Math 1060Q Lecture 10

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Finding factors and zeros of polynomials

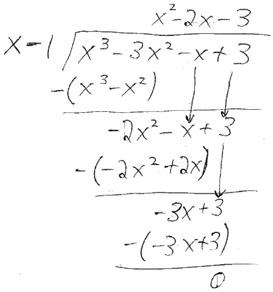
- Polynomial division
- Testing for possible zeros
- A procedure to find zeros and factor

The factored form for a polynomial is often useful, but not provided... so how do we get it?

Consider $p(x) = x^3 - 3x^2 - x + 3$. One of the roots is $x = 1$ (how can you check that this is true?) and so we must have

$$x^3 - 3x^2 - x + 3 = (x - 1)q(x),$$

where $q(x)$ is another polynomial of order 2. But what is $q(x)$? There is a procedure to find it called polynomial division.

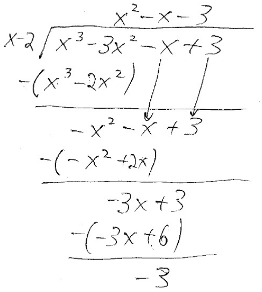


We can factor further in this case.

We see that $x^3 - 3x^2 - x + 3 = (x-1)(x^2 - 2x - 3)$. Can the quadratic be factored? One can always apply the quadratic formula to check for a quadratic. In this case, $x = 3$ and $x = -1$ are roots, and

$$p(x) = x^3 - 3x^2 - x + 3 = (x-1)(x-3)(x+1).$$

We could have divided $p(x)$ by any linear or quadratic polynomial, but in general there will be a remainder, e.g.



We may express polynomial division in an analogous way in general.

As a result from the last example, we may say

$$\frac{x^3 - 3x^2 - x + 3}{x - 2} = x^2 - x - 3 + \frac{-3}{x - 2}.$$

In this case, -3 is called the remainder. If you divide a polynomial by a *factor* of that polynomial, then the remainder will be zero.

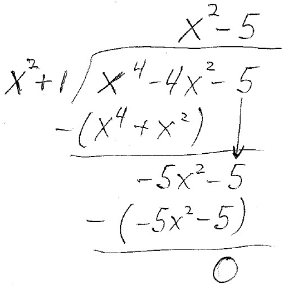
In fact, we will generally have

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)},$$

- $P(x)$: higher-order polynomial being divided by $D(x)$.
- $D(x)$: the divisor.
- $Q(x)$: the quotient.
- $R(x)$: the remainder.

We can also divide by a higher-order divisor.

Example L10.1: Divide $p(x) = x^4 - 4x^2 - 5$ by $d(x) = x^2 + 1$.
Solution: It turns out that $d(x)$ is a factor of $p(x)$;



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We can narrow down where to look for roots when the polynomial has rational coefficients.

Rational Root Test

1. If needed, multiply the polynomial through by the smallest integer necessary to make all coefficients into integers.
2. Call the constant term a and the lead coefficient b.
3. Then a rational number p/q is a possible root if p divides a and q divides b.

Example L10.2: Find all possible rational roots of

$$x^4 - \frac{7}{4}x^3 - \frac{29}{8}x^2 + \frac{7}{4}x - \frac{3}{2}.$$

Solution: First, multiply through by 8 to get

$$8x^4 - 14x^3 - 29x^2 + 14x - 12.$$

Then p/q are all possible integer divisors of $a = -12$ divided by all possible integer divisors of $b = 8$:

$$\frac{p}{q} = \frac{\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\ \pm 6,\ \pm 12}{\pm 1,\ \pm 2,\ \pm 4,\ \pm 8}.$$

A less daunting example...

Example L10.3: Find all possible rational roots of
$f(x) = x^3 - \frac{1}{3}x^2 - x + \frac{1}{3}$.
Solution: First multiply through by 3...

$$3f(x) = 3x^3 - x^2 - 3x + 1.$$

Now list all possible roots as all integer divisors of 1 divided by those of 3:

$$\frac{p}{q} = \frac{\pm 1}{\pm 1,\ \pm 3} = \pm 1,\ \pm\frac{1}{3}.$$

This yields only four possibilities.

- Check by plugging them in and see if you get zero.
- Generally, proceed with the easiest possibilities first.

$$3f(1) = 0,\quad 3f(-1) = 0,\quad 3f(1/3) = 0,\quad 3f(-1/3) = 16/27.$$

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If you can find a single zero, then upon factoring you may narrow down the remaining potential candidates and simultaneously get the factored form for a polynomial.

Example L10.4: Factor $p(x) = 2x^3 - \frac{5}{2}x^2 - \frac{23}{2}x + 3$.
Solution: Multiply through by 2;

$$2p(x) = 4x^3 - 5x^2 - 23x + 6.$$

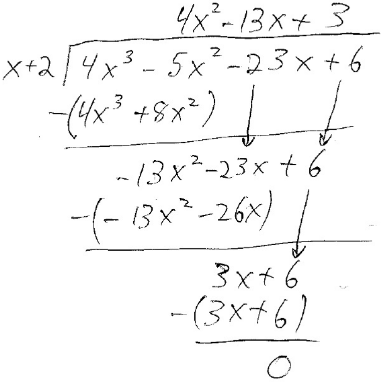
Thus the candidate rational roots are

$$\frac{p}{q} = \frac{\pm 1, 2, 3, 6}{\pm 1, 2, 4}$$

It turns out $x = -2$ works; plug it in...

$$\begin{aligned} 2p(-2) &= 4(-2)^3 - 5(-2)^2 - 23(-2) + 6 \\ &= -4 \cdot 8 - 5 \cdot 4 + 46 + 6 = -52 + 52 = 0. \end{aligned}$$

Divide $2p(x)$ by $x + 2$...



We see that $2p(x) = (x + 2)(4x^2 - 13x + 3)$. If the remaining factor were higher-order rather than quadratic, we could just use the rational root test for it to find another root, but at this point we apply the quadratic formula;

Apply the quadratic formula

$$x = \frac{13 \pm \sqrt{13^2 - 4 \cdot 3 \cdot 4}}{8} = \frac{13 \pm \sqrt{121}}{8} = \frac{13 \pm 11}{8}$$
$$\Rightarrow x = \frac{1}{4}, \quad \text{or} \quad x = 3.$$

Thus we may factor $4x^2 - 13x + 3 = a(x - 1/4)(x - 3)$, where a must be the same as the lead coefficient of the quadratic on the left, so

$$4x^2 - 13x + 3 = 4(x - 1/4)(x - 3) = (4x - 1)(x - 3)$$
$$\Rightarrow 2p(x) = (x + 2)(4x - 1)(x - 3)$$
$$\Rightarrow p(x) = \frac{1}{2}(x + 2)(4x - 1)(x - 3).$$

Practice!

Problem L10.1: Factor the polynomial

$$p(x) = \frac{1}{2}x^3 + \frac{7}{3}x^2 - \frac{23}{6}x + 1.$$