Math 1060Q Lecture 2

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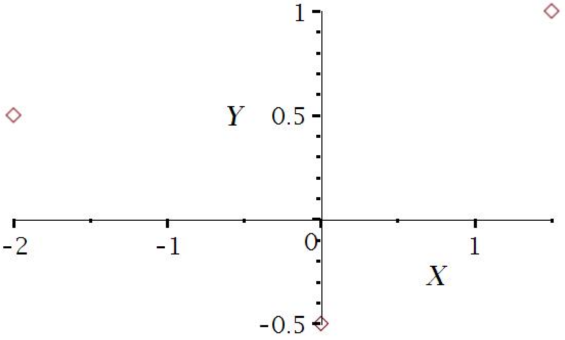
Today we explore sets of points in the X-Y plane

1. The coordinate plane
2. Distance between points
3. Circles: standard equation and graph
4. Completing the square

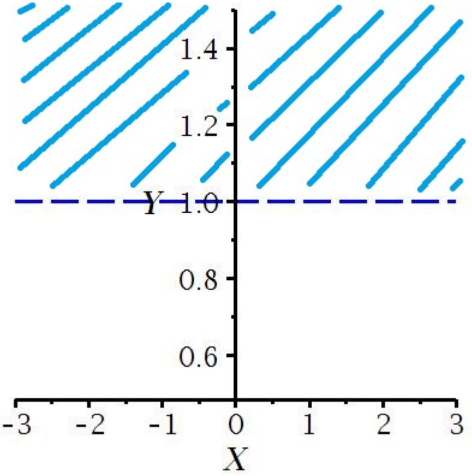
The coordinate plane is a way to visualize pairs of numbers and relationships between numbers

- A coordinate pair is denoted by (x, y); for example in the following plot are shown $\{(3/2, 1), (-2, 1/2), (0, -1/2)\}$.
- The x-value is the position along the horizontal axis.
- The y-value is the position along the vertical axis.



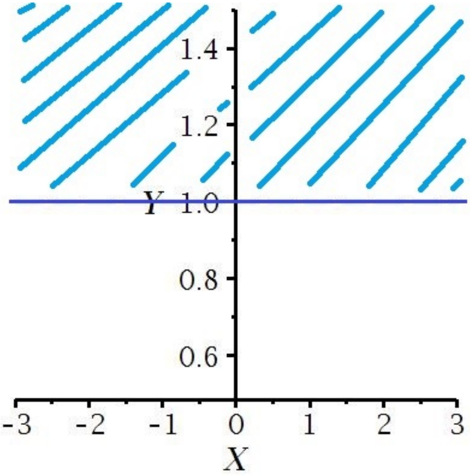
We shade in regions to denote sets of coordinates that satisfy certain criteria

Here we visualize $\{(x, y) \mid y > 1\}$. A dotted line reminds us that points with $y = 1$ are not included in this set.



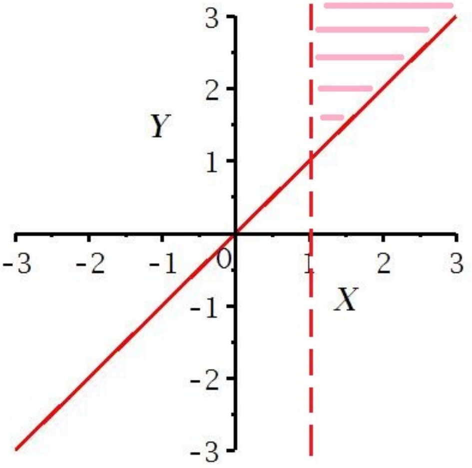
We shade in regions to denote sets of coordinates that satisfy certain criteria

Here we visualize $\{(x, y) \mid y \geq 1\}$. A solid line reminds us that points with $y = 1$ are included in this set.



We can specify more complicated regions, such as in this example.

Example L2.1: Shade the region of points $\{(x, y) \mid x > 1 \text{ and } x \leq y\}$.



1. The coordinate plane
2. Distance between points
3. Circles: standard equation and graph
4. Completing the square

Given two points with coordinates (x_1, y_1) and (x_2, y_2) there is a formula for the distance between the points.

The formula is: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example L2.2: Find the distance between points $(1, 1)$ and $(3, 2)$.
Solution: Take $(x_1, y_1) = (1, 1)$ and $(x_2, y_2) = (3, 2)$. If we plug these into the formula, we get

$$d = \sqrt{(3-1)^2 + (2-1)^2} = \sqrt{(2)^2 + (1)^2} = \sqrt{4+1} = \sqrt{5}.$$

Example L2.3: Find the distance between points $(-1, 0)$ and $(5, 4)$.
Solution: Take $(x_1, y_1) = (-1, 0)$ and $(x_2, y_2) = (5, 4)$. If we plug these into the formula, we get

$$d = \sqrt{(5-(-1))^2 + (4-0)^2} = \sqrt{(5+1)^2 + (4)^2} = \sqrt{36+16} = \sqrt{52}.$$

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A circle is a set of points that are all the same distance from the center (x_0, y_0).

- Let r mean the radius of the circle.
- Let (x, y) mean the coordinates of a point on the circle.
- Then the coordinates must satisfy the equation

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

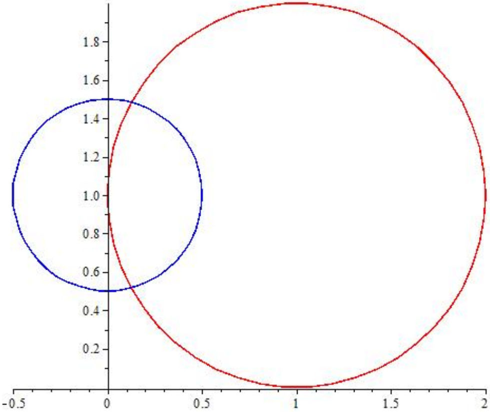
- This equation is in the standard form for a circle.

Example L2.4: Find the equation for a circle with radius 3 and center $(-1, 1)$.
Solution: We have $x_0 = -1$ and $y_0 = 1$ with $r = 3$. Plug these values into the standard-form equation for a circle:

$$(x - (-1))^2 + (y - 1)^2 = 3^2 \Rightarrow (x + 1)^2 + (y - 1)^2 = 9.$$

Here are two more examples of circles.

$$\begin{aligned}(x-1)^2+(y-1)^2 &= 1 \quad \text{red} \\ x^2+(y-1)^2 &= \frac{1}{4} \quad \text{blue}\end{aligned}$$



Given the equation for a circle in standard form, you should be able to find the center and radius.

Example L2.5: Find the center and radius of the circle with equation

$$(x+2)^2 + (y-4)^2 = 16.$$

Solution: In order for the first term to be of the form $(x - x_0)^2$, we would need $x_0 = -2$. Similarly, $y_0 = 4$ and the center of the circle is thus $(-2, 4)$. The right-hand side of the equation gives $r^2 = 16$ so that $r = 4$ is the radius.

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Sometimes equations for circles are NOT in standard form; we can complete the square to put the equation into standard form.

Here is an equation for a circle in disguise:

$$x^2 + ax + y^2 + by = c.$$

- Here a, b and c are just place-holders for three numbers.
- There is a restriction on the values c can have for this to be a circle, which we will soon see.
- We can put this equation into the standard form for a circle by following a series of steps called "completing the square".
- You must memorize this process and be able to perform it on demand. It is used all the time in practice.

How to complete the square.

1. Add the quantities $(a/2)^2$ and $(b/2)^2$ into the equation on BOTH SIDES:

$$x^2 + ax + \left(\frac{a}{2}\right)^2 + y^2 + by + \left(\frac{b}{2}\right)^2 = c + \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2.$$

2. The right-hand side is now r^2, so

$$r = \sqrt{c + \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}.$$

3. Imagine "grouping" the remaining terms as follows:

$$\left[x^2 + ax + \left(\frac{a}{2}\right)^2\right] + \left[y^2 + by + \left(\frac{b}{2}\right)^2\right] = r^2.$$

   This may be rewritten as

$$\left(x + \frac{a}{2}\right)^2 + \left(y + \frac{b}{2}\right)^2 = r^2.$$

4. Therefore, $x_0 = -a/2$ and $y_0 = -b/2$ gives the center.

For some people it may be easier to just memorize the formulas for the center and radius.

Short procedure:

1. Set $x_0 = -a/2$ and $y_0 = -b/2$.
2. Set according to the formula
$$r = \sqrt{c + \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2}.$$
3. Plug these into the standard form:
$$\begin{aligned}(x - (-a/2))^2 + (y - (-b/2))^2 &= (x + (a/2))^2 + (y + (b/2))^2 \\ &= c + \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2.\end{aligned}$$

This has been rather abstract; let's look at an example to watch how it works.

Example L2.6: Complete the square to get the standard form of the circle

$$x^2 + 6x + y^2 + 16y = 1.$$

Solution: Here we have $a = 6$ and $b = 16$. We cut these in half, square the results and add to both sides:

$$x^2 + 6x + 3^2 + y^2 + 16y + 8^2 = 1 + 3^2 + 8^2.$$

We have $r^2 = 1 + 9 + 64 = 74$, hence $r = \sqrt{74}$. Recall the left-hand side may be rewritten now:

$$(x + 3)^2 + (y + 8)^2 = 74.$$

Note that $x_0 = -3 = -a/2$ and $y_0 = -8 = -b/2$.

Another example to watch how it works.

Example L2.7: Complete the square to get the standard form of the circle

$$x^2 - 4x + y^2 + 2y = -3.$$

Solution: Here we have $a = -4$ and $b = 2$. We cut these in half, square the results and add to both sides:

$$x^2 - 4x + (-2)^2 + y^2 + 2y + 1^2 = -3 + (-2)^2 + 1^2.$$

We have $r^2 = -3 + 4 + 1 = 2$, hence $r = \sqrt{2}$. Recall the left-hand side may be rewritten now:

$$(x-2)^2 + (y+1)^2 = 2.$$

Note that $x_0 = 2 = -a/2$ and $y_0 = -1 = -b/2$.

Practice!

Problem L2.1: Sketch the region $\{(x, y) \mid -2 < y \leq 1\}$.

Problem L2.2: Calculate the distance between the points $(0, -6)$ and $(2, 3)$.

Problem L2.3: Find an equation for the circle with radius 5 and center $(-9, 2)$.

Problem L2.4: Find the center and radius of the circle satisfying

$$(x + 1)^2 + (y - 1)^2 = 3.$$

Problem L2.5: Complete the square to put the equation for the circle in standard form:

$$x^2 + 10x + y^2 - 4y = -28.$$