-9}) \end{aligned}$$

#2 The portion of a cone above the  $xy$ -plane is given as

$$z = \frac{1}{2} \sqrt{x^2 + y^2}.$$

Find the surface area lying above

$$D = \{(x, y) \mid x^2 + y^2 \leq 4\}.$$
$$z_x = \frac{2x}{4\sqrt{x^2 + y^2}} = \frac{x}{2\sqrt{x^2 + y^2}} \quad z_y = \frac{y}{2\sqrt{x^2 + y^2}}$$
$$z_x^2 + z_y^2 = \frac{1}{4}$$


$$S = \int_0^{2\pi} \int_0^2 \sqrt{1 + \frac{1}{4}} r dr d\theta$$
$$= \frac{\sqrt{5}}{2} \cdot 2\pi \int_0^2 r dr = \pi\sqrt{5} \left. \frac{r^2}{2} \right|_0^2$$
$$= \boxed{2\pi\sqrt{5}}$$