Math 1060Q Lecture 5

Jeffrey Connors

University of Connecticut

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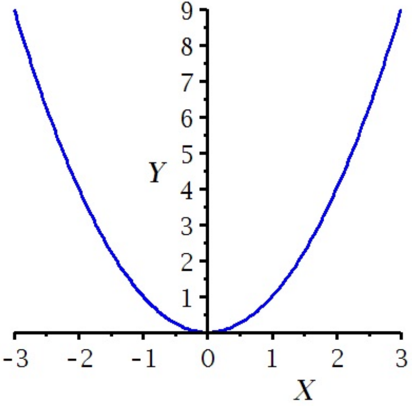
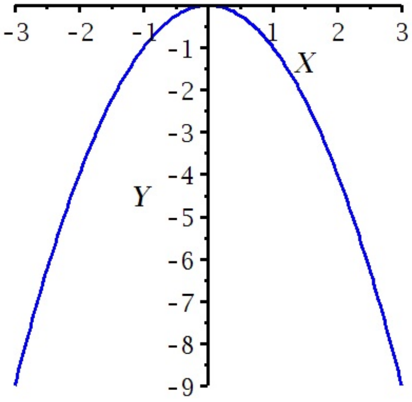
Today we discuss “quadratic” equations

1. What is a quadratic equation?
2. Vertical and horizontal shifts
3. Vertical stretching of graphs
4. Standard form of a quadratic equation
5. The quadratic formula

Quadratic equations look like $y = ax^2 + bx + c$.

Two simple examples: $y = x^2$ (left) and $y = -x^2$ (right).





Before discussing the general case, let us build up our intuition.

1. What is a quadratic equation?
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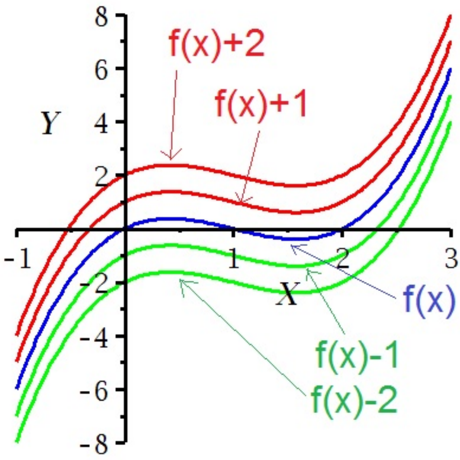
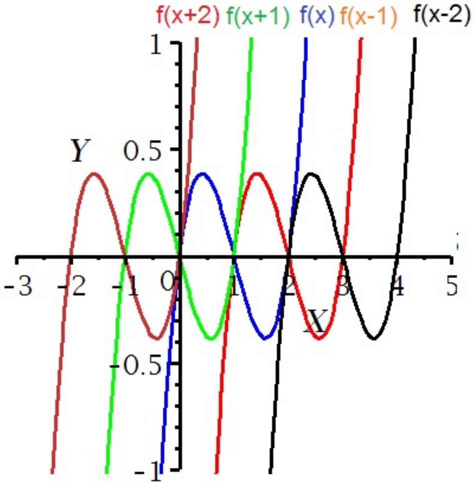
Consider a general graph of $f(x)$. Adding or subtracting from f creates a vertical shift.

- $g(x) = f(x) + a$ moves the graph of f up by $a > 0$ units.
- $g(x) = f(x) - a$ moves the graph of f down by $a > 0$ units.



Adding or subtracting from the argument x creates a horizontal shift.

- $g(x) = f(x - a)$ moves the graph of f right by $a > 0$ units.
- $g(x) = f(x + a)$ moves the graph of f left by $a > 0$ units.



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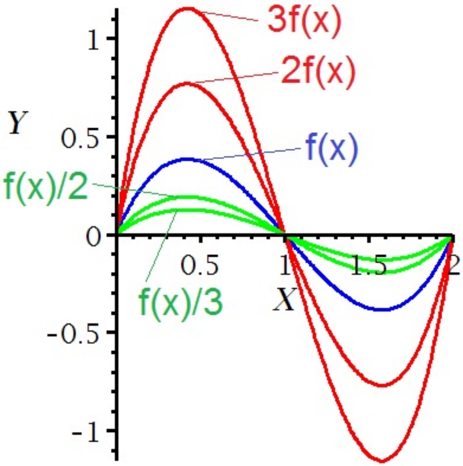
If we multiply the whole function $f(x)$ by a factor $a > 0$, it "stretches" or "shrinks" the graph

- If $a > 1$ then $|a\,f(x)| > |f(x)|$; the y-values get bigger.
- If $1 > a > 0$ then $|a\,f(x)| < |f(x)|$; the y-values get smaller.



Summary of graph operations

- $f(x) + a$: moves graph up by a units
- $f(x) - a$: moves graph down by a units
- $f(x - a)$: moves graph to the right by a units
- $f(x + a)$: moves graph to the left by a units
- $af(x)$: stretch (or shrink) vertically by factor of a

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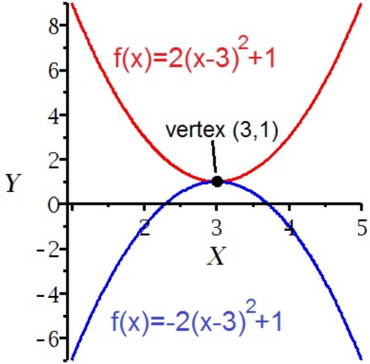
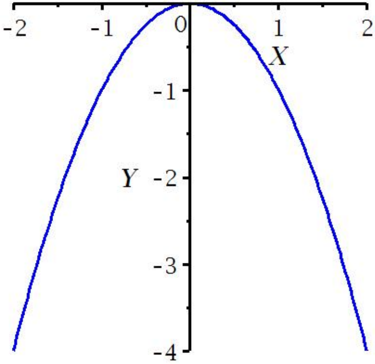
The standard form of a quadratic equation is more informative than $y = ax^2 + bx + c$.

- The standard form is $y = a(x - x_0)^2 + y_0$.
- Here (x_0, y_0) is the vertex or base of the parabola.
- If $a > 0$ the parabola "opens up"; if $a < 0$ the parabola "opens down".

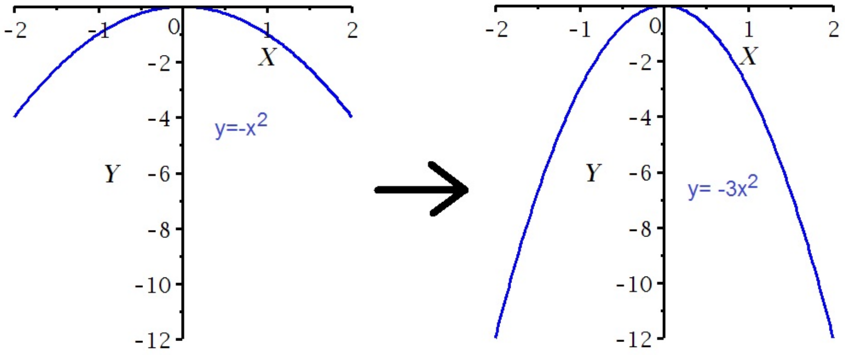


It is easy to understand graphs of quadratics from the standard form.

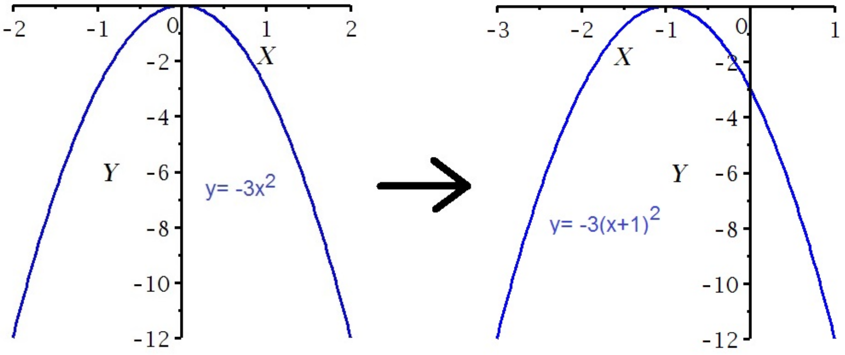
Example L5.1: Graph $f(x) = -3(x+1)^2 - 4$.
Solution: We will do this the "long" way for illustration. Start with $y = -x^2$, because the parabola we want opens down:



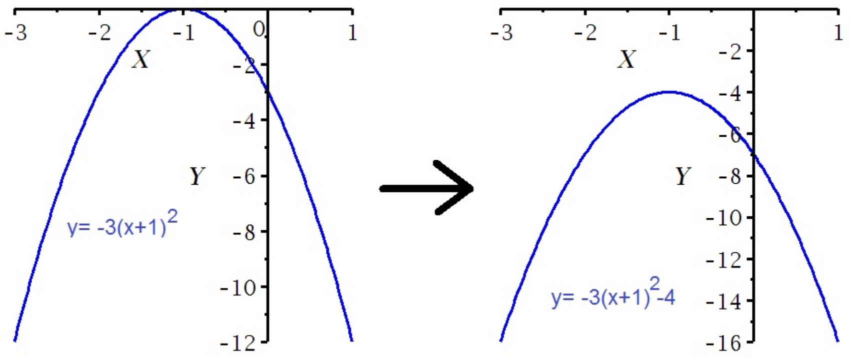
Solution to L5.1: Next, multiply (stretch vertically) by 3.



Solution to L5.1: Now shift left 1 unit by replacing x with $x + 1$.



Solution to L5.1: Last step: subtract 4 from the function to shift downwards.



Note that we can complete the square to get a quadratic into standard form.

We need a slight extension of the procedure: start with $y = ax^2 + bx + c$.

1. Factor a out of the first two terms: $y = a(x^2 + \frac{b}{a}x) + c$.
2. Inside the parentheses: add and subtract $(b/(2a))^2$:

$$y = a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c$$

3. Replace the first three terms inside the parentheses with $(x + b/2a)^2$:

$$y = a\left(\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c$$

Note that we can complete the square to get a quadratic into standard form.

4. Distribute the a through across the subtraction:

$$y = a\left(x + \frac{b}{2a}\right)^2 - a\left(\frac{b}{2a}\right)^2 + c$$
$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

Example L5.2: Find the standard form of the quadratic $f(x) = 2x^2 - 4x + 7$.
Solution: Follow the procedure...

$$\begin{aligned} 2x^2 - 4x + 7 &= 2\left(x^2 - 2x\right) + 7 \\ &= 2\left(x^2 - 2x + (-2/2)^2 - (-2/2)^2\right) + 7 \\ &= 2\left((x-1)^2 - 1\right) + 7 = 2(x-1)^2 - 2 + 7 \\ &= 2(x-1)^2 + 5. \end{aligned}$$

Example L5.3: Find the standard form of the quadratic $f(x) = -3x^2 - 24x - 41$.

Solution:

$$\begin{aligned} -3x^2 - 24x - 41 &= -3\left(x^2 + 8x\right) - 41 \\ &= -3\left(x^2 + 8x + 4^2 - 4^2\right) - 41 \\ &= -3\left((x+4)^2 - 16\right) - 41 \\ &= -3(x+4)^2 + 48 - 41 = -3(x+4)^2 + 7. \end{aligned}$$

What is the vertex of the parabola? It is $(-4, 7)$.

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What are the x-intercepts for a parabola? The quadratic formula answers this precisely!

Let us have a quadratic in the form $y = ax^2 + bx + c$. Then there is a formula for the x intercepts:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Note that the "$\pm$" sign means both add and subtract (possibly two answers).

- Call $D = b^2 - 4ac$ the discriminant.
- There are three scenarios to remember here: $D > 0$, $D < 0$ and $D = 0$.
  1. $D > 0$ means two distinct x-intercepts.
  2. $D = 0$ means exactly one x-intercept.
  3. $D < 0$ means no x-intercepts.

“Roots” of $f(x)$ are the x-values for which $f(x) = 0$.

The roots of $f(x)$ are where $y = f(x) = 0$, i.e. the x-intercepts of $f(x)$. Therefore, if $f(x)$ is a *quadratic*, then the roots are given by the quadratic formula.

Example L5.4: Find the roots of $f(x) = 2x^2 - 12x + 18$.
Solution: In this case we have $a = 2$, $b = -12$ and $c = 18$. The discriminant is $D = (-12)^2 - 4(2)(18) = 144 - 144 = 0$. Thus there is one root: $x = -b/2a = 12/4 = 3$.

Examples...

Example L5.5: Find the roots of $f(x) = x^2 + 2x + 2$.
Solution: The discriminant is $D = 2^2 - 4(1)(2) = 4 - 8 = -4 < 0$, therefore there are no roots.

Example L5.6: Find the roots of $f(x) = -3x^2 + 3x + 6$.
Solution: Here $a = -3$, $b = 3$ and $c = 6$ so that $D = 3^2 - 4(-3)(6) = 9 + 72 = 81$. Hence

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-3 \pm \sqrt{81}}{-6} = \frac{-3 \pm 9}{-6}.$$

The roots are $x_1 = 2$ and $x_2 = -1$.

Practice!

Problem L5.1: Sketch the parabola for $f(x) = (x+1)^2 - 2$.

Problem L5.2: Put the following into standard form:
$y = 6x^2 - 12x + 7$.

Problem L5.3: Find the roots of the quadratic: $y = 2x^2 - x - 1$.

Problem L5.4: Find the roots of the quadratic: $y = 4x^2 - 4x + 3$.

Problem L5.5: Find the roots of the quadratic: $y = 4x^2 - 4x + 1$.