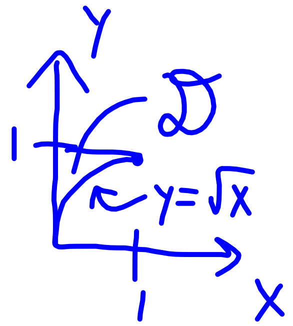
# MATH 2110Q Practice Exam 3

1 Find the volume of the region bounded by the following surfaces: $x=0$, $y=\sqrt{x}$, $y=1$, $z=0$, $z=\sqrt{x+y^2}$.




Surface $z=\sqrt{x+y^2}$ lies above $\mathcal{D}$.

$$V=\int_0^1\int_{\sqrt{x}}^1 \sqrt{x+y^2}\,dy\,dx$$ ? switch order...

$$V = \int_0^1 \int_0^{y^2} \sqrt{x+y^2}\, dx\, dy = \int_0^1 \int_0^{y^2} \frac{\partial}{\partial x}\left(\frac{2}{3}(x+y^2)^{3/2}\right) dx\, dy$$

$$= \int_0^1 \frac{2}{3}(x+y^2)^{3/2}\Big|_{x=0}^{x=y^2} dy$$

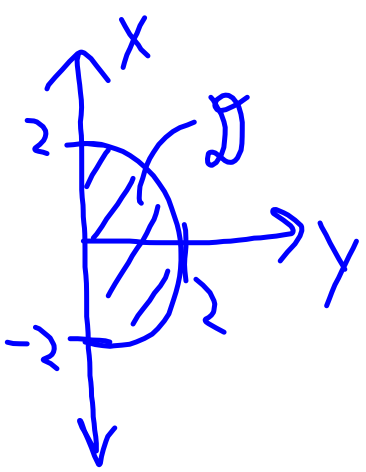
$$= \frac{2}{3}\int_0^1 2^{3/2}(y^2)^{3/2} - (y^2)^{3/2}\, dy = \frac{2}{3}\int_0^1 (2^{3/2}-1)y^3\, dy$$

$$= \frac{2}{3}(2^{3/2}-1)\frac{1}{4}y^4\Big|_0^1 = \boxed{\frac{1}{6}(2^{3/2}-1)}.$$

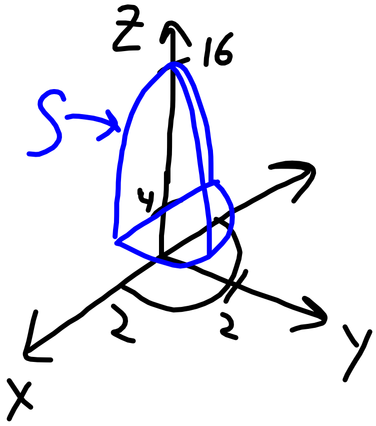
#2 Find the surface area of the portion of the surface $z=16-3x^2-3y^2$ above $z=4$ such that $y \geq 0$.

$z=4=16-3x^2-3y^2 \Rightarrow 3(x^2+y^2)=12$

$\Rightarrow x^2+y^2=4$




Surface lies above this region




$z = 16 - 3x^2 - 3y^2$

$z_x = -6x \quad z_y = -6y$

$1 + z_x^2 + z_y^2 = 1 + 36(x^2 + y^2)$

POLAR : $= 1 + 36r^2$

$\Rightarrow S = \int_0^{\pi} \int_0^2 \left(\sqrt{1+36r^2}\right) r\,dr\,d\theta = \left(\int_0^{\pi} d\theta\right)\left(\int_0^2 r\sqrt{1+36r^2}\,dr\right)$

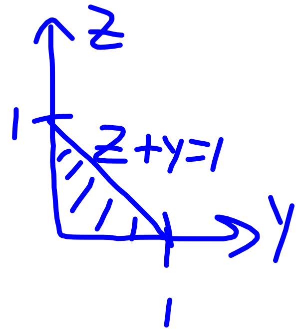
$u = 1 + 36r^2$

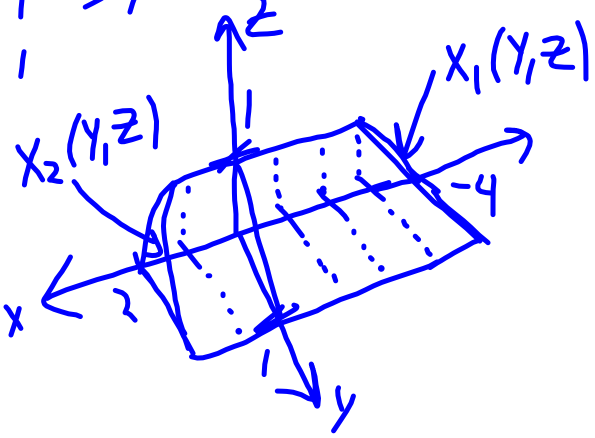
$du = 72r\,dr$

$= \frac{\pi}{72} \int_1^{145} u^{1/2}\,du = \frac{\pi}{72} \frac{2}{3} u^{3/2} \Big|_1^{145} = \frac{\pi}{108}\left(145^{3/2} - 1\right).$

#3 Find the volume of the region bounded by $\underbrace{y=0, z=0, z+y=1}$, $\underbrace{x=2-z^2,\ x+4=z^2}$.

Think $x_1(y,z)$ & $x_2(y,z)$ bounding surfaces.



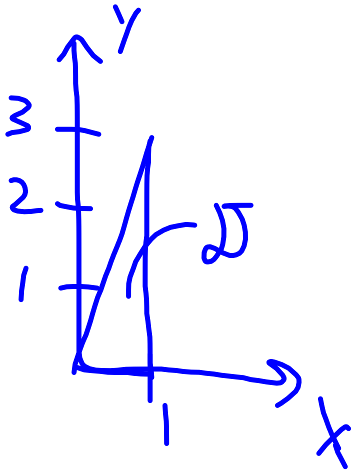
$$V = \int_0^1 \int_0^{1-y} \int_{z^2-4}^{2-z^2} dx\,dz\,dy = \int_0^1 \int_0^{1-y} 2-z^2-z^2+4\,dz\,dy$$

$$= \int_0^1 \int_0^{1-y} 6-2z^2\,dz\,dy = \int_0^1 \left[6z - \frac{2}{3}z^3\right]_0^{1-y} dy$$

$$= \int_0^1 6(1-y) - \frac{2}{3}(1-y)^3\,dy = \left[-\frac{6}{2}(1-y)^2 + \frac{2}{3}\frac{1}{4}(1-y)^4\right]_0^1$$

$$= 0 + 3 + 0 - \frac{1}{6} = \frac{18-1}{6} = \boxed{\frac{17}{6}}.$$

#4) Find the center of mass of a solid on the region $\mathcal{D}$ bounded by $y=0$, $y=3x$, $x=1$ if $\rho(x,y)=x+2y$.




$$m=\int_0^1\int_0^{3x} x+2y\,dy\,dx=\int_0^1\left[xy+y^2\right]_0^{3x}dx$$

$$=\int_0^1 3x^2+9x^2\,dx=12\int_0^1 x^2\,dx=\frac{12}{3}=4.$$

m

Need $M_x, M_y$ ...

$$M_x = \int_0^1 \int_0^{3x} y(x+2y)\,dy\,dx = \int_0^1 \int_0^{3x} yx+2y^2\,dy\,dx$$

$$= \int_0^1 \left[\frac{x}{2}y^2+\frac{2}{3}y^3\right]_{y=0}^{y=3x} dx = \int_0^1 \frac{9}{2}x^3+18x^3\,dx$$

$$= \frac{45}{2}\int_0^1 x^3\,dx = \frac{45}{8}.$$

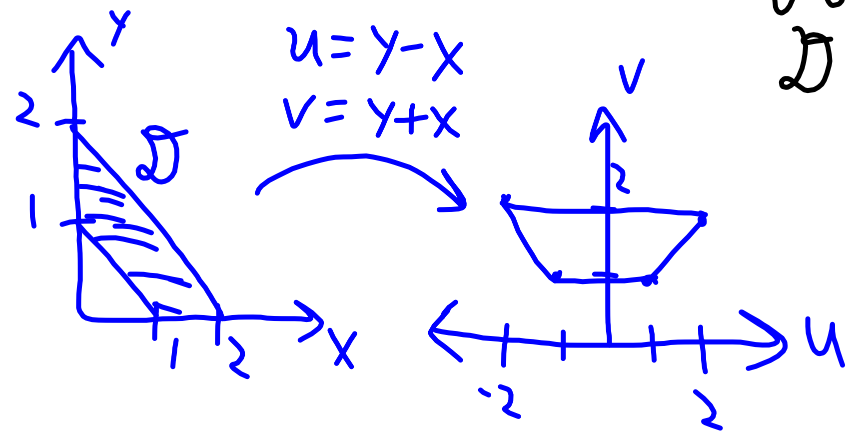
$$M_y = \int_0^1 \int_0^{3x} (x^2+2xy)\,dy\,dx = \int_0^1 \left[x^2y+xy^2\right]_0^{3x} dx$$

$$= \int_0^1 3x^3 + 9x^3\,dx = 12\int_0^1 x^3\,dx = \frac{12}{4} = 3.$$

Thus, $m = 4$, $M_x = \frac{45}{8}$, $M_y = 3$

$$\Rightarrow \boxed{\begin{aligned} \bar{x} &= \frac{M_y}{m} = \frac{3}{4} \\ \bar{y} &= \frac{M_x}{m} = \frac{45}{32}. \end{aligned}}$$

#5 Let $D$ be the trapezoidal region with corners $(1,0)$, $(2,0)$, $(0,1)$ & $(0,2)$. Use a transformation to find $\iint_D \cos\left(\frac{y-x}{y+x}\right) dA$.




When equations are linear, just plug in the 4 corners & connect the dots.

$$\left.\begin{aligned} x &= \tfrac{1}{2}(v-u) \\ y &= \tfrac{1}{2}(u+v) \end{aligned}\right\} \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

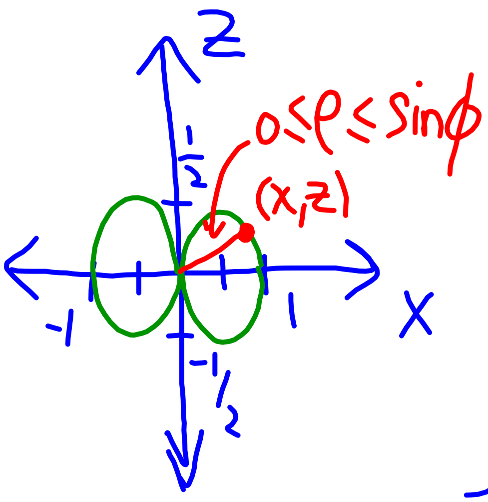
Thus $\displaystyle\iint_D \cos\left(\frac{y-x}{y+x}\right) dA = \int_1^2 \int_{-v}^{v} \cos(u/v) \left|-\frac{1}{2}\right| du\,dv$

$$= \frac{1}{2}\int_1^2 \int_{-v}^{v} \frac{\partial}{\partial u}\left(v \sin(u/v)\right) du\,dv = \frac{1}{2}\int_1 \left[v \sin\left(\frac{u}{v}\right)\right]_{-v}^{v} dv$$

$$= \frac{1}{2}\int_1^2 v\left(\sin(1) - \sin(-1)\right) dv = \sin(1)\int_1^2 v\,dv = \underline{\frac{3\sin(1)}{2}}.$$

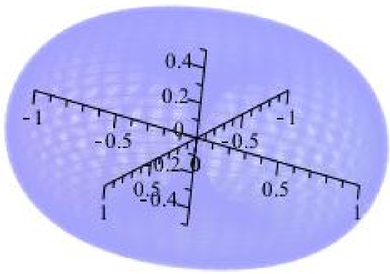
#6 Find the volume of the torus $\rho = \sin\phi$ (spherical).

Look at a cross-section...




$\theta$-independent so the green circles may be rotated around the z-axis...

This is a little easier to see on a computer screen; check out the slides online.



$$\Rightarrow V = \frac{2\pi}{3}\cdot\frac{1}{4}\cdot\int_0^{\pi} 1-2\cos 2\phi+\cos^2 2\phi \, d\phi$$

$$= \frac{\pi}{6}\left[\pi - \cancel{\sin 2\phi}\Big|_0^{\pi} + \int_0^{\pi} \cos^2 2\phi \, d\phi\right] \quad \left\} \begin{array}{l} \cos^2 2\phi \\ = \dfrac{1+\cos 4\phi}{4} \end{array}\right.$$

$$= \frac{\pi}{6}\left(\pi + \frac{1}{4}\int_0^{\pi} 1+\cos 4\phi \, d\phi\right)$$

$$= \frac{\pi}{6}\left(\pi + \frac{\pi}{4} + \frac{1}{16}\cancel{\sin 4\phi}\Big|_0^{\pi}\right) = \frac{\pi^2}{6}\cdot\frac{5}{4} = \boxed{\frac{5\pi^2}{24}}.$$

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right) \int_0^{\pi} \frac{1}{3}\rho^3 \sin\phi \Big|_0^{\sin\phi} d\phi$$

$$= \frac{2\pi}{3}\int_0^{\pi} \sin^4\phi \, d\phi$$

$$(\sin^2\phi)^2 = \left(\frac{1-\cos 2\phi}{2}\right)^2$$

$$= \frac{1}{4}\left(1 - 2\cos 2\phi + \cos^2 2\phi\right)$$