MATH 2110 - Review for final exam

* This review covers the main topics on the exam, but not necessarily all of the fine details.

* Be sure to study similar examples from past lectures, homework, quizzes and tests.

* The exam will have 10 questions. Roughly half the exam focuses on vector calculus.

Lines and planes

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

$\vec{v}$: direction vector

t: parameter (scalar)

$\vec{r}_0$: some point on the line

$$\vec{n} \cdot \vec{v} = 0 = \vec{n} \cdot \langle \underbrace{x - x_0, y - y_0, z - z_0} \rangle$$

$\vec{n}$: any normal to the plane

(x_0, y_0, z_0) : any point in the plane

We also have the scalar plane equation

$$ax+by+cz+d=0.$$

Recall $\langle a,b,c\rangle$ will be normal to the plane.

Tangent planes are found by deriving a normal vector at the point of tangency.

If $\vec{r}(u,v)$ describes a surface, then take $\vec{n}=\left(\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right)$.

"Natural" parameterization means

$z = f(x,y)$, so $\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$

$$\Rightarrow \vec{n} = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \left\langle -\frac{\partial f}{\partial x}, \frac{-\partial f}{\partial y}, 1 \right\rangle.$$

Also, if $F(x,y,z) = 0$ is a surface, then

$$\vec{n} = \nabla F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle.$$

__EX__: Find the line passing through $(-5,4,7)$ perpendicular to the lines

$$\vec{r}_1(t) = \langle t, 2t, 1-t \rangle$$

and $\vec{r}_2(t) = \langle 1, t, 4t-3 \rangle$.

Direction vectors are $\vec{v}_1 = \langle 1, 2, -1 \rangle$

$$\vec{v}_2 = \langle 0, 1, 4 \rangle$$

Direction vector: $\vec{v} = \vec{v}_1 \times \vec{v}_2 = \langle 9, -4, 1 \rangle$

Since $\langle 9,-4,1\rangle$ is perpendicular to both $\vec{r}_1$ & $\vec{r}_2$ and $(-5,4,7)$ is on the line we want,

$$\boxed{\begin{aligned}\vec{r}(t) &= \langle -5,4,7\rangle + t\langle 9,-4,1\rangle \\ &= \langle 9t-5,\ 4-4t,\ t+7\rangle.\end{aligned}}$$

<u>EX</u>: Find the tangent plane to the surface $f(x,y) = xy + x^2 + 1$ at $\underbrace{(2,-1)}$.

$f_x = y + 2x = -1 + 2\cdot 2 = 3$. $f(2,-1) = 3$

$f_y = x = 2$.

So $\underbrace{\langle -3, -2, 1 \rangle}_{\vec{n}} \cdot \langle x-2, y+1, z-3 \rangle = 0$,

or $\boxed{-3(x-2) - 2(y+1) + z - 3 = 0}$.

Derivatives

Recall the derivative of $f(x,y)$ in a direction $\vec{u}$ is $\boxed{D_{\vec{u}}f = \nabla f(x,y) \cdot \hat{u}}$

where $\hat{u} = \frac{1}{|\vec{u}|}\vec{u}$.

The maximum rate of increase of f at (x,y) is $|\nabla f|$, in direction ∇f.

EX: Let $f(x,y) = \sin(xy)$. Find the derivative in the direction $\langle -3, 1\rangle$ at the point $(\pi, 1)$. What is the maximum rate of increase at this point?

$$Df = \langle f_x, f_y \rangle = \langle y\cos(xy), x\cos(xy) \rangle$$

$$= \langle -1, -\pi \rangle,$$

and $\hat{u} = \frac{1}{\sqrt{10}}\langle -3, 1\rangle \Rightarrow D_{\vec{u}}f = \langle -1, -\pi\rangle \cdot \langle -3, 1\rangle \frac{1}{\sqrt{10}}$

so $D_{\hat{u}} f = \frac{1}{\sqrt{10}}(3-\pi)$.

Max. rate of increase is given by

$$|Df| = |\langle -1, -\pi \rangle| = \sqrt{1^2+\pi^2} = \sqrt{1+\pi^2}.$$

Chain Rule $f(x,y,z)$: let x,y,z all be functions of r,s,t.

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}.$$

$$f_s = f_x x_s + f_y y_s + f_z z_s.$$

$$f_t = f_x x_t + f_y y_t + f_z z_t.$$

EX: $f(x,y) = y\sqrt{x} + x^2$, $x = t\cos(s)$
$y = st$.

Find $\frac{\partial f}{\partial s}$ & $\frac{\partial f}{\partial t}$.

$$\frac{\partial f}{\partial s} = \left(\frac{y}{2\sqrt{x}} + 2x\right)\left(-t\sin(s)\right) + \left(\sqrt{x}\right)t.$$

$$\frac{\partial f}{\partial t} = \left(\frac{y}{2\sqrt{x}} + 2x\right)\left(\cos(s)\right) + \left(\sqrt{x}\right)s.$$

Implicit Differentiation

Given $F(x,y,z)=0$, if $z=z(x,y)$ locally then we can ask for $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$.

$$\frac{\partial}{\partial x}(F)=\frac{\partial}{\partial x}(0)=0$$

$$\Rightarrow \frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0 \Rightarrow \boxed{\frac{\partial z}{\partial x}=-\frac{\partial F/\partial x}{\partial F/\partial z}}.$$

Similarly, $\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y}=0 \Rightarrow \boxed{\frac{\partial z}{\partial y}=\frac{-\partial F/\partial y}{\partial F/\partial z}}.$

In practice, I prefer the following...

EX: Given $e^{xz} = (1+x)yz - 2$, assume $z = z(x,y)$ near $(0,1,3)$ and find $\frac{\partial z}{\partial y}$ at this point.

Differentiate through the relationship

$$xe^{xz}\frac{\partial z}{\partial y} = (1+x)z + (1+x)y\frac{\partial z}{\partial y}.$$

Plug in $(0,1,3)$...

$$0 = 3 + \frac{\partial z}{\partial y} \Rightarrow \boxed{\frac{\partial z}{\partial y} = -3}.$$

Lagrange multipliers

If you want to maximize or minimize a quantity

f(x,y), where the parameters (x,y) are constrained by some relationship g(x,y)=0, then one solves the system of equations below.

$$\nabla f = \lambda \nabla g \quad \begin{cases} \dfrac{\partial f}{\partial x} = \lambda \dfrac{\partial g}{\partial x} \\ \dfrac{\partial f}{\partial y} = \lambda \dfrac{\partial g}{\partial y} \end{cases}$$

$$g = 0$$

- Find all (x, y, λ) solutions
- Plug (x, y) into $f(x, y)$
- Take max./min. answer.

EX: A circular section S has radius r and angle θ (in radians):

The area is $A = \frac{1}{2}\theta r^2$.

The perimeter is $P = 2r + r\theta$.

Maximize the area if $\underbrace{P = \pi}$.

$g(r,\theta) = 2r + r\theta - \pi = 0$.

$f(r,\theta) = A = \frac{1}{2}\theta r^2 \ldots$

solve $\frac{\partial f}{\partial r} = \lambda \frac{\partial g}{\partial r}$, $\frac{\partial f}{\partial \theta} = \lambda \frac{\partial g}{\partial \theta}$, $g = 0 \ldots$

$r\theta = \lambda(2+\theta)$

$\frac{1}{2}r^2 = \lambda r \longrightarrow r^2 - 2\lambda r = \underbrace{r(r-2\lambda) = 0}$.

$2r + r\theta = \pi \quad (g=0)$ $\qquad r \neq 0$, so $r = 2\lambda$.

Plug $r = 2\lambda$ into first equation...

$2\lambda\theta = 2\lambda + \lambda\theta \Rightarrow \lambda\theta - 2\lambda = \lambda(\theta - 2) = 0$.

But $\lambda \neq 0$ or else $r = 2\lambda = 0$, so $\underline{\underline{\theta = 2}}$. Plug into constraint: $4r = \pi \Rightarrow r = \pi/4$

$\Rightarrow A = \frac{1}{2} \cdot 2 \cdot (\pi/4)^2 = \boxed{\frac{\pi^2}{16}}$.

Area, volume, double & triple integrals.

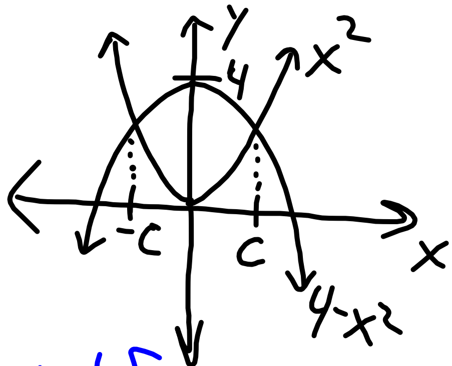
If $\mathcal{D} \subset \mathbb{R}^2$ then the area of $\mathcal{D}$ is

$$|\mathcal{D}| = \iint_{\mathcal{D}} 1 \, dA.$$

If $V \subset \mathbb{R}^3$, it has volume given by

$$|V| = \iiint_V 1 \, dV.$$

EX: Find the area bounded between $y=x^2$ & $y=4-x^2$.



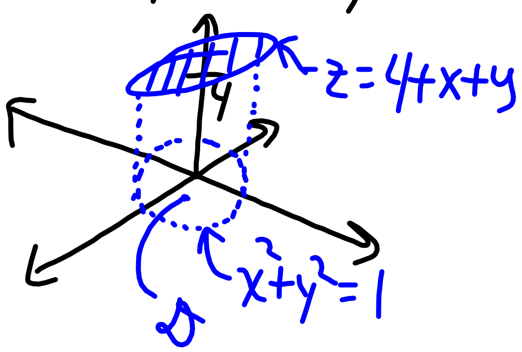
Find the intersection;

$4-x^2=x^2 \Rightarrow 4=2x^2$
$\Rightarrow 2=x^2 \Rightarrow x=\pm\sqrt{2}$.

$$\text{Area} = \int_{-\sqrt{2}}^{\sqrt{2}}\int_{x^2}^{4-x^2} dy\,dx = \int_{-\sqrt{2}}^{\sqrt{2}} 4-2x^2\,dx = \left[4x-\frac{2}{3}x^3\right]_{-\sqrt{2}}^{\sqrt{2}}$$

algebra... $= \boxed{\frac{16\sqrt{2}}{3}}$.

Ex: Find the volume above the region $x^2+y^2 \le 1$, $(z=0)$, and below $z=4+x+y$.



$$|V| = \iint_{\mathcal{D}} \int_0^{4+x+y} dz\, dA$$

Polar: $\displaystyle\int_0^{2\pi}\int_0^1\int_0^{4+r\cos\theta+r\sin\theta} r\, dz\, dr\, d\theta$

$$= \int_0^{2\pi}\int_0^1 4r + r^2(\cos\theta+\sin\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[2r^2 + \frac{1}{3} r^3 (\cos\theta + \sin\theta) \right]_{r=0}^{r=1} d\theta$$

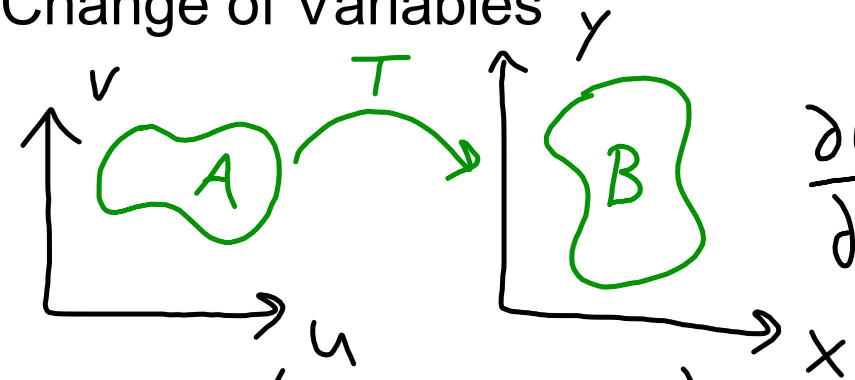
$$= \int_0^{2\pi} 2 + \frac{1}{3}(\cos\theta + \sin\theta)\, d\theta$$

$$= 4\pi + \frac{1}{3}\left[\sin\theta - \cos\theta\right]_0^{2\pi}$$

$$0$$

$$= \boxed{4\pi.}$$

Change of Variables



Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$T(u,v) = \big(x(u,v),\ y(u,v)\big).$$

$$\boxed{\iint_B f(x,y)\,dx\,dy = \iint_A f(u,v) \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\text{absolute value}} du\,dv.}$$

EX: Let $x = u(1+4v^2)$, $y = 2v$ map from $-1 \le u \le 1$, $-1 \le v \le 1$ to the set

$$B = \{(x,y) \mid -2 \le y \le 2,\ -1-y^2 \le x \le 1+y^2\}.$$

Find the area of B using a change of variables.

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1+4v^2 & 8uv \\ 0 & 2 \end{vmatrix} = 2+8v^2 \neq 0.$$

So $|B| = \iint_B 1 \, dx\,dy$

$$= \int_{-1}^{1}\int_{-1}^{1} 2+8v^2 \, du\,dv = 2\int_{-1}^{1} du \int_{-1}^{1} 1+4v^2 \, dv$$

$$= 4\left[v+\frac{4}{3}v^3\right]_{-1}^{1} = 4\left(2+\frac{8}{3}\right) = 4\cdot\frac{14}{3} = \boxed{\frac{56}{3}}.$$

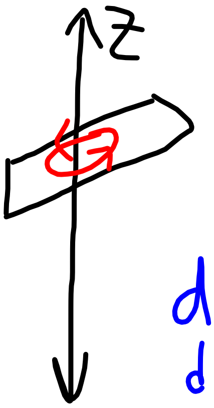
Line integrals

$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

$$\int_C \vec{F}\cdot d\vec{s} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$\vec{F} = \langle P, Q, R\rangle \Rightarrow \int_C \vec{F}\cdot d\vec{s} = \int_C P\,dx + Q\,dy + R\,dz.$$

EX: Find $\int_C 2dx + ydy - dz$, if C is the intersection of the cylinder $x^2+y^2=4$ and the plane $z=1+x+y$, oriented counter-clockwise when viewed from "above" (+z-axis).



$x = 2\cos(t)$
$y = 2\sin(t)$
$z = 1+x+y = 1+2(\cos(t)+\sin(t))$

$dx = -2\sin(t)\,dt$
$dy = 2\cos(t)\,dt$

$dz = 2(\cos(t)-\sin(t))\,dt$

$$\int_C 2\,dx + y\,dy - dz$$

$$= \int_0^{2\pi} \left[-4\underbrace{\sin(t)}_{0} + \underbrace{4\cos(t)\sin(t)}_{\substack{u=\sin(t) \\ du=\cos(t)dt}} - 2\cancel{\cos(t)}^{\,0} + 2\cancel{\sin(t)} \right] dt$$

$$= 4\int_{u=0}^{u=\sin(2\pi)=0} u\,du = \boxed{0}.$$

Conservative vector fields

* Check if $\vec{F} = \nabla f$ by verifying $\nabla \times \vec{F} = 0$ (3D) or $\vec{F} = \langle P, Q \rangle \Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.
* Know the procedure to find f.
* "Fundamental Theorem"

$$\vec{F} = \nabla f \Rightarrow \int_C \vec{F} \cdot d\vec{s} = f(B) - f(A).$$

B: end-point on C

A: starting point on C.

EX: Let $\vec{F} = \langle ye^{yx} - 1 + \frac{3}{2}y^2\sqrt{x},\ y + xe^{yx} + 2yx^{3/2} \rangle$.

Is $\vec{F}$ conservative? If so, find the potential function $f(x,y)$.

$P = ye^{yx} - 1 + \frac{3}{2}y^2\sqrt{x}$, $Q = y + xe^{yx} + 2yx^{3/2}$

$$\left.\begin{aligned} Q_x &= e^{yx} + xye^{yx} + 3y\sqrt{x} \\ P_y &= e^{yx} + xye^{yx} + 3y\sqrt{x} \end{aligned}\right\} \begin{aligned} &Q_x = P_y \\ &\Rightarrow \text{conservative.} \end{aligned}$$

$$\int P\,dx = \int ye^{yx} - 1 + \tfrac{3}{2}y^2\sqrt{x}\,dx$$
$$= e^{yx} - x + y^2x^{3/2} + g(y) = f(x,y).$$

Set y-derivative = $\vec{j}$-component, Q

$$Q = y + xe^{yx} + 2yx^{3/2} = \frac{\partial}{\partial y}\left(e^{yx} - x + y^2x^{3/2} + g(y)\right)$$
$$= xe^{yx} + 2yx^{3/2} + g'(y)$$

So $y = g'(y) \Rightarrow \frac{1}{2}y^2 = g(y)$. Plug in above to get

$$\boxed{f(x,y) = e^{yx} - x + y^2x^{3/2} + \tfrac{1}{2}y^2}.$$

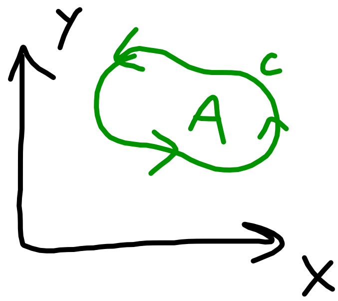
EX: Given $f(x,y,z) = x - y^2 + zy$, find $\int_C \nabla f \cdot d\vec{r}$, where $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ for $0 \leq t \leq \pi/2$ parameterizes C.

Note $\vec{r}(0) = \langle 1, 0, 0 \rangle$ & $\vec{r}(\frac{\pi}{2}) = \langle 0, 1, \pi/2 \rangle$.

Thus (Fundamental Theorem)

$$\int_C \nabla f \cdot d\vec{r} = f(0,1,\tfrac{\pi}{2}) - f(1,0,0) = -1 + \tfrac{\pi}{2} - 1 = \boxed{\frac{\pi}{2} - 1}.$$

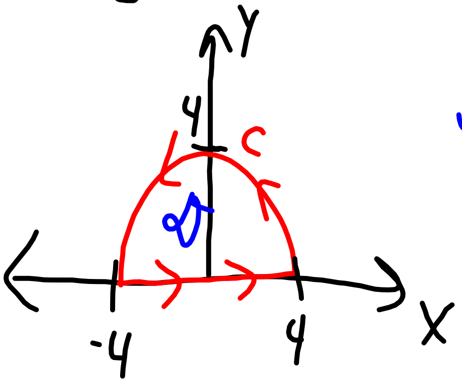
Green's Theorem



$$\int_C P\,dx + Q\,dy = \int_C \vec{F}\cdot d\vec{s}$$

$$= \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

EX: Let C be the boundary of the region $\{(x,y) \mid x^2+y^2 \le 16,\ y \geq 0\}$, with positive orientation. Find $\int_C y\,dx + xy\,dy$, using Green's Theorem.



$$\int_C P\,dx + Q\,dy = \iint_D Q_x - P_y\,dA$$

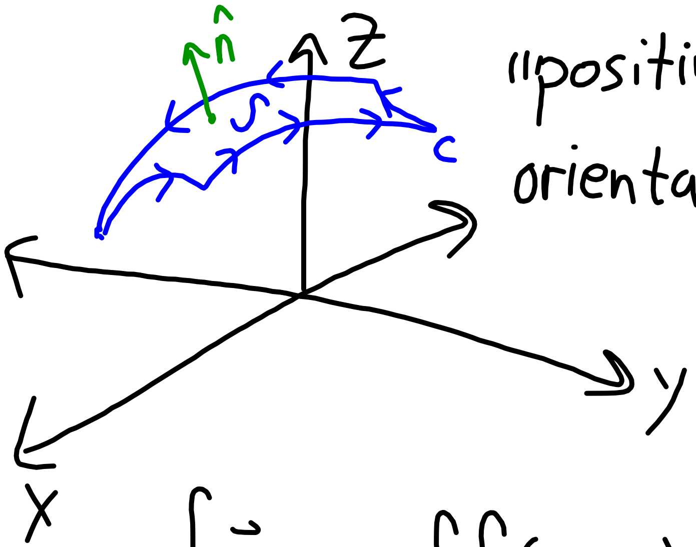
$$= \iint_D y - 1\,dA = \int_0^{\pi}\int_0^4 (r\sin\theta - 1)\,r\,dr\,d\theta$$

$$= \int_0^{\pi} \int_0^{4} r^2 \sin\theta - r \, dr \, d\theta = \int_0^{\pi} \frac{1}{3} \cdot 4^3 \sin\theta - \frac{1}{2} \cdot 4^2 \, d\theta$$

$$= \frac{64}{3} \left(-\cos\theta\right)\Big|_0^{\pi} - 8\pi$$

$$= \boxed{\frac{128}{3} - 8\pi}.$$

Stoke's Theorem



"positive" orientation

$$\int_{C=\partial S} \vec{F}\cdot d\vec{s} = \iint_S (\nabla \times \vec{F})\cdot \hat{n}\, dS .$$

Ex: Let S be the section of the surface $z=1+y^2$ that lies over $D=\{(x,y) \mid -1\leq x\leq 1, -1\leq y\leq 1\}$.
Given $\vec{F}=\langle y, yz^2, z^4y^2\rangle$, find $\int \vec{F}\cdot d\vec{s}$ if ∂S has positive orientation with ∂S respect to the upward-pointing normal on S.

Line integral requires 4 pieces, so let's apply Stoke's.

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ y & yz^2 & y^2z^4\end{vmatrix}=\langle 2yz^4-2yz,0,-1\rangle.$$

Recall $\vec{n}=\langle -z_x,-z_y,1\rangle=\langle 0,-2y,1\rangle=\vec{r}_x\times\vec{r}_y$

if we choose $\vec{r}(x,y)=\langle x,y,1+y^2\rangle$.

We get

$$\int_C \vec{F}\cdot d\vec{s} = \int_{-1}^{1}\int_{-1}^{1} (\nabla\times\vec{F})\cdot(\vec{r}_x \times \vec{r}_y)\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} \langle 2yz^4 - 2yz, 0, -1\rangle \cdot \langle 0, -2y, 1\rangle\,dy\,dx$$

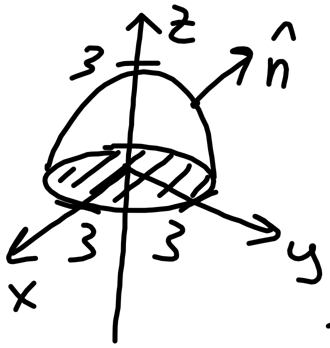
$$= \int_{-1}^{1}\int_{-1}^{1} -1\,dy\,dx = \boxed{-4}.$$

Divergence Theorem

$$\iint_{\partial V} \vec{F}\cdot\hat{n}\, dS = \iiint_{V} \nabla\cdot\vec{F}\, dV$$

where $\hat{n}$ points "outward".

Ex: Let S be the surface of the region $V=\{(x,y,z) \mid x^2+y^2+z^2 \leq 9,\ z \geq 0\}$, $\hat{n}$ outward, $\vec{F}=\langle x,y,z^2\rangle$. Find $\iint_{\partial V} \vec{F}\cdot\hat{n}\, dS$.



Surface integral requires two pieces and is tedious. Use Divergence Theorem:

$$\iint_{\partial V} \vec{F}\cdot\hat{n}\,dS = \iiint_V \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2)\,dV$$

$$= \iiint_V 2(1+z)\,dV = 2\underbrace{|V|}_{\frac{4}{3}\pi\cdot 3^3} + 2\underbrace{\iiint_V z\,dV}_{\text{use spherical coordinates}}.$$

$$= 36\pi + 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^3 \rho\cos\phi\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 36\pi + 2\int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi\cos\phi\, d\phi \int_0^3 \rho^3\, d\rho$$

$$= 36\pi + 4\pi \left[\tfrac{1}{2}\sin^2\phi\right]_0^{\pi/2} \left[\tfrac{1}{4}\rho^4\right]_0^3$$

$$= 36\pi + 4\pi \tfrac{1}{2}\cdot\tfrac{1}{4}\cdot 3^4 = 36\pi + \frac{81\pi}{2}$$

$$= \boxed{\frac{153\pi}{2}}.$$