# Math 1060Q Lecture 11

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# Rational functions

- What is a rational function?
- Horizontal and vertical asymptotes
- Slant asymptotes
- Nonlinear asymptotes

# A rational function includes a ratio of polynomials

Let $p(x)$ and $q(x)$ be polynomial functions. Then

$$r(x) = \frac{p(x)}{q(x)}$$

is a rational function. Note that the domain will be

$$\mathcal{D} = \{x \mid q(x) \neq 0\}.$$

For example,

$$r(x) = \frac{x^2 + 1}{-3x^3 + 5x - 2}$$

is rational. The denominator has three roots; it turns out that

$$\mathcal{D} = \left\{x \mid x \neq 1, \ \frac{-3 - \sqrt{33}}{6}, \ \frac{-3 + \sqrt{33}}{6}\right\}.$$

# Rational functions

- What is a rational function?
- Horizontal and vertical asymptotes
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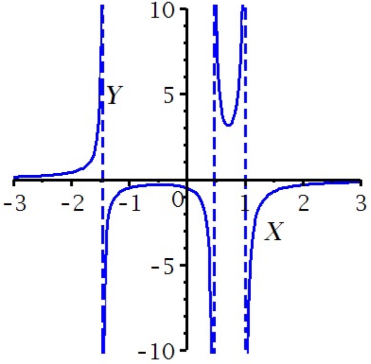
- Nonlinear asymptotes

## A horizontal asymptote is a horizontal line that the graph converges to as $x \to \pm\infty$

Consider the graph of the previous rational function:




- 3 vertical asymptotes
- 1 horizontal asymptote ($x$-axis)

# Two main examples of horizontal asymptotes.

1. The denominator of the rational function is higher-order than the numerator, e.g.

$$r(x) = \frac{x-2}{x^2+3x+1}.$$

   Then the denominator grows *faster* than the numerator as $|x| \to \infty$, thus the ratio goes to zero. Hence we get the horizontal asymptote $y = 0$.

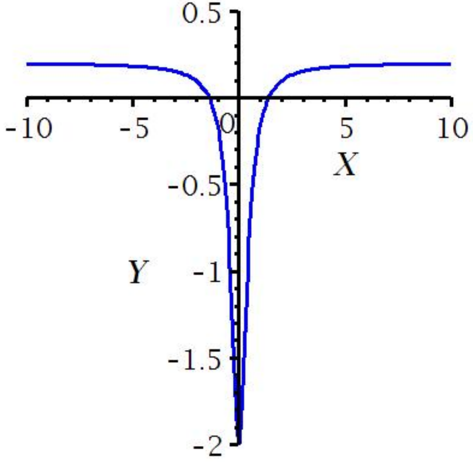
2. The denominator of the rational function is the same order as the numerator, e.g.

$$r(x) = \frac{x^2-2}{5x^2+1}.$$

   Then the growth rate as $|x| \to \infty$ is determined by the ratio of the leading terms on top and bottom, thus this ratio gives the horizontal asymptote. For example, in the above case we have $y = \frac{1}{5}$ is the horizontal asymptote.

# In some cases there are no vertical asymptotes.

Consider the graph of $\frac{x^2-2}{5x^2+1}$.



# You should be able to plot some of the simpler cases.

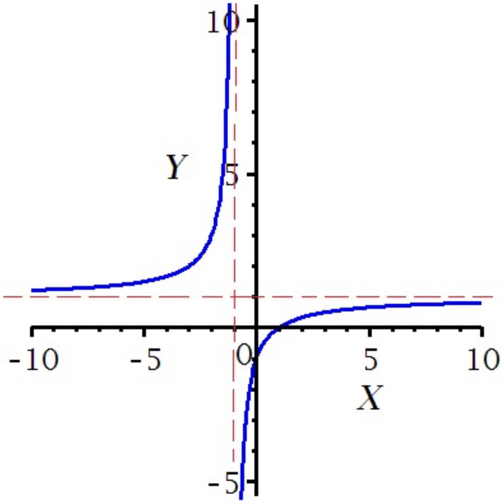
Example L11.1: Sketch the graph of

$$r(x) = \frac{x-1}{x+1}.$$

Solution: Recall our guidelines for sketching... find $x$ and $y$ intercepts and identify asymptotes. You can also plot a few points to help.

- Set $x = 0$; $r(0) = -1$.
- Set $r(x) = 0$ and solve for $x$... $x = 1$.
- Vertical asymptote at $x = -1$.
- Horizontal asymptote $y = x/x = 1$.
- Point to the left of the asymptote: $(-2, 3)$.

# The plot of the function.



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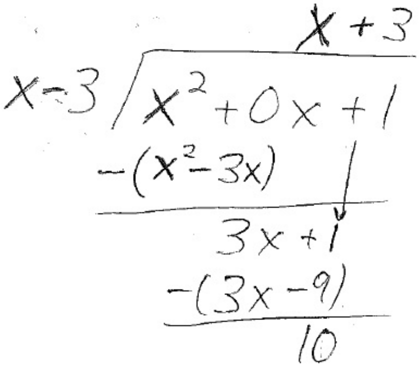
## A slant asymptote will occur when numerator is one order higher than the denominator.

A slant asymptote is a line with slope $m \neq 0$ (not horizontal). Consider

$$r(x) = \frac{x^2 + 1}{x - 3}$$

We find the slant asymptote for this by using polynomial division first:

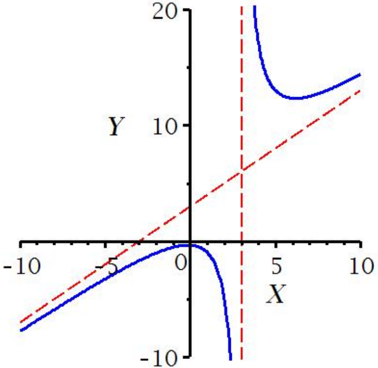


# Drop the remainder to find the slant asymptote.

We find that

$$r(x) = \frac{x^2+1}{x-3} = x + 3 + \frac{10}{x-3}.$$

We drop the last term on the right and what remains (on the right) is the equation for the slant asymptote; $y = x + 3$.



# Another example...

Example L11.2: Find all asymptotes of

$$r(x) = \frac{x^2 - x - 2}{2x + 4}$$

and sketch the graph.
Solution: There is a vertical asymptote when

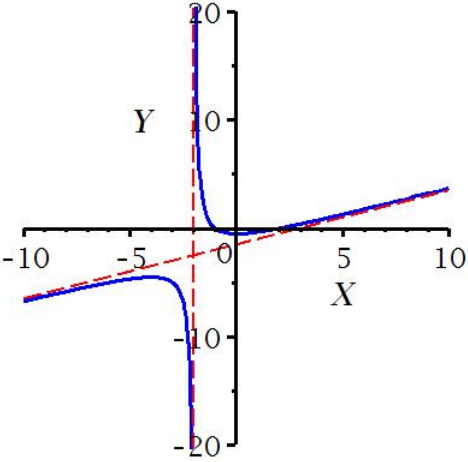
$$2x + 4 = 0 \;\Rightarrow\; x = -2.$$

There is a slant asymptote since the numerator is of one order higher than the denominator; we divide to get

$$r(x) = \frac{x^2 - x - 2}{2x + 4} = \frac{1}{2}x - \frac{3}{2} + \frac{4}{2x + 4}.$$

# Another example...

Thus $y = \frac{1}{2}x - \frac{3}{2}$ is the slant asymptote.



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## Consider the case that the numerator is at least two orders "higher" than the denominator.

For example,

$$r(x) = \frac{x^4 + x^3 - 2x^2 + x + 1}{x^2 + 6x + 1}.$$

For large $x$, the lead terms on the top and bottom determine the growth rate:

$$x \to \pm\infty \ \Rightarrow r(x) \approx \frac{x^4}{x^2} = x^2.$$

Thus the function $r(x)$ behaves like $y = x^2$ as $x \to \pm\infty$. We would say that $r \to x^2$ *asymptotically* as $x \to \pm\infty$. We will not discuss such cases further; these could be called *nonlinear asymptotes*.

# Summary to identify asymptotes

1. If the denominator is higher-order than the numerator, we get the horizontal asymptote $y = 0$.
2. If the denominator is the same order as the numerator, we get a non-zero horizontal asymptote $y = a/b$, with $a$, $b$ the lead coefficients on top and bottom, respectively.
3. If the denominator is ONE order lower than the numerator, we get a slant asymptote. One uses polynomial division to find this.
4. Any time the denominator is zero, we get a vertical asymptote.

# Practice!

Problem L11.1: Find all asymptotes of the function

$$r(x) = \frac{x+1}{x^2+3x}.$$

Problem L11.2: Find all asymptotes of the function

$$r(x) = \frac{x^2+1}{4x^2+3}.$$

Problem L11.3: Find all asymptotes of the function

$$r(x) = \frac{8x^2+1}{4x+3}.$$

Problem L11.4: Sketch the graph of the function

$$r(x) = \frac{x^2+x+1}{x}.$$