Math 1060Q Lecture 18

Jeffrey Connors

University of Connecticut

November 10, 2014

Inverse functions: how to "undo" the action of a function

- What is an inverse function?
- One-to-one functions and the horizontal line test.
- Graphing the inverse.
- Deriving formulas for the inverse.

The inverse function reverses the action of the original function.

If $y = f(x)$, where f is a function (remember vertical line test?) then we denote the inverse of $f(x)$ by

$$f^{-1}(y) = x.$$

- If $\mathcal{D}$ is the domain of f and $\mathcal{R}$ is the range of f, then f^{-1} has domain $\mathcal{R}$ and range $\mathcal{D}$.
- Note that $f^{-1}(y) = f^{-1}(f(x)) = x$; *i.e.* f^{-1} in some sense reverses the action of $f(x)$, bringing us back to x.
- One of the main uses is to solve something like $f(x) = 0$, in which case the answer may be $x = f^{-1}(0)$. It can therefore be useful to come up with formulas for inverse functions, so that they may be evaluated.

Example of an inverse function.

Let $f(x) = 2x + 5$ on the domain $[-1, 1]$. This is simply a "line"; note $f(-1) = 3$ and $f(1) = 7$, therefore the range is $[3, 7]$.

CLAIM: The inverse of this function exists and is given by

$$f^{-1}(x) = \frac{x - 5}{2},$$

with domain $[3, 7]$ and range $[-1, 1]$.

CHECK: Let x be any number in $[-1, 1]$. We find that

$$f^{-1}(f(x)) = f^{-1}(2x + 5) = \frac{(2x + 5) - 5}{2} = x.$$

Thus, if $y = f(x)$ is in the set $[3, 7]$, then $f^{-1}(y) = x$.

- What is an inverse function?
- One-to-one functions and the horizontal line test.
- Graphing the inverse.
- Deriving formulas for the inverse.

The inverse does not generally exist; we will consider only when the function is "1-to-1"

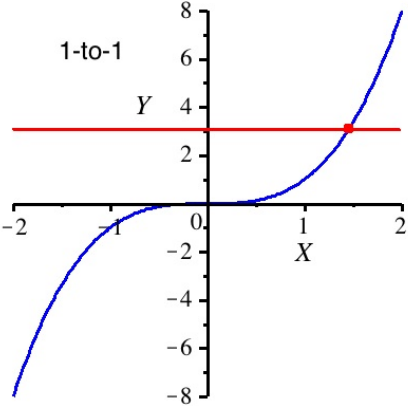
A function $f(x)$ with domain $\mathcal{D}$ is called 1-to-1 if it satisfies the following logical statement: Given any two numbers x_1 and x_2 in the domain $\mathcal{D}$,

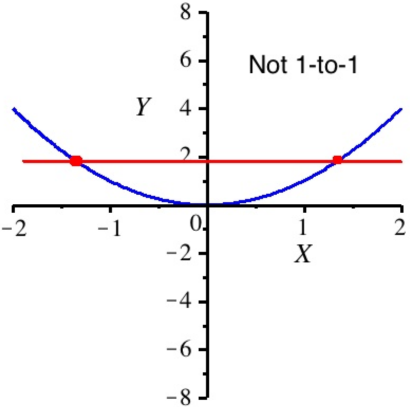
$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2).$$

- If $f(x)$ is 1-to-1, then $f^{-1}(x)$ exists.
- Note that this property may depend on the specified domain; *e.g.* think of $f(x) = x^2$ on $[-1, 1]$ versus on $[0, 1]$.
- If $f(x)$ is a *function* that is 1-to-1, then $f^{-1}(x)$ is also a 1-to-1 function.
- Unfortunately, existence of the inverse does not mean we can come up with a nice *formula* for the inverse; that is a different problem.

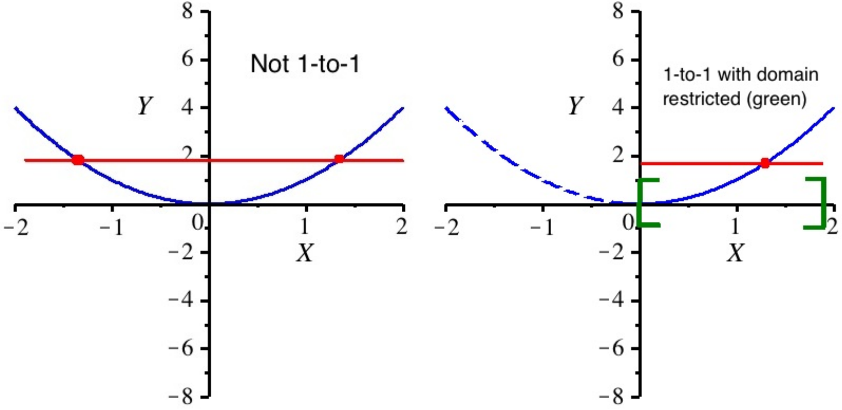
You can tell from the graph of $f(x)$ if it is 1-to-1 quite easily.

Horizontal line test: If an arbitrary horizontal line only intersects the graph of $f(x)$ once over it's domain, then it is 1-to-1.





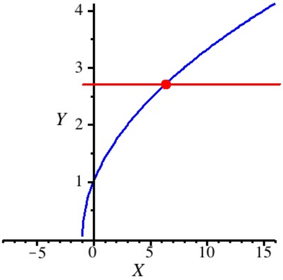
However, we can also restrict the domain of a function and it may then be 1-to-1



Example: you need to identify the domain before you talk about the inverse.

Example L18.1: Does the function $f(x) = \sqrt{x+1}$ have an inverse?

Solution: Note that the domain is where $x + 1 \geq 0$, so $[-1, \infty)$. Look at the graph and apply the horizontal line test OVER THE DOMAIN ONLY:



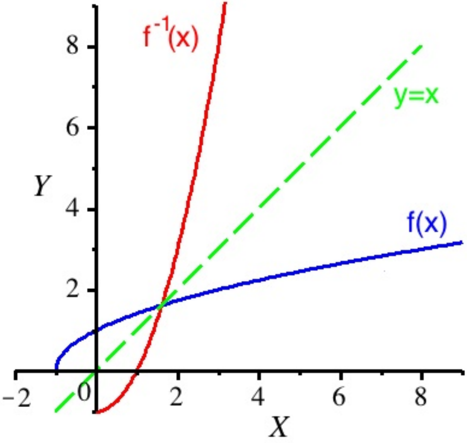
We see that the inverse exists.

- What is an inverse function?
- One-to-one functions and the horizontal line test.
- Graphing the inverse.
- Deriving formulas for the inverse.

To graph an inverse, flip across the line $y = x$.

Example L18.2: Graph the inverse of $y = \sqrt{x+1}$.

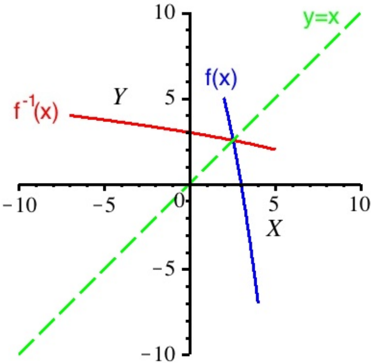
Solution: We "mirror" the graph across the line $y = x$; note that the domain and range get flipped.



Another example...

Example L18.3: Let $f(x) = 9 - x^2$ on the domain $[2, 4]$. Graph the inverse.

Solution: Note that the range is $[-7, 5]$, and the domain and range flip for the inverse;



- What is an inverse function?
- One-to-one functions and the horizontal line test.
- Graphing the inverse.
- Deriving formulas for the inverse.

There is a procedure to find a formula for the inverse, which sometimes works!

1. Given $f(x)$, set $y = f(x)$.
2. Switch x and y; write $x = f(y)$.
3. Solve for y in terms of x; this yields $y = f^{-1}(x)$.

Example L18.4: Find $f^{-1}(x)$ if $f(x) = -2x + 1$.

Solution: Set $x = f(y) = 1 - 2y$. Solve for y:

$$2y = 1 - x \Rightarrow y = \frac{1 - x}{2}.$$

It follows that $f^{-1}(x) = (1 - x)/2$.

More examples...

Example 18.5: Find the inverse of $f(x) = (1 - x)/(2x + 1)$.

Solution: Set $x = f(y)$ and solve for y... this will take some work.

$$x = \frac{1-y}{2y+1} \Rightarrow x(2y+1) = 1 - y.$$

Next, group the terms with a y on one side, throw other terms on the other side...

$$x(2y+1) = 2xy + x = 1 - y \Rightarrow 2xy + y = 1 - x.$$

Now factor the y out on the left and move x-terms to the right:

$$y(2x+1) = 1 - x \Rightarrow y = f^{-1}(x) = \frac{1-x}{2x+1}.$$

It turns out that $f^{-1}(x) = f(x)$. This happens sometimes, in which case we call $f(x)$ an *involution*.

More examples...

Example L18.6: Find $f^{-1}(x)$ if $f(x) = (x+1)/(7x-5)$.

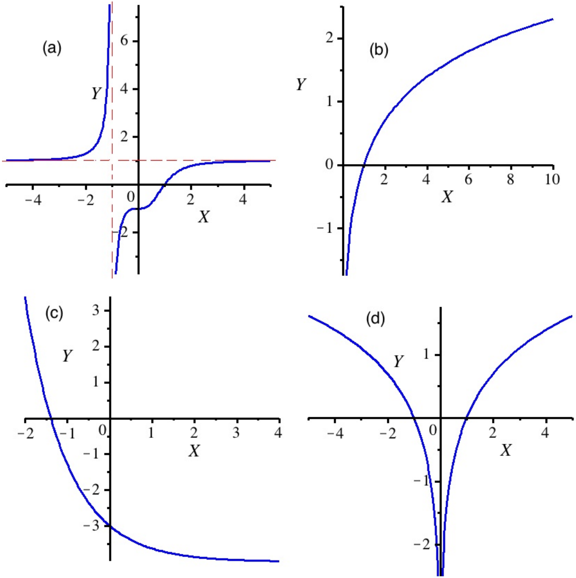
Solution: Set $x = f(y)$ and solve for y...

$$x = \frac{y+1}{7y-5} \Rightarrow x(7y-5) = y+1 \Rightarrow 7xy - 5x = y+1$$
$$\Rightarrow 7xy - y = 1 + 5x \Rightarrow y(7x-1) = 1+5x$$
$$\Rightarrow y = \frac{1+5x}{7x-1}.$$

In summary, $f^{-1}(x) = (1+5x)/(7x-1)$.

Practice!

Problem L18.1: Determine if the functions shown are invertible.



Practice!

Problem L18.2: Let $f(x) = x^3$. Sketch a graph of $f^{-1}(x)$.

Problem L18.3: Find the inverses of the following functions:

(a) $f(x) = 3x - 4$

(b) $f(x) = \dfrac{2}{x+3}$

(c) $f(x) = \dfrac{x-2}{2x+1}$