Math 1060Q Lecture 13

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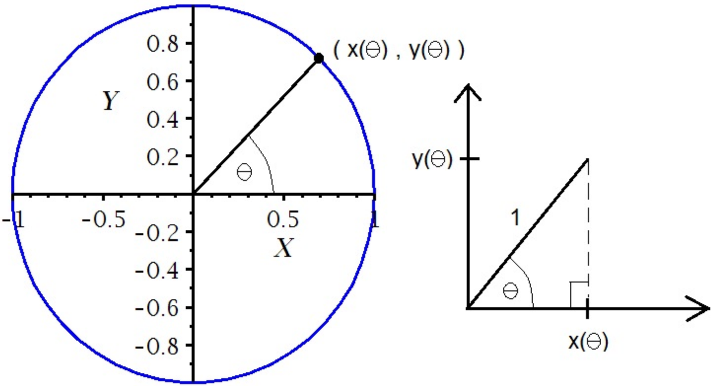
October 15, 2014

Sinusoidal functions

- Relationship of unit circle with $\sin(\theta)$ and $\cos(\theta)$
- The Pythagorean Identity
- Sinusoidal graphs
- Sinusoids are "periodic" functions

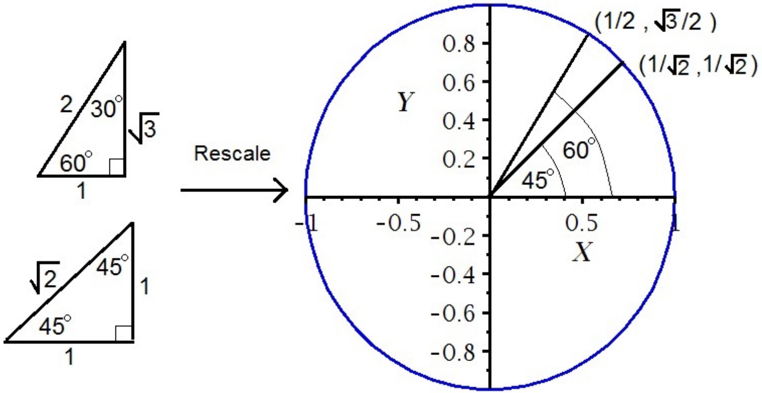
We can think of sin(θ) and cos(θ) as functions on the unit circle



$$\sin(\theta) = \frac{y(\theta)}{1} = y(\theta), \qquad \cos(\theta) = \frac{x(\theta)}{1} = x(\theta).$$

Examples in Quadrant I: x and y are both positive.

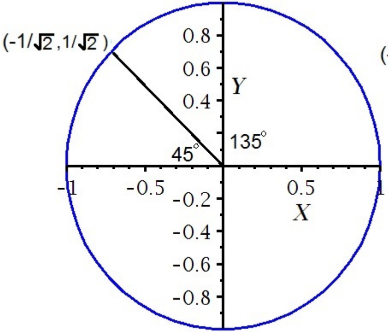
We can find $\sin(\theta)$ or $\cos(\theta)$ for certain θ values using special triangles.

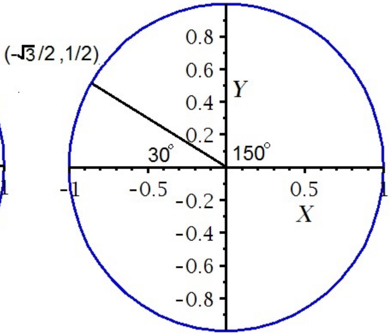


$$\cos(45^\circ) = \sin(45^\circ) = \frac{1}{\sqrt{2}}, \quad \cos(60^\circ) = \frac{1}{2}, \quad \sin(60^\circ) = \frac{\sqrt{3}}{2}.$$

Examples in Quadrant II: x becomes negative.

So $x(\theta) = \cos(\theta)$ becomes negative...

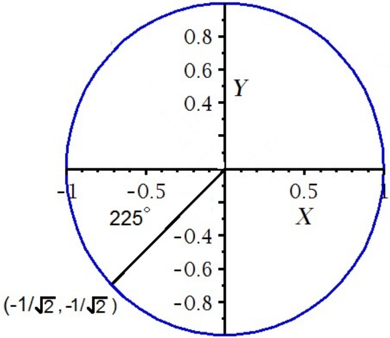


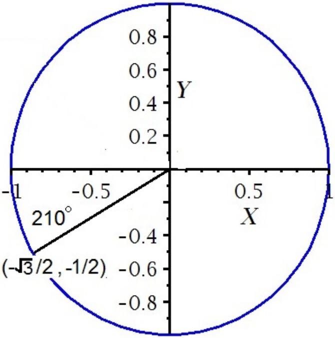


$$\cos(135^\circ) = \frac{-1}{\sqrt{2}}, \ \sin(135^\circ) = \frac{1}{\sqrt{2}}, \ \cos(150^\circ) = \frac{-\sqrt{3}}{2}, \ \sin(150^\circ) = \frac{1}{2}.$$

Examples in Quadrant III: x, y both are negative.

So now both $\sin(\theta)$ and $\cos(\theta)$ are negative.

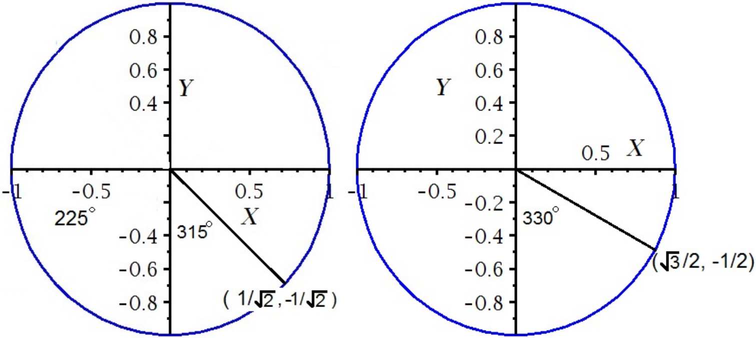




$$\cos(225^\circ) = \frac{-1}{\sqrt{2}}, \ \sin(225^\circ) = \frac{-1}{\sqrt{2}}, \ \cos(210^\circ) = \frac{-\sqrt{3}}{2}, \ \sin(210^\circ) = \frac{-1}{2}$$

Examples in Quadrant IV: only y is negative.

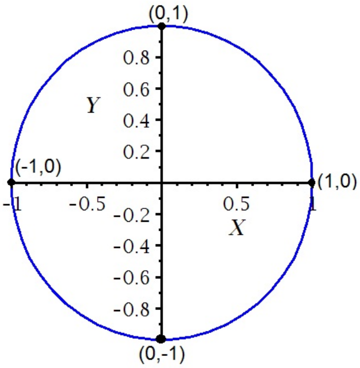
Then $\sin(\theta) < 0$ and $\cos(\theta) > 0$.



$$\cos(315^\circ) = \frac{1}{\sqrt{2}}, \ \sin(315^\circ) = \frac{-1}{\sqrt{2}}, \ \cos(330^\circ) = \frac{\sqrt{3}}{2}, \ \sin(330^\circ) = \frac{-1}{2}.$$

It is easy to find $\sin(\theta)$, $\cos(\theta)$ when θ is a multiple of 90°.

For $\theta = 90^\circ,\ 180^\circ,\ 270^\circ, 360^\circ$ we are on a coordinate axis.



θ	0°	90°	180°	270°	360°
$\cos(\theta)$	1	0	-1	0	1
$\sin(\theta)$	0	1	0	-1	0

Calculation problems.

Example L13.1: Find $\sin(5\pi/4)$.
Solution: We note that the angle $\theta = 5\pi/4$ is in Quadrant III and will have the same size y coordinate as for $\theta = \pi/4$ in Quadrant I, except with opposite *sign*.

$$\sin(\pi/4) = \frac{1}{\sqrt{2}} \Rightarrow \sin(5\pi/4) = -\frac{1}{\sqrt{2}}.$$

Example L13.2: Find $\cos(11\pi/6)$.
Solution: For this angle, which is in Quadrant IV, the corresponding point on the unit circle has x coordinate the same as for the angle $\theta = \pi/6$. Therefore,

$$\cos(11\pi/6) = \cos(\pi/6) = \frac{\sqrt{3}}{2}.$$

- Relationship of unit circle with $\sin(\theta)$ and $\cos(\theta)$
- The Pythagorean Identity
- Sinusoidal graphs
- Sinusoids are “periodic” functions

Recall the Pythagorean Theorem: $a^2 + b^2 = c^2$.

c is the length of the hypoteneuse of a right triangle and a, b are the lengths of the other sides.

- $x = \cos(\theta)$, $y = \sin(\theta)$ on the unit circle.
- We envision a right triangle with hypoteneuse 1 and sides of length x and y.
- It follows from the Pythagorean Theorem that

$$x^2 + y^2 = 1^2 = 1 \Rightarrow \cos^2(\theta) + \sin^2(\theta) = 1.$$

- Note that this holds for any angle θ.
- This trigonometric identity is called the Pythagorean Identity.
- It is useful to "reduce" expressions, because we often encounter $\cos^2(\theta) + \sin^2(\theta)$ in practice.

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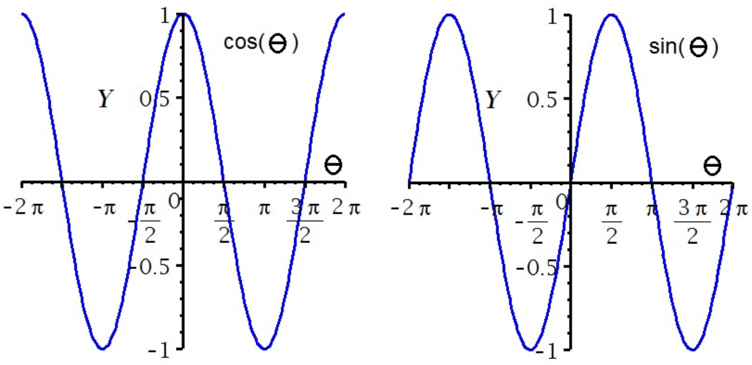
If we plot $\cos(\theta)$ and $\sin(\theta)$ versus θ, we get the following.



- Domain is $(-\infty, \infty)$.
- Range is $[-1, 1]$.
- $\cos(\theta)$ is EVEN.
- $\sin(\theta)$ is ODD.

- Relationship of unit circle with $\sin(\theta)$ and $\cos(\theta)$
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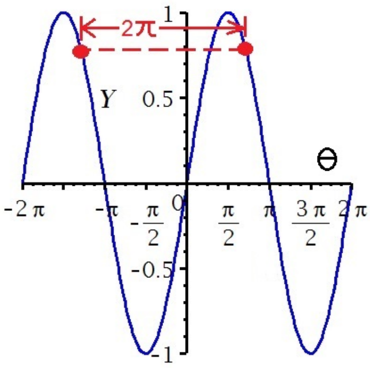
Sinusoids are periodic, meaning the graph repeats itself as θ increases.

- A function $f(\theta)$ is periodic if there is a number τ such that $f(\theta + \tau) = f(\theta)$ holds for all θ.
- τ is called the period of the function.
- For $\sin(\theta)$ and $\cos(\theta)$, $\tau = 2\pi$.



To check if something is periodic, check if it satisfies the definition for some period τ.

Example L13.3: Show that $f(z) = \sin(\pi z)$ is periodic and find the period τ.

Solution: Since we have multiplied the argument for the periodic function $\sin(\theta)$ by π, the new period is found by dividing the period of $\sin(\theta)$ by π: $\tau = 2\pi/\pi = 2$. To check, plug $z + \tau$ into $f(z)$:

$$f(z+\tau) = \sin(\pi(z+\tau)) = \sin(\pi z+\pi\tau) = \sin(\pi z+2\pi) = \sin(\pi z) = f(z).$$

We have shown that $f(z + \tau) = f(z)$ where $\tau = 2$, so we are done.

Practice!

Problem L13.1: Fill in the following table:

θ	π	$\pi/4$	$7\pi/6$	$3\pi/2$	2π
$\cos(\theta)$					
$\sin(\theta)$					

Problem L13.2: Find all values $0 \leq \theta \leq 2\pi$ such that $\cos(\theta) = \sqrt{3}/2$.

Problem L13.3: Show that the function $f(\theta) = \sin(\theta) + \cos(\theta)$ is periodic. What is the period of f?