Math 1060Q Lecture 8

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Today's topic: forming compositions of functions

- What is a composition of functions?
- Domain and range for a composition of functions
- Some examples of graphs

A composition is formed by taking the output of one function and using as the input of another.

Given $f(x)$ and $g(x)$, we denote the composition $f(g(x))$ as $(f \circ g)(x)$.

- The output of g is used as the input of f.
- For example, if $g(1) = -2$, and $f(-2) = 4$, then

$$(f \circ g)(1) = f(g(1)) = f(-2) = 4.$$

Example L8.1: Let $f(x) = 3/x$ and $g(x) = x^2 + 1$. Find both $f \circ g$ and $g \circ f$.

Solution: Replace the x in $f(x)$ with $x^2 + 1$:

$$(f \circ g)(x) = f(x^2 + 1) = \frac{3}{x^2 + 1}.$$

It often helps to use parentheses:

$$(g \circ f)(x) = g(3/x) = \left(\frac{3}{x}\right)^2 + 1 = \frac{9}{x^2} + 1.$$

We have already seen some compositions

Example L8.2: Write the function $h(x) = |x+3|$ as a composition of functions.
Solution: Note that the function $g(x) = x + 3$ is inside the fence posts, so if $f(x) = |x|$, we have $f \circ g = h$.

Example L8.3: Write the function $h(x) = \sqrt{2x-3}$ as a composition of functions.
Solution: So write $g(x) = 2x - 3$ and take $f(x) = \sqrt{x}$. It follows that $f \circ g = h$.

Now we can also look at more complicated compositions:
Example L8.4: Let $f(x) = -x^2 + 3$ and $g(x) = |x|$. Find $g \circ f$.
Solution: $(g \circ f)(x) = g(-x^2 + 3) = |-x^2 + 3|$.

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Compositions can have "smaller" domains and ranges than the component functions f and g

Example L8.5: Given $f(x) = \sqrt{x}$ and $g(x) = 1 - 5x^2$, find the domain and range of $f \circ g$.
Solution: First, note that the composition is $(f \circ g)(x) = \sqrt{1 - 5x^2}$. Thus, since the argument of $f(x)$ needs to be non-negative, we must have

$$1 - 5x^2 \geq 0 \Rightarrow x^2 \leq \frac{1}{5} \Rightarrow -\sqrt{\frac{1}{5}} \leq x \leq \sqrt{\frac{1}{5}}.$$

Thus, $\mathcal{D} = [-\sqrt{1/5}, \sqrt{1/5}]$. The smallest value $f \circ g$ can have is 0. The largest is the square root of the largest value for the parabola $1 - 5x^2$ for $-\sqrt{1/5} \leq x \leq \sqrt{1/5}$. This occurs at the vertex of the parabola; $(x = 0, y = 1)$. Thus, $\mathcal{R} = [0, 1]$.

Another example...

Example L8.6: Given $f(x) = \sqrt{x}$ and $g(x) = |1 - 5x^2|$, find the domain and range for both $f \circ g$ and $g \circ f$.

Solution: We note $(f \circ g)(x) = \sqrt{|1 - 5x^2|}$. We need the argument for $f(x)$ to be non-negative, but since the range of g is non-negative it works out and the domain is $\mathbb{R}$. Furthermore, the range of g is $[0, \infty)$, which is the full domain of $f(x)$, thus the range of $f \circ g$ is also $[0, \infty)$.

Now note that $(g \circ f)(x) = |1 - 5(\sqrt{x})^2|$. We are restricted to a domain of $[0, \infty)$ since $\sqrt{x}$ requires non-negative inputs. Therefore, we could simplify: $(g \circ f)(x) = |1 - 5x|$. The range of this will be $[0, \infty)$ due to the absolute value.

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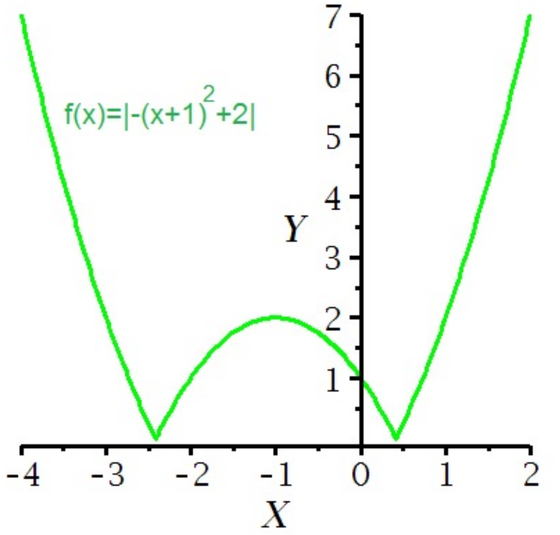
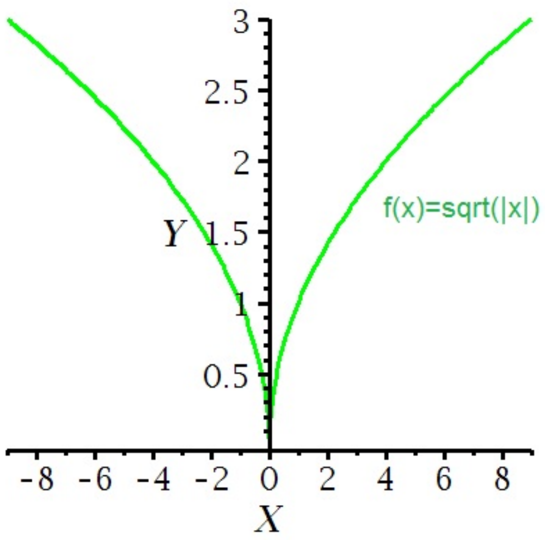
Note that $|x|$ makes things non-negative; on a graph it takes all y-values and flips them above the x-axis



One could also look at $\sqrt{|x|}$, which has domain $\mathbb{R}$ now.

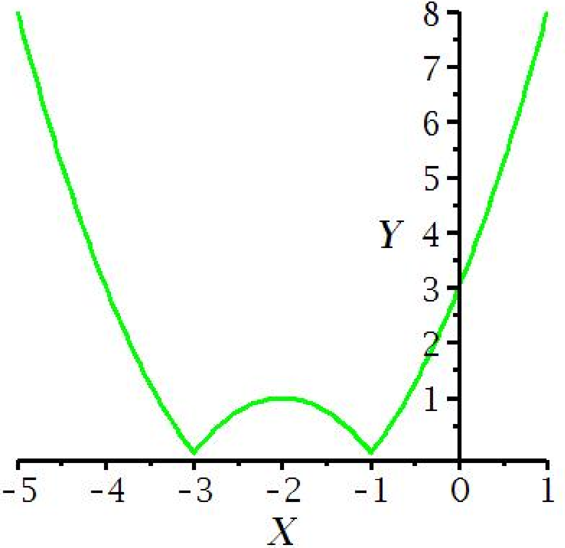


Think about the graph to figure out domain and range.

Example L8.7: Find the domain and range of $f \circ g$ if $f(x) = 1/x$ and $g(x) = |(x+2)^2 - 1|$.
Solution: Let us plot $f \circ g$. First, we plot $g(x)$, which is found by plotting the parabola $y = (x+2)^2 - 1$ and then flipping any negative y-values up above the x-axis.

Think about the graph to figure out domain and range.

Example L8.7: Find the domain and range of $f \circ g$ if $f(x) = 1/x$ and $g(x) = |(x+2)^2 - 1|$.



Think about the graph to figure out domain and range.

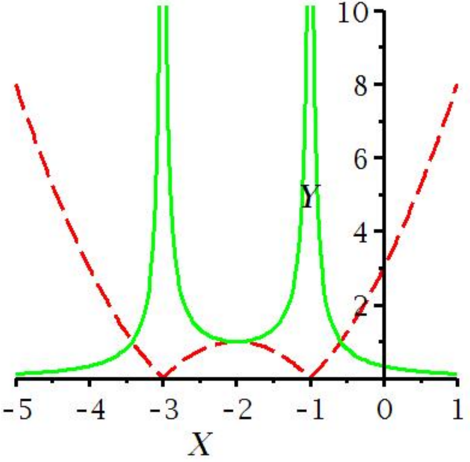
Example L8.7: Find the domain and range of $f \circ g$ if $f(x) = 1/x$ and $g(x) = |(x+2)^2 - 1|$.
Now we note that $f \circ g$ is the reciprocal of g, so we apply our guidelines to plot the reciprocal:

- Wherever $g \to \pm\infty$, we have $f \circ g \to 0$.
- Note $f \circ g = g$ whenever $g = 1$.
- $f \circ g$ and g always have the same sign (positive).
- $f \circ g$ is very big when g is very small and vice-versa.
- $f \circ g$ is undefined where $g = 0$; at these points we get vertical asymptotes.

Think about the graph to figure out domain and range.

Example L8.7: Find the domain and range of $f \circ g$ if $f(x) = 1/x$ and $g(x) = |(x+2)^2 - 1|$.

The domain is $(-\infty, -3) \cup (-3, -1) \cup (-1, \infty)$. The range is $(0, \infty)$.



Practice!

Problem L8.1: Evaluate $(f \circ g)(1)$ if $g(x) = x\sqrt{3}$ and $f(x) = \sqrt{1 + x^2}$.

Problem L8.2: Find the domain of $f \circ g$ if $f(x) = \sqrt{x - 2}$ and $g(x) = 2x^2 - 4x + 2$.

Problem L8.3: Find two functions $f(x)$ and $g(x)$ such that $(f \circ g)(x) = \sqrt{x + 12}$.

Problem L8.4: Let $f(x) = x^2 + 1$. What is $f \circ f$? What is $f \circ f \circ f$?