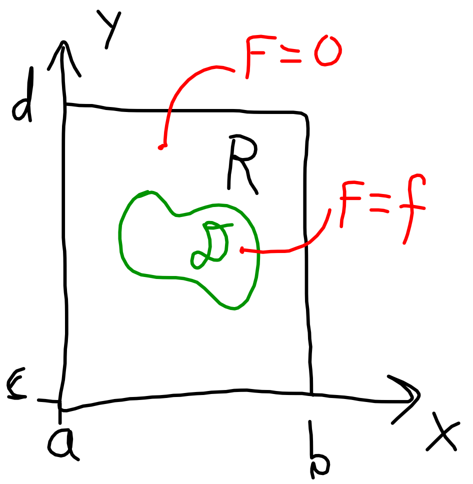
Integration over general regions.

Let $f(x,y)$ have domain $\mathcal{D}$ and define



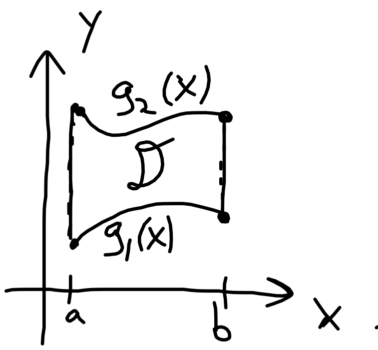
$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ in } \mathcal{D} \\ 0, & \text{otherwise} \end{cases}.$$

Then

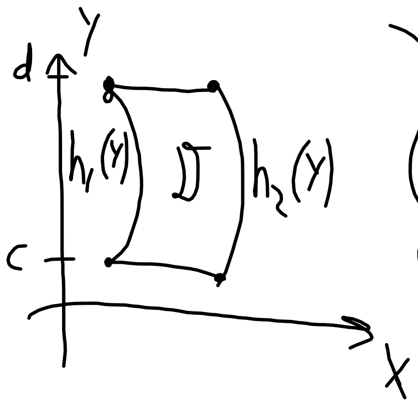
$$\iint_{\mathcal{D}} f(x,y)\,dA = \underbrace{\iint_{R} F(x,y)\,dA}.$$

This can be defined using double Riemann sums, same as before.

We often identify domains as being bounded by certain curves.

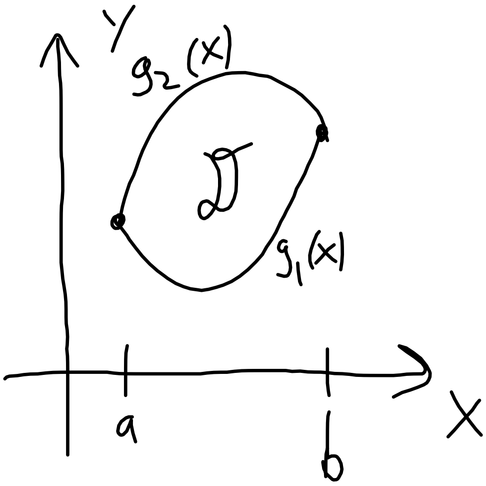


$$\iint_D f(x,y)\,dA = \int_a^b \underbrace{\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy}_{\text{depends on } x}\,dx .$$



$$\iint_D f(x,y)\,dA = \int_c^d \underbrace{\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx}_{\text{depends on } y}\,dy .$$

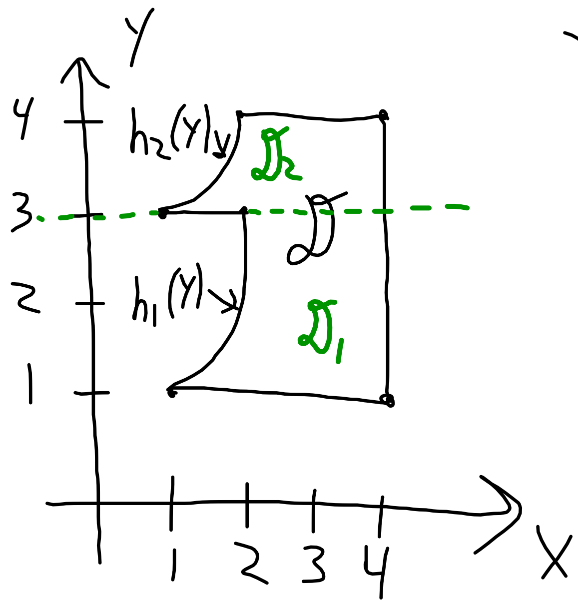
Often you can choose the order of integration; this is a matter of preference usually.



$$\iint_{\mathcal{D}} f\,dA = \underbrace{\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx}$$

Integration in the opposite order may be possible, but perhaps trickier.

Sometimes it makes sense to break the integral up by breaking up the domain.

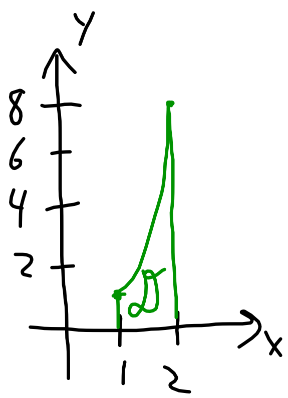


$$\iint_{D} f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$$

$$= \int_{1}^{3}\int_{h_1(y)}^{4} f(x,y)\,dx\,dy + \int_{3}^{4}\int_{h_2(y)}^{4} f(x,y)\,dx\,dy.$$

EX: Let $f(x,y) = e^{y/x}$ on $D = \{(x,y) \mid 1 \le x \le 2,\ 0 \le y \le x^3\}$.

Find $\displaystyle\iint_D f(x,y)\,dA$.



$$= \int_1^2 \int_0^{x^3} e^{y/x}\,dy\,dx = \int_1^2 \left[x e^{y/x} \right]_{0=y}^{y=x^3} dx$$

$$= \int_1^2 \left(x e^{x^2} - x \right) dx = \int_1^2 x e^{x^2}\,dx - \int_1^2 x\,dx$$

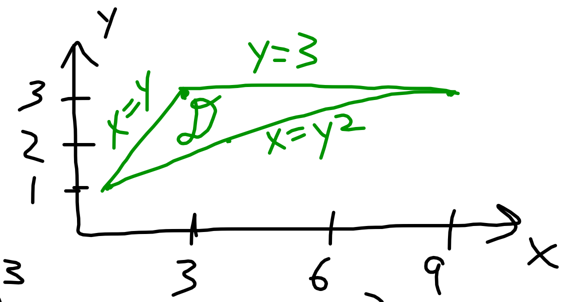
$u = x^2$

$du = 2x\,dx$

$$= \int_{u=1}^{u=4} \frac{1}{2} e^u\,du - \int_1^2 x\,dx = \frac{1}{2}\left(e^4 - e\right) - \frac{1}{2}\left(4 - 1\right).$$

EX: Let $f(x,y)=xy$ on $D=\{(x,y)\mid y\le x\le y^2,\ 1\le y\le 3\}$.

Find $\iint_D f(x,y)\,dA$.

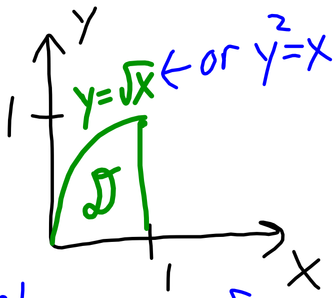


$$=\int_1^3\int_y^{y^2} xy\,dx\,dy=\frac{1}{2}\int_1^3 \left(x^2y\right)\Big|_{x=y}^{x=y^2}dy=\frac{1}{2}\int_1^3\left(y^5-y^3\right)dy$$

$$=\frac{1}{2}\left[\frac{1}{6}y^6-\frac{1}{4}y^4\right]_1^3=\frac{1}{2}\left[\frac{1}{6}\left(3^6-1\right)-\frac{1}{4}\left(3^4-1\right)\right].$$

EX: Let $f(x,y)=ye^x$ on $\mathcal{D}=\{(x,y) \mid 0\le x\le 1,\ 0\le y\le\sqrt{x}\}$.

Find $\iint_{\mathcal{D}} f\,dA$.



We'll show 2 ways...

① $\displaystyle\int_0^1\int_0^{\sqrt{x}} ye^x\,dy\,dx = \int_0^1 \left[\frac{1}{2}y^2 e^x\right]_{y=0}^{y=\sqrt{x}} dx = \frac{1}{2}\int_0^1 (xe^x)\,dx$

Integrate by parts...

$$=\frac{1}{2}\int_0^1 x\frac{d}{dx}(e^x)\,dx = \frac{1}{2}\int_0^1\left(\frac{d}{dx}(xe^x) - \left[\frac{d}{dx}(x)\right]e^x\right)dx$$

$$= \frac{1}{2}\left[xe^x\Big|_0^1 - \int_0^1 e^x dx\right] = \frac{1}{2}\left(e - (e-1)\right) = \boxed{\frac{1}{2}}.$$

② $$= \int_0^1 \int_{x=y^2}^{x=1} ye^x \, dx\, dy = \int_0^1 \left[ye^x\right]_{x=y^2}^{x=1} dy$$

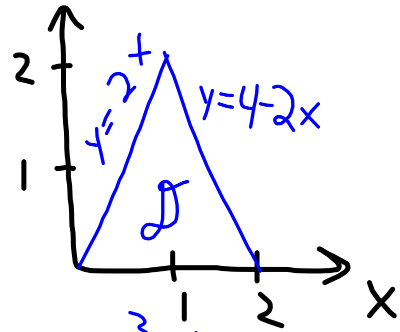
$$= \int_0^1 \left(ey - ye^{y^2}\right) dy = e\int_0^1 y\, dy - \int_0^1 ye^{y^2} dy$$

$u = y^2$

$\frac{1}{2}du = y\,dy$

$$= \frac{e}{2} - \int_0^1 \frac{1}{2} e^u du = \frac{e}{2} - \left(\frac{e}{2} - \frac{1}{2}\right) = \boxed{\frac{1}{2}}.$$

EX: Let $f(x,y)=2x-4y$. Integrate this over $\mathcal{D}$ as shown:



Two approaches

(1) $\displaystyle\int_0^1\int_0^{2x}(2x-4y)\,dy\,dx+\int_1^2\int_0^{4-2x}(2x-4y)\,dy\,dx$

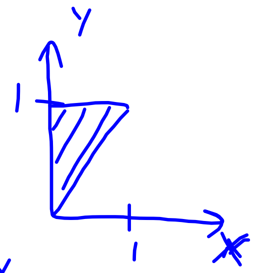
(2) $\displaystyle\int_0^2\int_{y/2}^{\frac{4-y}{2}}(2x-4y)\,dx\,dy=\int_0^2\left[x^2-4xy\right]_{x=\frac{y}{2}}^{x=\frac{4-y}{2}}dy$

$$= \int_0^2 \left[\frac{1}{4}(4-y)^2 - \frac{1}{4}y^2 - 2y(4-y) + 2y^2 \right] dy$$

$$= \left[\frac{-1}{4\cdot 3}(4-y)^3 - \frac{1}{12}y^3 - 4y^2 + \frac{2}{3}y^3 + \frac{2}{3}y^3 \right]_0^2$$

$$= -\frac{2}{3} + \frac{16}{3} - \frac{8}{12} - 16 + \frac{32}{3} = -12 + \frac{32}{3} = \frac{32-36}{3} = \boxed{-\frac{4}{3}}.$$

EX: Find $\int_0^1 \int_x^1 \underbrace{\sin(y^2)\,dy}_{\int \sin(y^2)dy = ?}\,dx = ?$



Switch order of integration...

$$= \int_0^1 \int_0^y \sin(y^2)\,dx\,dy = \int_0^1 \Big[x\sin(y^2)\Big]_{x=0}^{x=y} dy$$

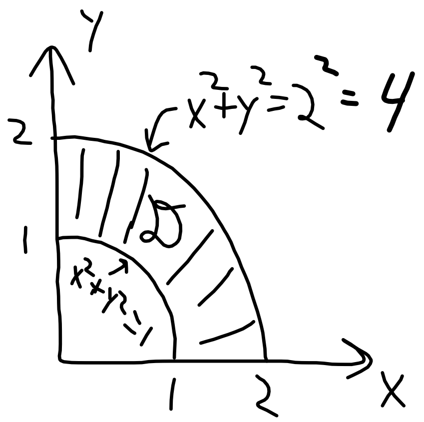
$$= \int_0^1 \Big[y\sin(y^2) - 0\Big]dy = \frac{1}{2}\int_0^1 \sin(u)\,du = \frac{1}{2}\Big[-\cos(1) + \cos(0)\Big]$$

$u = y^2$

$\frac{1}{2}du = y\,dy$

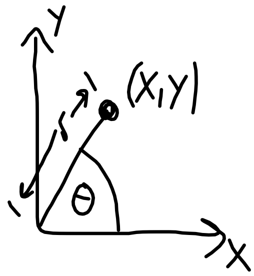
$$= \boxed{\frac{1}{2}\big(1 - \cos(1)\big)}.$$

Polar coordinates



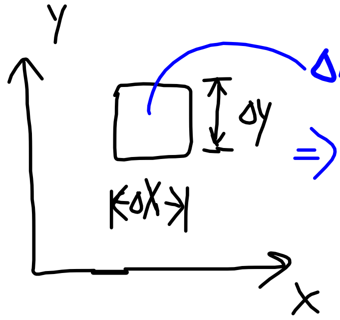
This should be handled using polar coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$



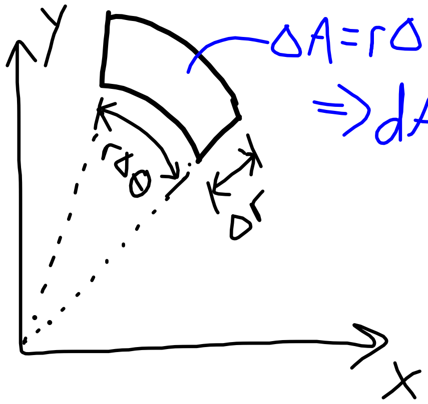
$$\int_{r=1}^{r=2}\int_{\theta=0}^{\theta=\pi/2} \underbrace{f(x,y)}_{\substack{x=r\cos\theta \\ y=r\sin\theta}}\ \underbrace{dA}_{\text{need to discuss this further}} = ?$$

Differential area in polar coordinates



$\Delta A = \Delta x \Delta y$

$\Rightarrow dA = dx\,dy$



$\Delta A = r \Delta r \Delta \theta$

$\Rightarrow dA = r\,dr\,d\theta$

Back to the integral...

$$\int_{r=1}^{r=2} \int_{\theta=0}^{\theta=\pi/2} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta = \iint_{D} f(x,y)\, dA.$$

don't forget!

EX: Find $\iint_D f(x,y)\,dA$ if $D = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{2}\}$,

$f = x - 4y^2$. $f(r,\theta) = r\cos\theta - 4r^2\sin^2\theta$

$$\int_0^{\pi/2}\int_1^2 \left(r\cos\theta - 4r^2\sin^2\theta\right) r\,dr\,d\theta = \int_0^{\pi/2}\int_1^2 \left(r^2\cos\theta - 4r^3\sin^2\theta\right) dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{3}r^3\cos\theta - r^4\sin^2\theta\right]_1^2 d\theta = \int_0^{\pi/2} \left[\frac{1}{3}(8-1)\cos\theta - (16-1)\sin^2\theta\right] d\theta$$

$$= \int_0^{\pi/2} \left(\frac{7}{3}\cos\theta - 15\sin^2\theta\right) d\theta$$

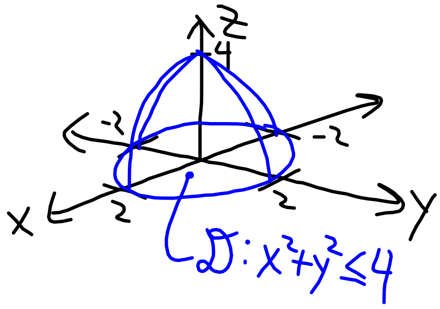
Use $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$

$$\Rightarrow \int_0^{\pi/2} \left[\frac{7}{3}\cos\theta - \frac{15}{2}(1-\cos(2\theta))\right] d\theta$$

$$= \frac{7}{3}\sin\theta\Big|_0^{\frac{\pi}{2}} - \frac{15}{2}\cdot\frac{\pi}{2} + \frac{15}{2}\left(\frac{1}{2}\sin(2\theta)\right)\Big|_0^{\pi/2} \rightarrow 0$$

$$= \frac{7}{3} - \frac{15\pi}{4}.$$

EX: Find the volume under the surface $z=4-x^2-y^2$ & above $z=0$.

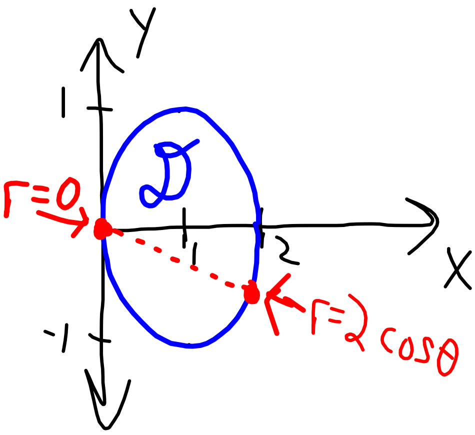


$$V=\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2}(4-r^2\cos^2\theta-r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^2(4r-r^3)\,dr\,d\theta=\int_0^{2\pi}\left(2r^2-\frac{1}{4}r^4\right)\Big|_0^2\,d\theta$$

$$=\int_0^{2\pi}(8-4)\,d\theta=4\cdot 2\pi=\boxed{8\pi}.$$

EX: Let D be the region inside the circle $(x-1)^2+y^2=1$. Find $\iint_D xy\,dA$.



On boundary, we have

$(r\cos\theta-1)^2+r^2\sin^2\theta=1$

$r^2\cos^2\theta-2r\cos\theta+1+r^2\sin^2\theta=1$

$r^2-2r\cos\theta+1=1$

$r(r-2\cos\theta)=0$

$\Rightarrow r=0$ or $r=2\cos\theta$.

$0\le r\le 2\cos\theta$

$$\iint_{\mathcal{D}} xy\,dA = \int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{2\cos\theta} r\cos\theta\, r\sin\theta\, r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3 \sin\theta\cdot\cos\theta\,dr\,d\theta$$

error, $-\frac{2}{3}$

$$= \int_{-\pi/2}^{\pi/2} \sin\theta\cos\theta\,\frac{1}{4}r^4\Big|_0^{2\cos\theta}\,d\theta = \int_{-\pi/2}^{\pi/2} 4\sin\theta\cos^5\theta\,d\theta$$

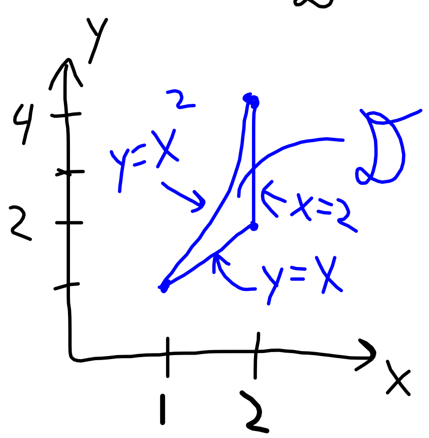
$$= 4\int_{-\pi/2}^{\pi/2} -\frac{1}{6}\frac{d}{d\theta}\cos^6\theta\,d\theta = \frac{-1}{24}\left(\cos^6\frac{\pi}{2} - \cos^6\left(\frac{-\pi}{2}\right)\right)$$

$\cos^6\frac{\pi}{2} \to 0$, $\cos^6\left(\frac{-\pi}{2}\right) \to 0$

$= 0.$

Practice!

(#1) Find $\iint_D (x^2+y)\,dA$, with D given in the diagram:



$$= \int_1^2 \int_x^{x^2} (x^2+y)\,dy\,dx$$

$$= \int_1^2 \left[x^2 y + \frac{1}{2}y^2\right]_{y=x}^{y=x^2} dx$$

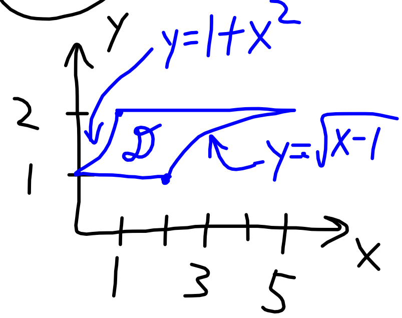
$$= \int_1^2 \left(x^4 - x^3 + \frac{1}{2}x^4 - \frac{1}{2}x^2\right) dx$$

$$= \int_1^2 \left(\frac{3}{2}x^4 - x^3 - \frac{1}{2}x^2\right) dx$$

$$= \left[\frac{3}{10}x^5 - \frac{1}{4}x^4 - \frac{1}{6}x^3\right]_1^2$$

$$\frac{3}{10}(32-1) - \frac{1}{4}(16-1) - \frac{1}{6}(8-1)$$

#2 Let $\mathcal{D}$ be as in the diagram; find $\iint_{\mathcal{D}} x\,dA$.



$1 \le y \le 2$

$\sqrt{y-1} \le x \le y^2+1$

$$\int_1^2 \int_{\sqrt{y-1}}^{y^2+1} x\,dx\,dy$$

$$= \frac{1}{2}\int_1^2 x^2 \Big|_{\sqrt{y-1}}^{y^2+1} dy$$

$$= \frac{1}{2}\int_1^2 (y^2+1)^2 - (y-1)\,dy = \frac{1}{2}\int_1^2 \left(y^4+2y^2+2-y\right)dy$$

$$= \frac{1}{2}\left[\frac{1}{5}y^5 + \frac{2}{3}y^3 + 2y - \frac{1}{2}y^2\right]_1^2$$

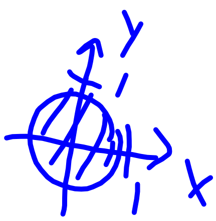
$$= \frac{1}{2}\left(\frac{32-1}{5} + \frac{2(8-1)}{3} + 2(2-1) - \frac{1}{2}(4-1)\right)$$

$$= \frac{1}{2}\left(\frac{31}{5} + \frac{14}{3} + 2 - \frac{3}{2}\right) \ldots$$

#3 Find $\iint_D x^2+y^2\,dA$; $D=\{(x,y) \mid x^2+y^2\le 1\}$.

$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2(\cos^2\theta+\sin^2\theta) = r^2$$



$$\int_0^{2\pi}\int_0^1 r^2\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{4} r^4 \Big|_0^1 \, d\theta = \frac{\pi}{2}$$

#4 Find the volume V under $z = 1 + y + x$, above $\mathcal{D} = \{(x,y) \mid x^2 + y^2 \le 4\}$ in the xy-plane.

$$|V| = \int_0^{2\pi}\int_0^2 \left(1 + r\cos\theta + r\sin\theta\right) r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left(r + r^2(\cos\theta + \sin\theta)\right) dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2} r^2 + \frac{1}{3} r^3 (\cos\theta + \sin\theta) \right]_0^2 d\theta$$

$$= \int_0^{2\pi} \left(2 + \underbrace{\frac{8}{3} (\cos\theta + \sin\theta)} \right) d\theta$$

$$= 2 \cdot 2\pi + 0 = 4\pi$$