Math 1060Q Lecture 22

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Models of exponential growth and decay.

- General form of exponential growth and decay models.
- Example 1: Compound interest
- Example 2: Population growth
- Example 3: Radioactive decay

Exponential models have been found to represent many growth and decay behaviors quite well.

- Let $Q(t)$ mean the amount of a quantity "Q" at time t.
- Often the rate of increase/decrease of some Q is observed to be proportional to amount of Q present (the value of Q). In this case, the growth/decay behavior is represented well by

$$Q(t) = Q_0 e^{kt}.$$

- Here $Q_0 = Q(t = 0)$, the "initial amount" of Q.
- $k > 0 \Rightarrow$ exponential growth.
- $k < 0 \Rightarrow$ exponential decay.

Let us explore some examples.

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Consider first compounding interest only periodically.

Consider first an investment of $\$100,000$ that accrues 2% interest, compounded annually. After a year, the investment is worth

$$\$100,000 + 0.02 \cdot \$100,000 = 1.02 \cdot \$100,000 = \$102,000.$$

In two year, the investment is worth

$$\begin{aligned}\$102,000 + 0.02 \cdot \$102,000 &= 1.02 \cdot \$102,000 \\ &= (1.02)(1.02) \cdot \$100,000 = (1.02)^2 \cdot \$100,000.\end{aligned}$$

After 3 years, the investment is worth $(1.02)^3 \cdot \$100,000$ and we see that in general, after n years, the investment is worth

$$(1.02)^n \cdot \$100,000.$$

We may generalize this result further.

Now let $Q(t)$ be the investment value at time $t \geq 0$, where t is the number of years. Let r be the annual interest rate, converted to decimal, and Q_0 be the initial investment. Then

$$Q(t) = (1 + r)^t Q_0.$$

This is the general formula for annually-compounded interest. If, instead, the interest were compounded monthly, then after 1 month we have

$$Q\left(t = \frac{1}{12}\right) = \left(1 + \frac{r}{12}\right) Q_0.$$

After 2 months,

$$Q\left(t = \frac{2}{12}\right) = \left(1 + \frac{r}{12}\right)^2 Q_0.$$

In general ,

$$Q\left(t\right) = \left(1 + \frac{r}{12}\right)^{12t} Q_0.$$

Periodically and continuously compounded interest.

Our choice of monthly compounding was arbitrary; if we compounded n-times per annum, then we would have found

$$Q(t) = \left(1 + \frac{r}{n}\right)^{nt} Q_0.$$

Now consider what would happen if we let $n \to \infty$. It turns out that this precise behavior is modeled by the formula

$$Q(t) = Q_0 e^{rt}.$$

This is called *continuously*-compounded interest.

Examples of monthly- and continuously-compounded interest.

Example L22.1: If an initial investment of \$20,000 is made with 6% interest, compounded monthly, find the value of the investment after 5 years.

Solution: We choose $Q_0 = 20,000$, $r = 0.06$, $n = 12$ and $t = 5$ in the formula for periodically-compounded interest. We find that

$$Q(5) = \left(1 + \frac{0.06}{12}\right)^{12\cdot 5} 20,000 = 1.005^{60} \cdot 20,000 \approx 26,977.$$

Example L22.2: If an initial investment of \$20,000 is made with 6% interest, compounded continuously, find the value of the investment after 5 years.

Solution: We choose $Q_0 = 20,000$, $r = 0.06$ and $t = 5$ in the formula for continuously-compounded interest. We find that

$$Q(5) = 20,000e^{0.06\cdot 5} = 20,000e^{0.3} \approx 26,997.$$

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Sometimes exponential models approximate population growth.

Example L22.3: A culture of bacteria cells quadruples in size after 2 days. If it grows exponentially fast, how much will there be in 10 days?

Solution: We apply an exponential growth model, where $Q(t)$ means the number of bacteria cells, Q_0 is the (unknown) initial size and let k be the (also unknown) growth rate. If we let t be measured in days, then we know that

$$Q(t) = Q_0 e^{kt} \Rightarrow Q(2) = 4Q_0 = Q_0 e^{2k}.$$

Note that we may cancel;

$$4 = e^{2k} \Rightarrow \ln(4) = \ln\left(e^{2k}\right) = 2k.$$

In this way we have found the growth rate: $k = \ln(4)/2$. Given any initial amount Q_0, it follows that

$$Q(t) = Q_0 e^{t \ln(4)/2}.$$

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The half-life of a radioactive material is how long it takes before half of an initial sample has decayed.

Example L22.4: Strontium-90 has a half-life of 29.1 years. How much time will pass before only 1% of an initial amount is left?

Solution: This problem can be solved without knowing the initial amount. We note that $Q(29.1) = Q_0/2$ (definition of half-life) and

$$Q(29.1) = Q_0 e^{29.1k} = \frac{1}{2} Q_0 \Rightarrow e^{29.1k} = \frac{1}{2}.$$

Take the natural logarithm:

$$29.1k = \ln\left(\frac{1}{2}\right) \Rightarrow k = \frac{1}{29.1} \ln\left(\frac{1}{2}\right).$$

Note that this rate is negative, which is more easily seen by recalling that

$$\ln\left(\frac{1}{2}\right) = \ln\left(2^{-1}\right) = -\ln(2).$$

Example L22.4 ...

Therefore, $k = -\ln(2)/29.1$. The rate must be negative, since the material is decaying over time. Now, we want to solve

$$Q(t) = Q_0 e^{-t\ln(2)/29.1} = 0.01Q_0 \Rightarrow e^{-t\ln(2)/29.1} = 0.01.$$

Again, we use the natural logarithm:

$$-t\ln(2)/29.1 = \ln(0.01) \Rightarrow t = -29.1\frac{\ln(0.01)}{\ln(2)}.$$

Note that $\ln(0.01) = \ln(100^{-1}) = -\ln(100)$. Thus,

$$t = 29.1\frac{\ln(100)}{\ln(2)}.$$

This is the time when only 1% of the initial sample remains, approximately 193.34 years.

Practice!

Problem L22.1: If $\$50,000$ is invested in a portfolio and it ends up increasing in value at a rate of 1% annually, compounded continuously, what would the value be after 10 years?

Problem L22.2: How long does it take for a sample of Strontium-90 to decay to 20% of the original amount?