Review for Exam 2

Math 2110Q

1. Find the limit or else show that it does not exist:

(a) $$\lim_{(x,y)\to(1,2)} \frac{x+y-2}{x+y-3}$$ ← get "1" ← looks like "0"

This "blows up" somehow. Take $x=1$

$$\Rightarrow \frac{1+y-2}{1+y-3} = \frac{y-1}{y-2}$$ positive as $y\to 2^+$, negative as $y \to 2^-$

So this goes to $\pm\infty$ depending on the way we approach $\Rightarrow$ D.N.E.

(b) $$\lim_{(x,y)\to(-1,1)} \frac{x^2+y^3}{x-y+\sin(x+y)}.$$

So $(-1,1)$ is in the domain here...

- cite continuity
- plug in point to evaluate

$$\lim_{(x,y)\to(-1,1)} \frac{x^2+y^3}{x-y+\sin(x+y)} = \frac{1+1}{-1-1+0} = \frac{2}{-2} = \boxed{-1}$$

(c) $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x)}{\sin(y)\cos(y)}$ } "$\frac{0}{0}$"

- Try $x=0, y\neq 0 \Rightarrow \dfrac{\sin(0)}{\sin(y)\cos(y)} = 0$
- Look for a way to get a different result; $y=x \Rightarrow \dfrac{\cancel{\sin}(x)}{\cancel{\sin}(x)\cos(x)}$

$= \dfrac{1}{\cos(x)} \xrightarrow[x\to 0]{} 1 \neq 0$ (DNE)

2. Given that $xy+xz+e^z=2$
find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ at $(1,1,0)$.

$$\frac{\partial}{\partial x}\left(xy+xz+e^z\right)=\frac{\partial}{\partial x}(2)=0$$

$$\Rightarrow y+z+x\frac{\partial z}{\partial x}+e^z\frac{\partial z}{\partial x}=0$$

(plug in $(1,1,0)$...)

$$\Rightarrow 1+0+\frac{\partial z}{\partial x}+\underbrace{e^0}_{1}\frac{\partial z}{\partial x}=0=1+2\frac{\partial z}{\partial x}$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{1}{2}.$$

$$\frac{\partial}{\partial y}\left(xy + xz + e^{z}\right) = x + x\frac{\partial z}{\partial y} + e^{z}\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow 1 + \frac{\partial z}{\partial y} + \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial y} = -\frac{1}{2}.$$

3. Let $F(x,y,z)=e^x(2y+4z)$, with $x=t+s$, $y=2r-3t$, $z=r-t+s$. Use the Chain Rule to find $\frac{\partial F}{\partial t}$ with $(r,s,t)=(1,0,1)$.

$F_t = F_x x_t + F_y y_t + F_z z_t$

$$\left.\begin{array}{ll} F_x = e^x(2y+4z) & x_t = 1 \\ F_y = 2e^x & y_t = -3 \\ F_z = 4e^x & z_t = -1 \end{array}\right\} \text{still need } (x,y,z)$$

$$x = 1+0 = 1$$
$$y = 2-3 = -1$$
$$z = 1-1+0 = 0$$

$$\Longrightarrow F_x = e(2(-1)+4(0))$$
$$= -2e$$
$$F_y = 2e, \quad F_z = 4e$$

Thus, $F_t = -2e + 2e(-3) + 4e(-1)$
$$= (-2-6-4)e = \underline{-12e}.$$

4. Let $f(x,y)=4xy+x^2-2y^2$

(a) Find the derivative of f in the direction $\langle 4,3\rangle$ at $(-1,0)$.

$$|\langle 4,3\rangle| = \sqrt{16+9} = \sqrt{25} = 5 \Rightarrow \hat{u} = \frac{1}{5}\langle 4,3\rangle.$$

$$\nabla f = \langle 4y+2x,\ 4x-4y\rangle \Big\}$$ plug in $x=-1$, $y=0$

$$\nabla f(-1,0) = \langle -2,-4\rangle$$

$$\Rightarrow D_{\hat{u}} f = \frac{1}{5}\langle -2,-4\rangle \cdot \langle 4,3\rangle = \frac{-8-12}{5} = \boxed{-4}.$$

(b) Find $D_{\hat{u}} f$ at $(2,-1)$ if $\hat{u}$ points in the direction of fastest increase.

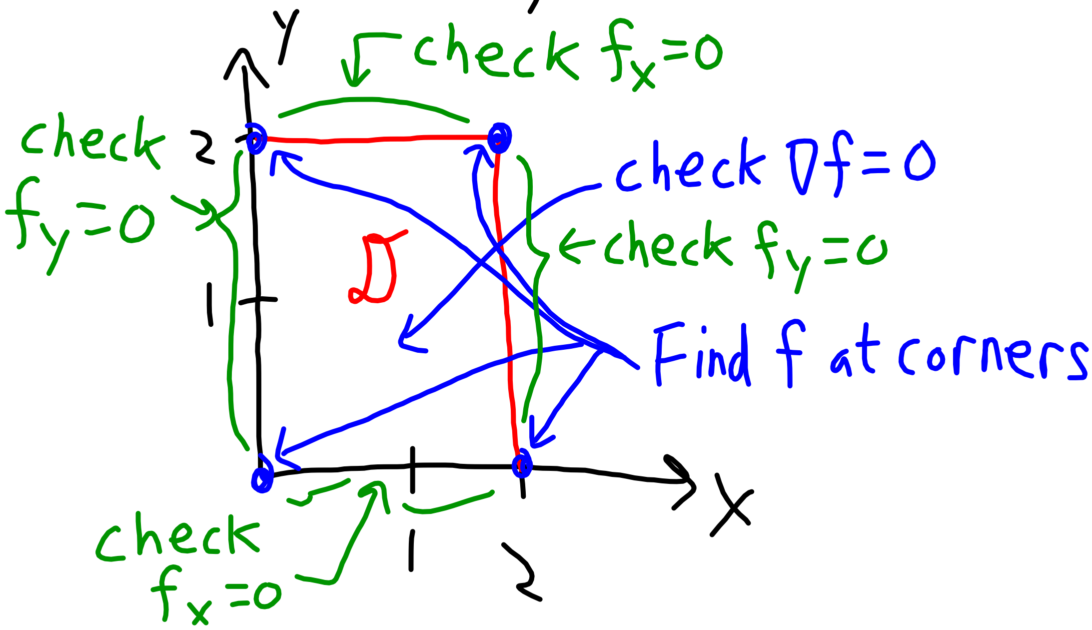
Then $\hat{u} = \dfrac{\nabla f}{|\nabla f|}$ and $D_{\hat{u}} f = |\nabla f|$

$$\nabla f\Big|_{(2,-1)} = \langle 4y+2x,\, 4x-4y \rangle\Big|_{(2,-1)}$$

$$= \langle -4+4,\, 8+4 \rangle = \langle 0, 12 \rangle$$

$$\Rightarrow |\nabla f| = D_{\hat{u}} f = 12.$$

5. Given $f(x,y)=e^{-x^2+2x-2y^2+4y-3}$ find the absolute max/min values for f on $0 \le x \le 2$, $0 \le y \le 2$.



$$f_x = 0 = (-2x+2)\underbrace{e^{-x^2+2x-2y^2+4y-3}}_{\text{never} = 0}$$

$$\Rightarrow x = 1$$

Candidate

$$f_y = 0 = (-4y+4)e^{(\ldots)} \Rightarrow y = 1$$

So check $f(1,1) = e^{-1+2-2+4-3} = e^0 = 1.$

Check corners of $\mathcal{D}$...

$f(0,0) = e^{-3}$ $\qquad f(2,0) = e^{-4+4-3} = e^{-3}$

$f(0,2) = e^{-8+8-3} = e^{-3}$ $\qquad f(2,2) = e^{-3}$

So far, max $=1$, min $= e^{-3}$. Now we check $f_x = 0$ along $y=0$, $y=2$...

$f_x = 0 \Rightarrow x = 1$ (already solved this)

So now check f at $(1,0)$ & $(1,2)$

$f(1,0) = e^{-1+2-3} = e^{-2}$

$f(1,2) = e^{-1+2-8+8-3} = e^{-2}$

Similarly, $f_y = 0 \Rightarrow y = 1$, so check $f(0,1)$, $f(2,1)$...

$f(0,1) = e^{-2+4-3} = e^{-1}$ & $f(2,1) = e^{-1}$

MAX $=1$, MIN $= e^{-3}$.

6. The volume of a cylinder in terms of radius r and height h is given by $V=\pi r^2 h$. The surface area is

$$S=2\pi r^2+2\pi rh.$$

Maximize the volume if $S=4$, fixed.

Solve $\begin{cases} \nabla V = \lambda \nabla S \\ S = 4 \end{cases}$

for r, h, λ.

1) $V_r = 2\pi r h = \lambda S_r = \lambda(4\pi r + 2\pi h)$

2) $V_h = \pi r^2 = \lambda S_h = \lambda(2\pi r)$

3) $S = 2\pi r^2 + 2\pi r h = 4$

$(r \neq 0)$

$(2) \Rightarrow r^2 = 2\lambda r \Rightarrow r(r - 2\lambda) = 0 \Rightarrow r = 2\lambda$

Insert in (1)... $4\pi\lambda h = \lambda(8\pi\lambda + 2\pi h)$

$\Rightarrow 2\lambda h = 4\lambda^2 + \lambda h$

$\Rightarrow 0 = 4\lambda^2 - \lambda h = \lambda(4\lambda - h)$.

$\Rightarrow \cancel{\lambda = 0}$ or $4\lambda = h$

Put $r=2\lambda$, $h=4\lambda$ into $S=4$:

$$2\pi(2\lambda)^2+2\pi(2\lambda)(4\lambda)=4$$

$$8\pi\lambda^2+16\pi\lambda^2=4$$

$$24\pi\lambda^2=4$$

positive root

$$\lambda^2=\frac{4}{24\pi}=\frac{1}{6\pi}\Rightarrow \overbrace{\lambda=\frac{1}{\sqrt{6\pi}}}$$

$$\Rightarrow r=\frac{2}{\sqrt{6\pi}},\ h=\frac{4}{\sqrt{6\pi}}$$

$$\Rightarrow V=\pi r^2h=\pi\frac{4}{6\pi}\cdot\frac{4}{\sqrt{6\pi}}=\boxed{\text{MAX } V\ \ \frac{8}{3\sqrt{6\pi}}}$$