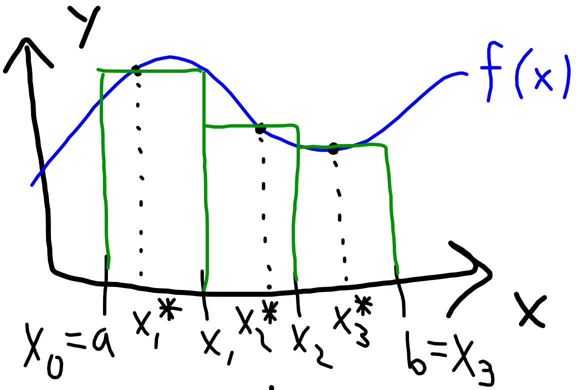
Double integrals

First, let's recall Riemann sums for integrals with a single variable...



- Height of rectangle is $f(x_j^*)$
- Width of rectangle Δx
- Area of " " $\Delta x \cdot f(x_j^*)$

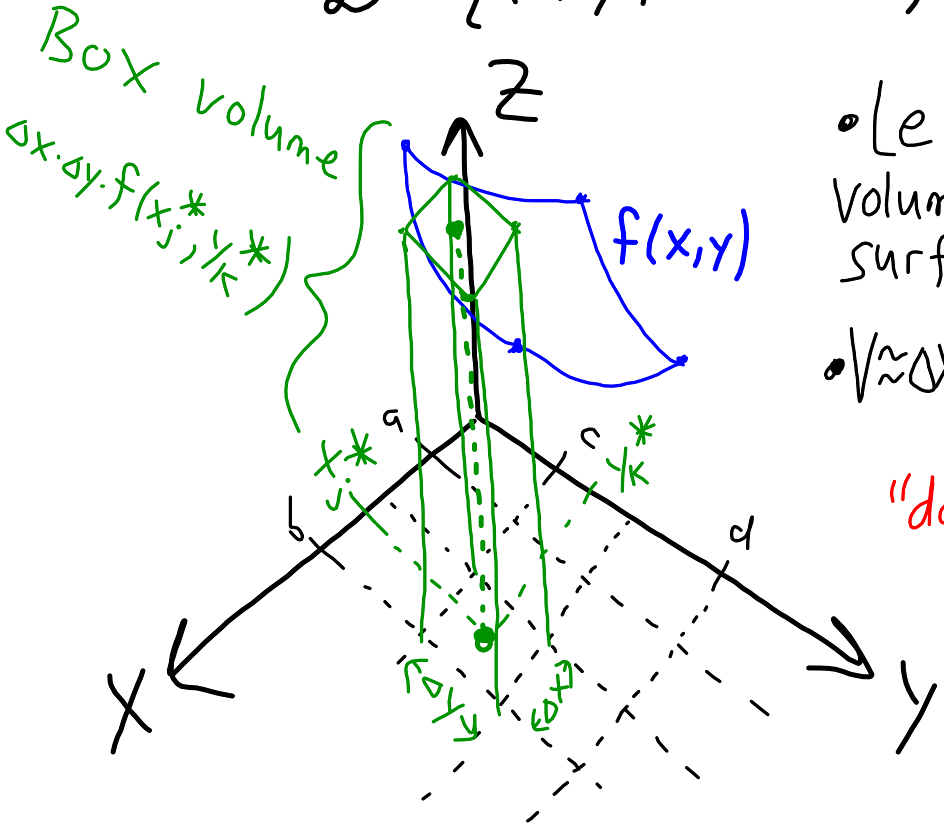
$$\int_a^b f(x)\,dx \approx \Delta x \sum_{j=1}^{3} f(x_j^*)$$

The approximation gets better with more rectangles, and in fact

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \underbrace{\Delta x \sum_{j=1}^{N} f(x_j^*)}_{\text{Riemann sum}}.$$

Now consider using boxes for the volume under a surface...

let $f(x,y)$ be defined on a domain
$$\mathcal{D}=\{(x,y)\mid a\le x\le b,\ c\le y\le d\}.$$



- let V be the volume under the surface on $\mathcal{D}$
- $V\approx\Delta x\cdot\Delta y\sum_{j=1}^{N}\sum_{k=1}^{N}f(x_j^*, y_k^*)$

"double Riemann sum"

Now think of using many boxes to get a better estimate... let the number of boxes tend to ∞:

area differential

domain of integration

$$V = \underbrace{\iint_{\mathcal{D}} f(x,y)\, dA}_{\text{double integral}} = \lim_{N\to\infty} \Delta x \cdot \Delta y \sum_{j=1}^{N} \sum_{k=1}^{N} f(x_j^*, y_k^*).$$

We also write

$$\iint_D f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

y-limits, y-integral

x-limits, x-integral

or

$$\iint_D f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

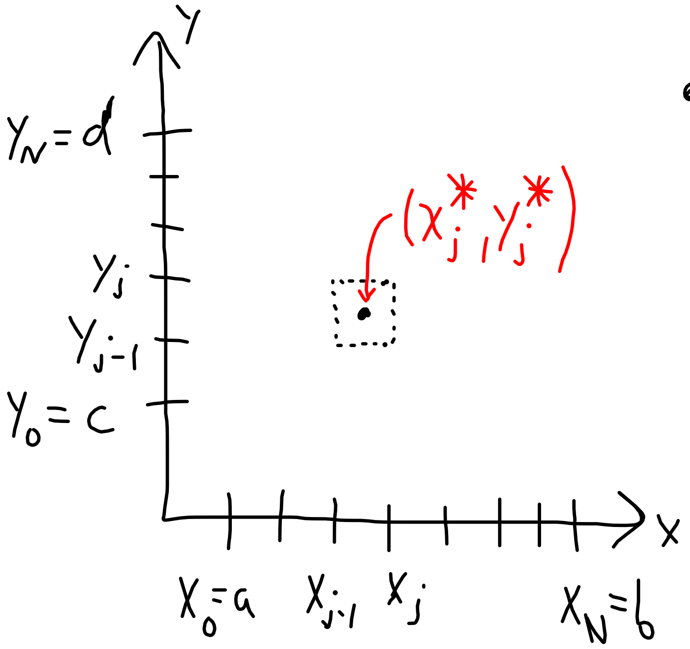
There are some caveats here, to be discussed in a few more slides.

Some comments:

(1) Note that we can estimate double integrals using double Riemann sums. While this is tedious by hand, computers do this type of thing routinely.

(2) The interpretation as volume is only for surfaces $f(x,y)>0$; otherwise we get regions with "negative" volume. Volume calculation is not the motivation anyway, just a way to start thinking about the Riemann sum.

Midpoint rule: Use the centers of rectangles as sample points in the double Riemann sum.



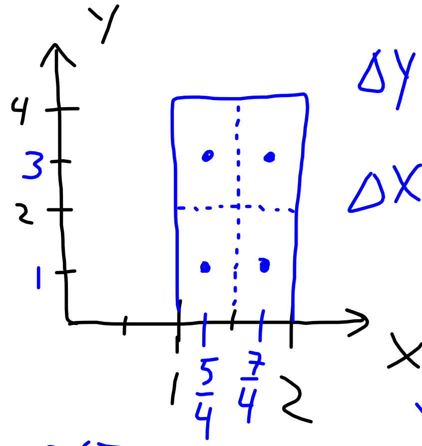
- Use $x_j^* = \dfrac{x_{j-1} + x_j}{2}$

  $y_j^* = \dfrac{y_{j-1} + y_j}{2}$

- Easy to "code" and leads to reasonably good integral estimates.

EX: Use a midpoint rule with $N=2$ to estimate

$$\int_1^2 \int_0^4 (x^2+y)\,dy\,dx.$$



$\Delta y = 2$

$\Delta x = 1/2$

$$\int_1^2 \int_0^4 (x^2+y)\,dy\,dx$$

$$\approx \frac{1}{2}\cdot 2\cdot\left(f\left(\tfrac{5}{4},1\right)+f\left(\tfrac{5}{4},3\right)+f\left(\tfrac{7}{4},1\right)+f\left(\tfrac{7}{4},3\right)\right)$$

$$= \frac{25}{16}+1+\frac{25}{16}+3+\frac{49}{16}+1+\frac{49}{16}+3$$

$$= 8+\frac{148}{16} = \frac{69}{4}.$$

Average value Recall the average value of $f(x)$ over $a \leq x \leq b$ is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx .$$

Similarly, for $f = f(x,y)$ we have average value over $D = \{(x,y) \mid a \leq x \leq b,\ c \leq y \leq d\}$ is

$$\bar{f} = \frac{1}{|D|}\int_a^b \int_c^d f(x,y)\,dy\,dx .$$

$|D| = (b-a)(d-c)$: size of domain D.

Calculation of integrals: iterated integrals.

We estimated $\int_1^2 \int_0^4 (x^2+y)\,dy\,dx$ ... what is the exact value?

$$\int_1^2 \underbrace{\left[\int_0^4 (x^2+y)\,dy\right]}_{\text{hold } x \text{ fixed, integrate over } y\ldots} dx$$

$$= \int_1^2 \left[x^2 y + \frac{1}{2}y^2\right]_0^4 dx = \int_1^2 (4x^2+8)\,dx$$

$$= \left[\frac{4}{3}x^3 + 8x\right]_1^2 = \frac{32}{3} + 16 - \frac{4}{3} - 8$$

$$= \frac{28}{3} + 8 = \frac{52}{3}$$

* Midpoint approximation was

$$\frac{69}{4} = 17.25\ldots \qquad \frac{52}{3} = 17.3333\ldots$$

* We calculate one integral before then another for each variable, called "iterated" integrals.

EX: Calculate $\int_0^3 \int_{-2}^1 xy\,dx\,dy$.

$$= \int_0^3 \left[\int_{-2}^1 xy\,dx \right] dy = \int_0^3 \left[\frac{1}{2}x^2 y \right]_{-2}^1 dy$$

$$= \int_0^3 \left(\frac{1}{2}y - 2y \right) dy = -\frac{3}{2} \int_0^3 y\,dy$$

$$= -\frac{3}{2} \cdot \frac{1}{2} y^2 \Big|_0^3$$

$$= \boxed{\frac{-27}{4}}.$$

Fubini's Theorem: If $f(x,y)$ is CTS on $\mathcal{D} = \{(x,y) \mid a \le x \le b, c \le y \le d\}$ then

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

*So you can switch the order of integration for convenience... very useful sometimes.

*Here f(x,y) can be discontinuous on a finite set of curves so long as it is also bounded and the iterated integrals exist.

EX: Find $\int_0^1 \int_0^1 xe^{xy}\,dx\,dy$.

You can try integrating over x first... integration by parts for example, but it gets ugly. So we just switch the order of integration:

$$= \int_0^1 \left[\int_0^1 xe^{xy}\,dy\right] dx = \int_0^1 \left[\int_0^1 \frac{\partial}{\partial y}\left(e^{xy}\right) dy\right] dx$$

$$= \int_0^1 \left[e^{xy}\right]_{y=0}^{y=1} dx = \int_0^1 \left(e^x - 1\right) dx = e^1 - e^0 - 1 = \boxed{e-2}.$$

EX: Find the volume above the xy-plane and below $f(x,y) = x^2 + \frac{1}{9}y^2 + 1$, over the domain $\mathcal{D} = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 1\}$.

$$V = \int_0^2 \int_0^1 x^2 + \frac{1}{9}y^2 + 1 \, dy\, dx = \int_0^2 \left[x^2 y + \frac{1}{27}y^3 + y \right]_0^1 dx$$

$$= \int_0^2 \left(x^2 + \frac{28}{27} \right) dx = \left(\frac{1}{3}x^3 + \frac{28}{27}x \right) \Big|_0^2$$

$$= \frac{8}{3} + \frac{56}{27} = \boxed{\frac{128}{27}}.$$

Practice!

#1 Find $\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy\, dx$.

$$= \int_1^4 \left[x\ln(y) + \frac{1}{x}\frac{1}{2}y^2 \right]_1^2 dx$$

$$= \int_1^4 x\ln(2) + \frac{3}{2x}\, dx = \left[\frac{1}{2}\ln(2)x^2 + \frac{3}{2}\ln(x) \right]_1^4$$

$$= \frac{15}{2}\ln(2) + \frac{3}{2}\ln(4)$$

$$= \frac{15}{2}\ln(2) + \frac{3}{2}\ln(2^2) = \frac{21}{2}\ln(2).$$

#2 Find $\int_0^1 \int_{-1}^1 \frac{xy^2}{x^2+1}\,dy\,dx$.

$$\int_0^1 \left[\frac{x}{3(x^2+1)} y^3\right]_{-1}^{1} dx = \int_0^1 \frac{2x}{3(x^2+1)}\,dx$$

$$= \int_0^1 \frac{1}{3}\frac{d}{dx}\ln(x^2+1)\,dx$$

$$= \frac{1}{3}\ln(x^2+1)\Big|_0^1 = \frac{1}{3}\ln(2).$$

$*$ Note... $\displaystyle\int_0^1 \frac{2x}{3(x^2+1)}\,dx$ $\Bigg\}$ Set $u = x^2+1$
$\Rightarrow du = 2x\,dx$

$$= \int_{u=1}^{u=2} \frac{1}{3u}\,du = \frac{1}{3}\ln(u)\Big|_1^2$$

$$= \frac{1}{3}\ln(2).$$

#3 $\int_0^1 \int_0^1 \frac{x}{1+xy}\,dx\,dy = ?$

Hint: $\frac{d}{du}\left(u\ln(u)-u\right)=\ln(u)$.

$$= \int_0^1\int_0^1 \frac{x}{1+xy}\,dy\,dx = \int_0^1 \ln(1+xy)\Big|_0^1\,dx$$

$$= \int_0^1 \ln(1+x)\,dx = \Big[(x+1)\ln(x+1)-(x+1)\Big]_0^1$$

$$= 2\ln(2)-1.$$