Math 1060Q Lecture 21

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Logarithm function: another important special function

- Definition of a logarithmic function
- The graph of $\log_a(x)$.
- Properties of logarithmic functions.
- The natural logarithmic function.

The logarithm function is the inverse of the exponential function.

The statement

$$f(x) = \log_a(x) = y$$

means precisely that

$$a^y = x.$$

We take $a > 1$; this is the "base" of the logarithm.

Examples:

$$\log_2(4) = 2$$
$$\log_2(8) = 3$$
$$\log_2(16) = 4$$
$$\log_3(9) = 2$$

Some attributes of these functions.

- Note that $\log_a(1) = 0$, always.
- These functions can have negative values (when x is a fraction). For example,

$$\log_2\left(\frac{1}{2}\right) = \log_2\left(2^{-1}\right) = -1.$$

- The domain is restricted to $x > 0$;

$$\log_a(x) = y \Rightarrow a^y = x \leq 0$$

is not possible, since $a^y > 0$ (remember?).

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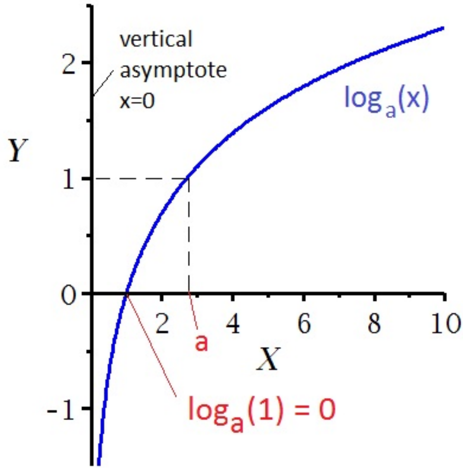
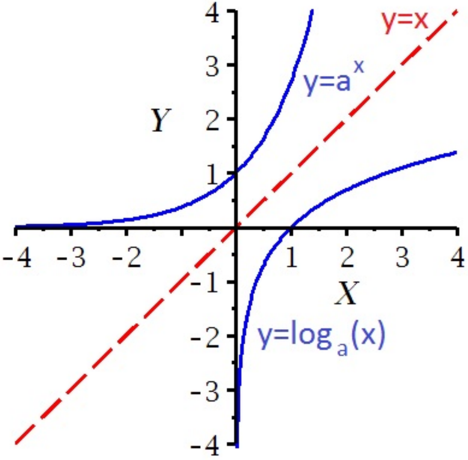
Take careful note of the details in the graph.



For graphing, it may help to remember a^x and $\log_a(x)$ are inverses.



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The following are quite useful identities.

$$\begin{aligned}
\log_a(1) &= 0 \\
\log_a(a) &= 1 \\
\log_a(x_1 x_2) &= \log_a(x_1) + \log_a(x_2) \\
\log_a(x_1/x_2) &= \log_a(x_1) - \log_a(x_2) \\
\log_a(x^r) &= r \log_a(x)
\end{aligned}$$

Some examples:

$$\begin{aligned}
\log_4(1) &= 0 \\
\log_7(7) &= 1 \\
\log_8(3 \cdot 5) &= \log_8(3) + \log_8(5) \\
\log_2(5/3) &= \log_2(5) - \log_2(3) \\
\log_3(8^4) &= 4 \log_3(8)
\end{aligned}$$

Some further examples to illustrate applications.

Example L21.1: Solve $\log_3(x+1) = -1$.

Solution: By definition of the logarithm function, we must have

$$3^{-1} = \frac{1}{3} = x + 1 \Rightarrow x = \frac{1}{3} - 1 = -\frac{2}{3}.$$

Example L21.2: Solve $\log_2(x-2) + \log_2(x) = 3$.

Solution: We may combing these logarithms;

$$\log_2(x-2) + \log_2(x) = \log_2((x-2)x) = 3 \Rightarrow 2^3 = 8 = (x-2)x.$$

Solve for x (it is quadratic)

$$x^2 - 2x - 8 = 0 = (x-4)(x+2),$$

so $x = -2$ or $x = 4$. **CHECK** if these solve the original equation!
Note that $x = 4$ is fine, but $x = -2$ does not work!

- Definition of a logarithmic function
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The natural logarithm is the inverse of the natural exponential function.

$$y = \ln(x) \iff e^y = x.$$

Note the special notation $\log_e(x) = \ln(x)$. Also note that

- $\ln(e^x) = x$ always holds.
- $e^{\ln(x)} = x$ for $x > 0$, since the domain of $\ln(x)$ is $x > 0$.

Example L21.3: Solve $\ln(5x + 2) = 1$.

Solution: It follows that $e^1 = e = 5x + 2$. Therefore, $x = (e - 2)/5$.

Note here how we may use logarithms to solve equations involving exponential functions.

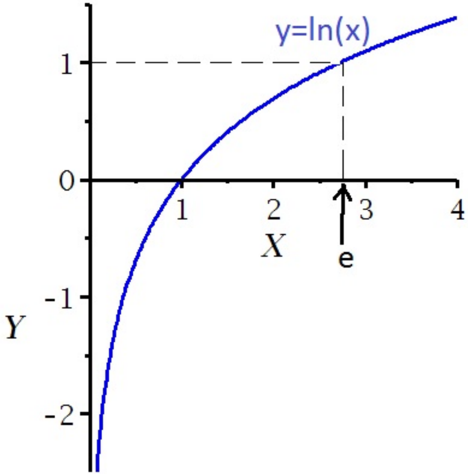
Example L21.4: Solve $2e^{x^2-1} = 10$.

Solution: First, divide through by 2: $e^{x^2-1} = 5$. We may take the natural logarithm of both sides of the equation; they must be equal.

$$\ln\left(e^{x^2-1}\right) = x^2 - 1 = \ln(5).$$

Now we simply solve for x: $x = \pm\sqrt{1+\ln(5)}$. Both of these are valid solutions (plug into original equation to check), since the domain of e^x is not restricted.

The graph is easy to understand; we are just talking about using the base $a = e$ for $\log_a(x)$.



It is often convenient to convert from base a to base e, as shown below.

To convert an exponential:

$$a^x = e^{x \ln(a)}.$$

To convert a logarithm:

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

Examples:

$$2^x = e^{x \ln(2)}$$

$$\log_3(x) = \frac{\ln(x)}{\ln(3)}.$$

Practice!

Problem L21.1: Evaluate the following exactly:

$\log_3(3^7)$	
$\log_2(32)$	
$\ln(e)$	
$\ln(e^{1/4})$	
$\log_{10}(\frac{1}{100})$	
$e^{\ln(6)}$	

Problem L21.2: Solve for x: $\log_2(x+1) = -1$.

Problem L21.3: Solve for x: $\ln(x-1) + \ln(x+1) = \ln(8)$.

Problem L21.4: Solve for x: $\log_3(5x+10) - \log_3(x-2) = 2$.