# Math 1060Q: Final Exam Review

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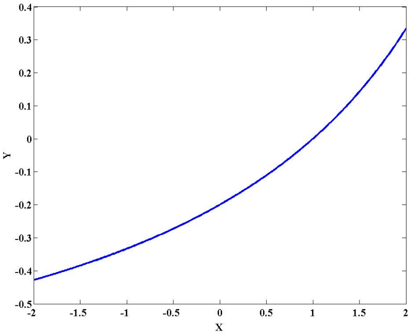
# Studying for the final exam.

- The final exam will contain 9 problems.
- The five problems covered today will cover five of the problems on the final.
- To prepare for the remaining four problems, study the two midterm exams and practice exams, along with similar problems.

# Problem 1

Let $f(x) = \frac{1-x}{x-5}$ on the domain $[-2, 2]$, as shown in the graph.




(a) What are the domain and range of $f^{-1}(x)$?
(b) Find the formula for $f^{-1}(x)$ and sketch the curve.

# Problem 1

The domain of $f^{-1}$ will be the range of $f$, which we may find by calculating $f(-2)$ and $f(2)$:

$$f(-2) = \frac{1+2}{-2-5} = -\frac{3}{7}, \quad f(2) = \frac{1-2}{2-5} = \frac{1}{3}.$$

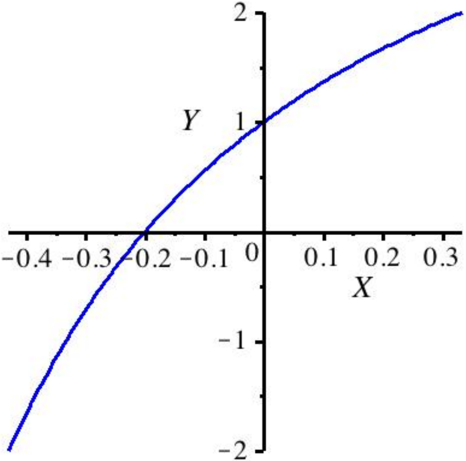
The domain of $f^{-1}$ is $[-3/7, 1/3]$ and the range is $[-2, 2]$.

To find $f^{-1}(x)$:

$$x = f(y) = \frac{1-y}{y-5} \Rightarrow x(y-5) = xy - 5x = 1 - y$$
$$\Rightarrow xy + y = 1 + 5x \Rightarrow y(x+1) = 1 + 5x$$
$$\Rightarrow y = f^{-1}(x) = \frac{1+5x}{x+1}.$$

# Problem 1

Graph of $f^{-1}(x)$:



# Problem 2

Find the exact value for each expression and fill in the table

| | |
|---|---|
| $\sin(\arcsin(1/\sqrt{2})) =$ | |
| $\cos(\arccos(1/\sqrt{3})) =$ | |
| $\arcsin(\sin(5\pi/4)) =$ | |
| $\arctan(\tan(\pi)) =$ | |
| $\arccos(\cos(\pi/6)) =$ | |

# Problem 2

The first two entries are simple, because

$$\sin(\arcsin(x)) = x \quad \text{and} \quad \cos(\arccos(x)) = x$$

are always true;

| | |
|---|---|
| $\sin(\arcsin(1/\sqrt{2})) =$ | $1/\sqrt{2}$ |
| $\cos(\arccos(1/\sqrt{3})) =$ | $1/\sqrt{3}$ |
| $\arcsin(\sin(5\pi/4)) =$ | |
| $\arctan(\tan(\pi)) =$ | |
| $\arccos(\cos(\pi/6)) =$ | |

# Problem 2

When solving $\arcsin(\sin(5\pi/4)) = y$, we seek $\sin(5\pi/4) = \sin(y)$ with $-\pi/2 \leq y \leq \pi/2$. The answer is thus $y = -\pi/4$.
Analogously, we find

| | |
|---|---|
| $\sin(\arcsin(1/\sqrt{2})) =$ | $1/\sqrt{2}$ |
| $\cos(\arccos(1/\sqrt{3})) =$ | $1/\sqrt{3}$ |
| $\arcsin(\sin(5\pi/4)) =$ | $-\pi/4$ |
| $\arctan(\tan(\pi)) =$ | $0$ |
| $\arccos(\cos(\pi/6)) =$ | $\pi/6$ |

# Problem 3

State explicitly the domain and range of each function and sketch the function:

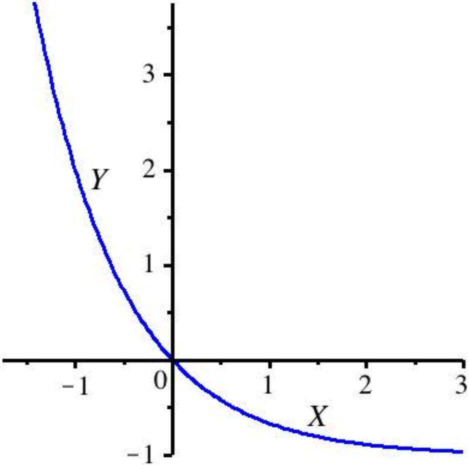
(a) $f(x) = 3^{-x} - 1$

(b) $f(x) = \log_5(x - 1)$

# Problem 3

(a) $f(x) = 3^{-x} - 1$

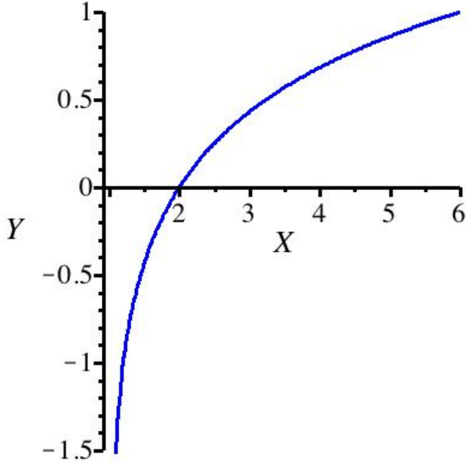
This is an exponential-type function, so the domain is $\mathbb{R}$ and the range is $(-1, \infty)$, due to the vertical shift. Graph:



# Problem 3

(b) $f(x) = \log_5(x - 1)$

The logarithmic function requires a positive argument, hence the domain is $(1, \infty)$. The range is $\mathbb{R}$. Graph:



# Problem 4

Solve the following equations for $x$:
(a) $e^{3x-1} = e^{6x+2}$
(b) $\ln(2-4x) + \ln(2x) = \ln(1-2x)$

# Problem 4

(a) $e^{3x-1} = e^{6x+2}$

The exponents must be the same:

$$3x - 1 = 6x + 2 \Rightarrow -1 - 2 = 6x - 3x \Rightarrow -3 = 3x,$$

so the answer is $x = -1$.

(b) $\ln(2 - 4x) + \ln(2x) = \ln(1 - 2x)$

Combine the logarithms on the left:

$$\ln(2 - 4x) + \ln(2x) = \ln((2 - 4x)2x) = \ln(1 - 2x).$$

# Problem 4

Now the arguments of the natural logarithm function on left and right must match:

$$(2-4x)2x = 1-2x \Rightarrow 4x - 8x^2 = 1-2x \Rightarrow 0 = 8x^2 - 6x + 1.$$

Factor or use the quadratic formula:

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(8)(1)}}{2(8)} = \frac{6 \pm \sqrt{36-32}}{16} = \frac{6 \pm 2}{16},$$

so that either $x = 1/2$ or $x = 1/4$. But note that $x = 1/2$ will not work, since in the original equation we would have an argument

$$2 - 4\left(\frac{1}{2}\right) = 0,$$

which is not in the domain of a logarithm function. The answer is just $x = 1/4$.

# Problem 5

Let the initial mass of a sample of Thorium-234 (half-life of 578 hours) be 1000 grams.
(a) Find $M(t)$, the mass of the sample at time $t > 0$.
(b) How much Thorium is left after 100 hours?
(c) What is the time when the mass has reduced to 1% of the original mass?

# Problem 5

(a) Find $M(t)$, the mass of the sample at time $t > 0$.

We apply the model

$$M(t) = 1000e^{kt},$$

with $t$ measured in hours. Half the original sample size would be 500, which is $M(578)$, so we solve

$$500 = 1000e^{578k} \Rightarrow \frac{1}{2} = e^{578k} \Rightarrow \ln\left(\frac{1}{2}\right) = 578k,$$

so $k = \ln(1/2)/578 = -\ln(2)/578$. Thus,

$$M(t) = 1000e^{-t\ln(2)/578}.$$

# Problem 5

(b) How much Thorium is left after 100 hours?

We evaluate

$$M(100) = 1000e^{-100\ln(2)/578}.$$

(c) What is the time when the mass has reduced to 1% of the original mass?

We solve

$$10 = 1000e^{-t\ln(2)/578} \Rightarrow \frac{1}{100} = 0.01 = e^{-t\ln(2)/578}.$$

# Problem 5

Take the natural logarithm of both sides of the equation:

$$\ln(0.01) = -t\frac{\ln(2)}{578} \Rightarrow t = -\frac{578\ln(0.01)}{\ln(2)}.$$

If desired, you could insert

$$\ln(0.01) = \ln(10^{-2}) = -2\ln(10)$$

so that

$$t = \frac{2 \cdot 578\ln(10)}{\ln(2)}.$$