Math 1060Q Lecture 9

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October 1, 2014

Today we discuss polynomial functions, which are very important

- What is a polynomial?
- Behavior for large values of x
- Roots of polynomials
- Factored form of a polynomial
- Graphing polynomials

$f(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$ is called a "polynomial"

Examples:

$$f(x) = 2x - 3$$
$$f(x) = x^2 - x + 5$$
$$f(x) = x^3 - 6x^2 + x - 1$$
$$f(x) = -2x^4 + x^3 + x^2 + x + 1$$

- The term without any power of x is the constant term.
- The number multiplying the highest power of x is the lead coefficient.
- The highest power of x in the polynomial is the order or degree of the polynomial.

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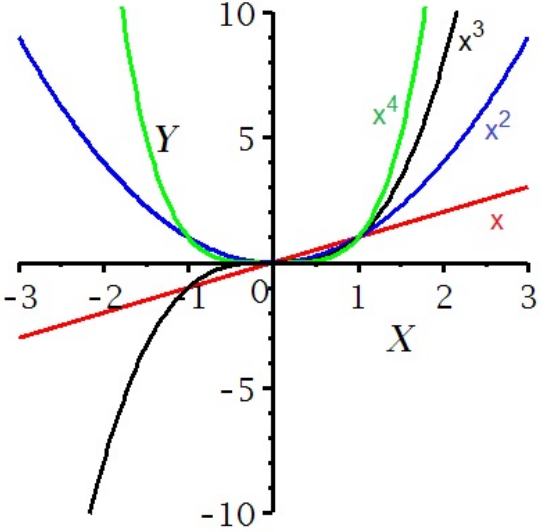
As $|x|$ gets big, higher powers of x grow faster than lower powers. Also note the difference between even and odd powers as $x \to -\infty$.



You should remember the behavior of polynomials as $x \to \pm\infty$, summarized as follows.

Large $|x|$ behavior of polynomials with a positive lead coefficient:

	$x \to \infty$	$x \to -\infty$
even order	$f \to \infty$	$f \to \infty$
odd order	$f \to \infty$	$f \to -\infty$

Large $|x|$ behavior of polynomials with a negative lead coefficient:

	$x \to \infty$	$x \to -\infty$
even order	$f \to -\infty$	$f \to -\infty$
odd order	$f \to -\infty$	$f \to \infty$

Example L9.1: Determine the behavior of each polynomial for large $|x|$.

Polynomial	$x \to \infty$	$x \to -\infty$
$f(x) = x^4 + 2x^3 - x + 1$	$f \to \infty$	$f \to \infty$
$f(x) = x^3 + 6x^2 + 10$	$f \to \infty$	$f \to -\infty$
$f(x) = -2x^3 + 6x^2 + 10$	$f \to -\infty$	$f \to \infty$
$f(x) = -4x^4 + 2x^3 - x + 1$	$f \to -\infty$	$f \to -\infty$

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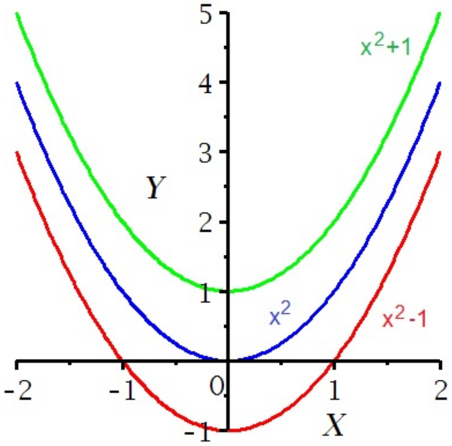
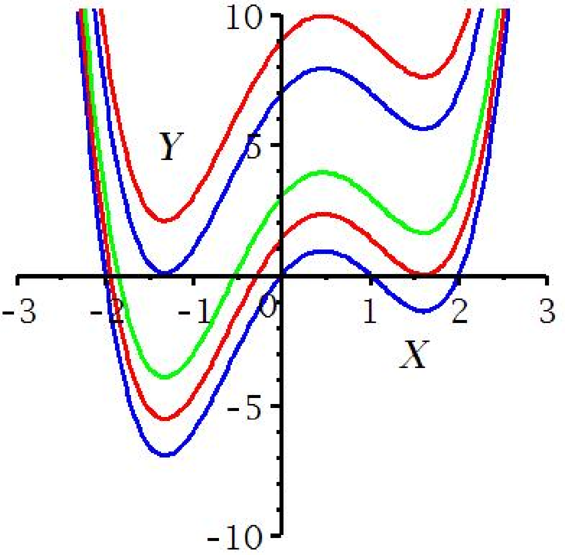
Recall: the roots of $f(x)$ are the values of x for which $f(x) = 0$.

Generally, a polynomial of order n may have between 0 and n roots.

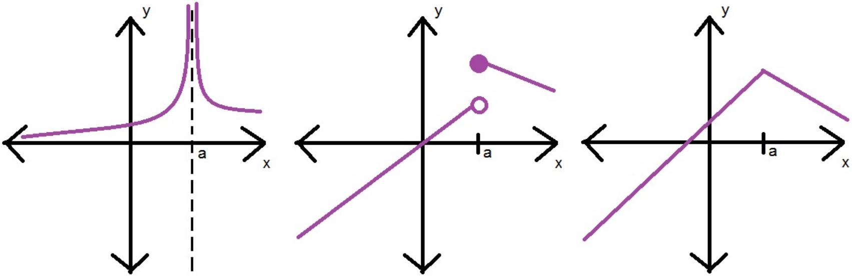


In this graph, polynomials of order 4 are shown having any number of roots between 0 and 4.

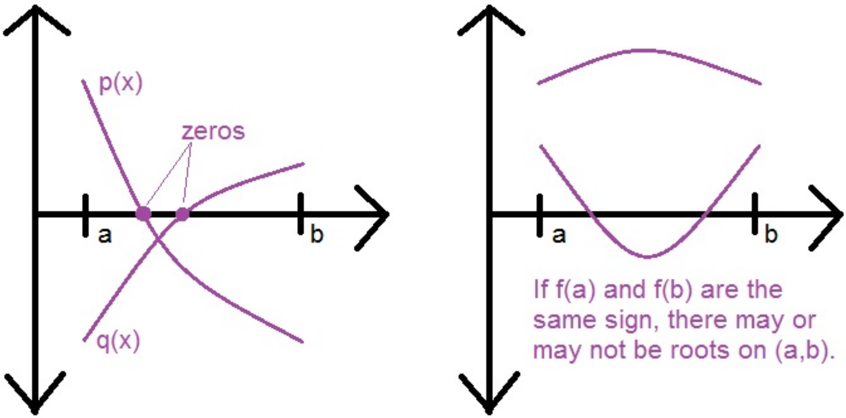


Polynomials are continuous, meaning their graphs don't suddenly jump or break

The left and center figures are NOT continuous at $x = a$; the right graph is continuous at $x = a$.



As a result, we can find zeros of polynomials by looking for where they change SIGN.



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First, recall factoring for quadratics.

We know that we may write

$$x^2 - 5x + 6 = (x - 3)(x - 2).$$

The right-hand side shows the factored form for this polynomial. Furthermore, we see from this that $x = 2, 3$ are roots. If a third-order polynomial $p(x)$ has roots $x = -1, 1, 2$ we may write

$$p(x) = c(x + 1)(x - 1)(x - 2).$$

Here c is not known without further information, such as a point that the graphs of $p(x)$ goes through. For example, if $p(0) = 4$, then

$$p(0) = 4 = c(0 + 1)(0 - 1)(0 - 2) = 2c \Rightarrow c = 2.$$

Thus $p(x) = 2(x + 1)(x - 1)(x - 2)$, which is in factored form.

If a polynomial of order n has exactly n different, real roots, we may write it as a product of linear factors.

Example L9.2: Write $p(x) = -2x^4 + 4x^3 + 2x^2 - 4x$ in factored form.

Solution: Note that $x = 0$, $x = -1$, $x = 1$ and $x = 2$ are all roots of the polynomial. Thus we must have

$$p(x) = cx(x-1)(x+1)(x-2),$$

for some value c. Now note that the fourth-order term must be cx^4, which must match $-2x^4$. Therefore, $c = -2$. The factored form is

$$p(x) = -2x(x-1)(x+1)(x-2).$$

In case the polynomial has less that n roots, the factored form is not all linear factors.

Recall, for example, that $p(x) = x^2 + 1$ has no roots, so it cannot be factored. Now consider making a cubic polynomial by multiplying by a linear factor, e.g.

$$q(x) = (x + 3)(x^2 + 1).$$

This is third-order but (by construction) only has ONE root; $x = -3$. It is in factored form. We will discuss more details about factoring at a later time.

We sometimes think of having the same root more than once; this is called multiplicity.

For example, $y = (x-1)^2$ has one root: $x = 1$. Also, this is already in factored form, but we think of the root occuring *twice*;

$$y = (x-1)(x-1)$$

and say the root $x = 1$ has multiplicity two. For example, consider

$$p(x) = (x-1)(x+1)^2(x-3)^4.$$

We say that the multiplicities of the roots are as follows:

Root	Multiplicity
$x = 1$	1
$x = -1$	2
$x = 3$	4

Examples...

Example L9.3: Find a polynomial $p(x)$ of order 5 such that the only real roots are $x = 0$ and $x = 4$.
Solution: Use $(x - 0) = x$ and $(x - 4)$ as linear factors. Then we have

$$p(x) = x(x - 4)q(x),$$

where $q(x)$ is any polynomial of order 3 that could only have $x = 0$ or $x = 4$ as roots, or could have no roots. For example, let $q(x) = x^3$, so $p(x) = x^4(x - 4)$.

Example L9.4: Find a polynomial $p(x)$ of order 3 with roots $x = -2$, $x = -1$ and $x = 5$.
Solution: $p(x) = (x + 2)(x + 1)(x - 5)$.

Example L9.5: Find a polynomial $p(x)$ of order 9 that has root $x = 6$ with multiplicity 5 and $x = -1$ with multiplicity 3.
Solution: $p(x) = x(x + 1)^3(x - 6)^5$.

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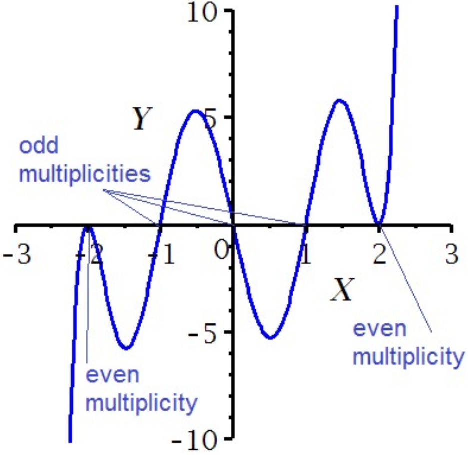
Multiplicities help to understand polynomial graphs.

Graphs cross the horizontal axis where the root has ODD multiplicity and do not cross (just touch) where the root has EVEN multiplicity.



You should be able to roughly sketch polynomial graphs now.

Example L9.6: Sketch the graph of $y = x(x-1)^2(x+2)^3$.
Solution: You need the behavior for $x \to \pm\infty$ and the root multiplicity. Since the polynomial is even order and you can tell the lead coefficient would be 1 (positive), $y \to \infty$ as $x \to \pm\infty$. Also, the graph will cross the x-axis at $x = 0$ and $x = -2$ only. At $x = 1$ it touches the x-axis.

You should be able to roughly sketch polynomial graphs now.

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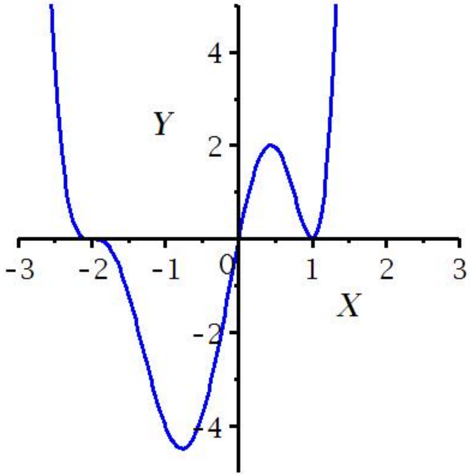

Practice!

Problem L9.1: Find $f(x)$ if it is a polynomial of order 8 with roots and multiplicities as follows, and such that $f(0) = 6$.

Root	Multiplicity
$x = 1$	3
$x = -1$	2
$x = 2$	3

Problem L9.2: Sketch the graph of $p(x) = (x + 1)(x - 1)^2$.