Math 1060Q Lecture 17

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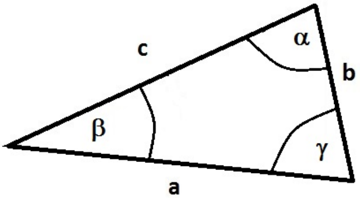
Today we cover 5 more important trigonometry formulas.

1. Law of Cosines
2. Law of Sines
3. Sine-combination formula
4. Cosine-combination formula
5. Heron's formula

The Law of Cosines generalizes the Pythagorean Theorem to general triangles!



$$a^2 = b^2 + c^2 - 2bc\cos(\alpha).$$

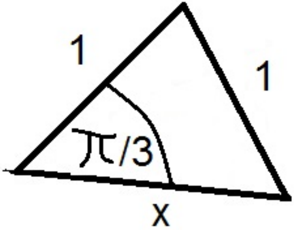
Notice that the angle α could have been any of the three interior angles. Indeed,

$$b^2 = a^2 + c^2 - 2ac\cos(\beta), \quad c^2 = a^2 + b^2 - 2ab\cos(\gamma).$$

The key idea is that the equation involves all three sides and one angle.

If you have three out of four of the things used in the formula, you can find the remaining piece of information.

Example L17.1: Solve for x shown in the diagram.



Solution:

$$1^2 = 1^2 + x^2 - 2 \cdot x \cdot 1 \cdot \cos(\pi/3) = 1 + x^2 - 2x \cdot \frac{1}{2} = x^2 - x + 1.$$

Solving for x, we see that $x^2 - x = x(x-1) = 0$. We know $x \neq 0$, thus $x = 1$.

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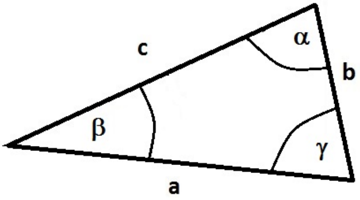
The Law of Sines requires slightly different information.

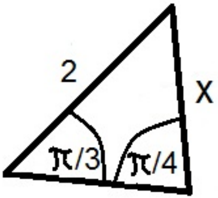


$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}.$$

To apply this formula, one requires information about two angles and two sides.

The application of the Law of Sines is straight-forward.

Example L17.2: Solve for x as shown in the diagram.



Solution:

$$\frac{\sin(\pi/4)}{2} = \frac{\sin(\pi/3)}{x} \Rightarrow \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}/2}{x} \Rightarrow x = 2\sqrt{2} \cdot \frac{\sqrt{3}}{2} = \sqrt{6}.$$

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The sine-combination formula is used to write a sum involving both $\sin(x)$ and $\cos(x)$ in terms of a single sine-function.

$$A\sin(x) + B\cos(x) = C\sin(x+\delta)$$
$$A^2 + B^2 = C^2$$
$$\tan(\delta) = \frac{B}{A}.$$

Here, the last two equations are normally used to calculate C and δ, given both A and B. Then these values are inserted into the right-hand side of the first equation.

Here is the standard application of this formula.

Example L17.3: Write $\sqrt{3}\sin(x) + \cos(x)$ in terms of a single sine-function.

Solution: Here, we take $A = \sqrt{3}$ and $B = 1$, thus

$$A^2 + B^2 = 3 + 1 = 4 = C^2 \Rightarrow C = \sqrt{4} = 2.$$

To find δ, note that $\tan(\delta) = 1/\sqrt{3}$. We know that this is true for $\delta = 30^\circ$... we use radians so $\delta = \pi/6$. Thus

$$\sqrt{3}\sin(x) + \cos(x) = 2\sin(x + \pi/6).$$

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The cosine-combination formula is analogous, but combines into a single cosine-function instead.

$$A\sin(x) + B\cos(x) = C\cos(x - \delta)$$
$$A^2 + B^2 = C^2$$
$$\tan(\delta) = \frac{A}{B}.$$

Note that there are three differences compared with the sine-combination formula. The most obvious is the appearance of "cos" on the right-hand side. Note also that we subtract δ now: $\cos(x - \delta)$. Finally, $\tan(\delta) = A/B$ is the reciprocal of the ratio used for sine.

Aside from minor differences in the formula, application of the cosine-combination formula is the same as for sine.

Example L17.4: Write $\sin(x) + \cos(x)$ in terms of a single cosine-function.

Solution: Here $A = 1 = B$, so

$$A^2 + B^2 = 1 + 1 = 2 = C^2 \Rightarrow C = \sqrt{2}.$$

We have in this case that $\tan(\delta) = 1/1 = 1$ so clearly $\delta = \pi/4$. It follows that

$$\sin(x) + \cos(x) = \sqrt{2}\cos(x - \pi/4).$$

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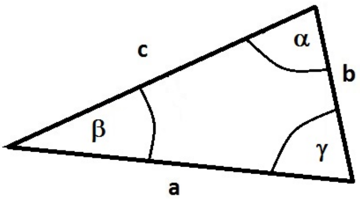
Heron's formula gives the area of any triangle.



$$\text{Area} = \frac{1}{4}\sqrt{P(P-2a)(P-2b)(P-2c)},$$

where $P = a + b + c$ is the length around the perimeter of the triangle. This formula is especially nice when you want the area of a triangle that is not a right-triangle.

So just calculate the perimeter P and plug it in...

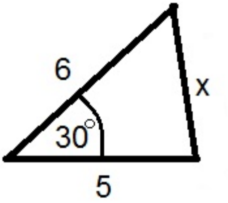
Example L17.5: Find the area of a triangle with sides of length 1, 2 and 5/4.

Solution: Then $P = 1 + 2 + 5/4 = 17/4$. It follows that

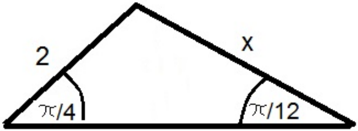
$$\begin{aligned}\text{Area} &= \frac{1}{4}\sqrt{\frac{17}{4}\left(\frac{17}{4}-2\right)\left(\frac{17}{4}-4\right)\left(\frac{17}{4}-\frac{5}{2}\right)} \\ &= \frac{1}{4}\sqrt{\frac{17}{4}\left(\frac{17}{4}-\frac{8}{4}\right)\left(\frac{17}{4}-\frac{16}{4}\right)\left(\frac{17}{4}-\frac{10}{4}\right)} \\ &= \frac{1}{4}\sqrt{\frac{17}{4}\frac{9}{4}\frac{1}{4}\frac{7}{4}} = \frac{3\sqrt{119}}{64}.\end{aligned}$$

Practice!

Problem L17.1: Find x...



Problem L17.2: Find x... (hint: recall we derived $\sin(\pi/12) = (\sqrt{3} - 1)/(2\sqrt{2})$.)



Practice!

Problem L17.3: Find the area of a triangle with sides of lengths 6, 7 and 11.

Problem L17.4: Write $(\sqrt{3} - 1)\sin(x) + (\sqrt{3} + 1)\cos(x)$ as a single cosine function. HINT: $\tan(\pi/12) = (\sqrt{3} - 1)/(\sqrt{3} + 1)$.