SINGULAR INTEGRALS ON SIERPINSKI GASKETS

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Abstract. We construct a class of singular integral operators associated with homogeneous Calderón-Zygmund standard kernels on d-dimensional, $d < 1$, Sierpinski gaskets E_d. These operators are bounded in $L^2(\mu_d)$ and their principal values diverge μ_d almost everywhere, where μ_d is the natural (d-dimensional) measure on E_d.

1. Introduction

Given a Radon measure μ on $\mathbb{R}^n$ and a continuously differentiable kernel $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y) : x = y\} \to \mathbb{R}$ that satisfies the antisymmetry condition

$$K(x,y) = -K(y,x) \text{ for } x,y \in \mathbb{R}^n,\ x \neq y,$$

the singular integral operator T associated with K and μ is formally given by

$$T(f)(x) = \int K(x,y) f(y) d\mu y.$$

Notice that the above integral does not usually exist when $x \in \text{spt}\mu$. The truncated singular integral operators T_ε, $\varepsilon > 0$;

$$T_\varepsilon(f)(x) = \int_{|x-y|>\varepsilon} K(x,y) f(y) d\mu y,$$

are considered in order to overcome this obstacle. In the same vein one considers the maximal operator T^*

$$T^*(f)(x) = \sup_{\varepsilon>0} |T_\varepsilon(f)(x)|$$

and the principal values of $T(f)$ at every $x \in \mathbb{R}^n$ which, if they exist, are given by

$$\text{p.v.}T(f)(x) = \lim_{\varepsilon \to 0} T_\varepsilon(f)(x).$$

The singular integral operator T associated with μ and K is said to be bounded in $L^2(\mu)$ if there exists some constant $C > 0$ such that for $f \in L^2(\mu)$ and $\varepsilon > 0$

$$\int |T_\varepsilon(f)|^2 d\mu \leq C \int |f|^2 d\mu.$$

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The natural question as to whether the $L^2(\mu)$-boundedness of the operator T forces its principal values to exist μ almost everywhere has been considered in many papers (see e.g. [MM], [MMV], [T], [D2], [Hu], [MV] and [Pr]). Even when μ is an m-dimensional Ahlfors-David (AD) regular measure in $\mathbb{R}^n$:

$$C^{-1}r^m \leq \mu(B(x,r)) \leq Cr^m \text{ for } x \in \text{spt}\mu, 0 < r < \text{diam}(\text{spt}\mu),$$

and K is any of the coordinate Riesz kernels:

$$R_i^m(x,y) = \frac{x_i - y_i}{|x-y|^{m+1}} \text{ for } i = 1, ..., n$$

the question remains open for $m > 1$.

When $m = 1$, or equivalently in the case of the Cauchy transform, the above question has a positive answer by the results of Mattila, Melnikov and Verdera (see [MM] and [MMV]). Later on, in [T], Tolsa improved the afore mentioned results by removing the Ahlfors-David regularity assumption.

In different settings the answer to the above question can be negative. Let C be the 1-dimensional four corners Cantor set and μ its natural (1-dimensional Hausdorff) measure. David in [D2], constructed Calderón-Zygmund standard, or simply CZ standard, kernels that define operators bounded in $L^2(\mu)$ whose principal values fail to exist μ almost everywhere. Although David's kernels can be chosen odd or even are not homogeneous of degree -1.

In this note we consider classical plane Sierpinski gaskets of Hausdorff dimension d, $0 < d < 1$. For each of these d-AD regular sets E_d, we find families of CZ standard, smooth, and antisymmetric kernels of the form

$$K(x,y) = \frac{\Omega((x-y)/|x-y|)}{h(|x-y|)} \tag{1.1}$$

where h is some increasing C^∞ function satisfying the homogeneity condition

$$h(r) \simeq r^d$$

for $0 < r < \text{diam}(E_d)$ and Ω is odd on the unit circle S^1. If μ_d is the restriction of the d-dimensional Hausdorff measure on E_d, these kernels define singular integral operators bounded in $L^2(\mu_d)$ whose principal values diverge μ_d almost everywhere. The proof is based on the $T(1)$-theorem of David and Journé, proved in [DJ], and the symmetry properties of Sierpinski gaskets allowing heavy cancelations.

Remark 1.1. Unfortunately if in the above kernels we replace the function $h(r)$ by r^d, where in this case the kernels would be d-homogeneous in the classical sense, we cannot say if the corresponding operators are bounded (or not) in L^2. However their principal values diverge μ_d almost everywhere, as the proof of Section 4 goes through with no changes.

Remark 1.2. Modified slightly, the proof can be applied to many other symmetric self similar sets, e.g. the four corners Cantor sets with Hausdorff dimension less than 1 or the self similar sets discussed in [D2]. The dimensional restriction is essential for the proof and it is not known to us if there exist CZ standard kernels, of the same form as in (1.1), satisfying the homogeneity condition $h(r) \approx r$, that define singular integral operators bounded in L^2 but whose principal values diverge almost everywhere.

2. Notation and Setting

Let $\lambda \in (0, 1/3)$ and consider the following three similitudes (depending on λ) $s_1^\lambda, s_2^\lambda, s_3^\lambda : \mathbb{R}^2 \to \mathbb{R}^2$

- $s_1^\lambda(x, y) = \lambda(x, y)$
- $s_2^\lambda(x, y) = \lambda(x, y) + (1 - \lambda, 0)$
- $s_3^\lambda(x, y) = \lambda(x, y) + (\frac{1-\lambda}{2}, \frac{\sqrt{3}}{2}(1 - \lambda))$.

Let $I = \{1, 2, 3\}$ and $I^* = \bigcup\limits_{n \geq 1} I^n$. The set I^* can be partially ordered in the following way, for $\alpha, \beta \in I^*$,

$$\alpha \prec \beta \Leftrightarrow \alpha \in I^n, \beta \in I^k \; k \geq n \text{ and } \beta\lfloor n = \alpha.$$

Where $\beta\lfloor n$ denotes the restriction of β in its first n coordinates. For $\alpha \in I^n$, say $\alpha = (i_1, ..., i_n)$, define $s_\alpha^\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ through iteration

$$s_\alpha^\lambda = s_{i_1}^\lambda \circ s_{i_2}^\lambda \circ ... \circ s_{i_n}^\lambda.$$

Let A be the equilateral triangle with vertices $(0, 0), (1, 0), (1/2, \sqrt{3}/2)$. Denote $s_\alpha^\lambda(A) = S_\alpha^\lambda$, $I^0 = \{0\}$ and $s_0^\lambda = id$. The limit set of the iteration

$$E_\lambda = \bigcap_{j \geq 0} \bigcup_{\alpha \in I^j} S_\alpha^\lambda$$

is self similar and in fact it is an λ-Sierpinski triangle with Hausdorff dimension

$$d_\lambda = \dim_{\mathcal{H}} E_\lambda = -\frac{\log 3}{\log \lambda}.$$

Notice that for $\lambda \in (0, 1/3)$, $d_\lambda \in (0, 1)$. As a general property of self similar sets the measures $\mu_\lambda = \mathcal{H}^{d_\lambda}\lfloor E_\lambda$ are d_λ-AD regular. Hence there exists a constant C_λ, depending only on λ, such that for $x \in E_\lambda$ and $0 < r \leq 1$,

$$C_\lambda^{-1} r^{d_\lambda} \leq \mu_\lambda(B(x, r)) \leq C_\lambda r^{d_\lambda}.$$

The spaces $(E_\lambda, \rho, \mu_\lambda)$, where ρ is the usual Euclidean metric, are simple examples of spaces of homogeneous type (See [Ch] for definition). We want to find Calderón-Zygmund standard kernels on $E_\lambda \times E_\lambda \setminus \{(x, y) : x = y\}$ that define bounded singular integral operators on $L^2(\mu_\lambda)$. In that direction, for $\lambda \in (0, 1/3)$, we need to define two auxiliary families of functions.

The functions Ω_λ: For any pair $(x,y)\in\mathbb{R}^2\times\mathbb{R}^2$, $x\neq y$, denote by $\theta_{(x,y)}\in[0,2\pi)$ the angle formed by the vectors $y-x$ and $e_1=(1,0)$. For every $\lambda\in(0,1/3)$ there exists some positive number ε_λ such that

(i) For all $x,y\in E_\lambda$, $x\neq y$
$$\theta_{(x,y)}\in(\frac{k\pi}{3}-\varepsilon_\lambda,\frac{k\pi}{3}+\varepsilon_\lambda)\text{ for some }k\in\{0,1,..,5\}.$$
(ii) The intervals $(\frac{k\pi}{3}-\varepsilon_\lambda,\frac{k\pi}{3}+\varepsilon_\lambda)$ are disjoint for $k\in\{0,1,..,5\}$.
(iii) For any $n\in\mathbb{N}=\{1,2,...\}$ and $\alpha,\beta,\gamma\in I^n$, $\alpha\neq\beta\neq\gamma$, such that $\alpha\lfloor n-1=\beta\lfloor n-1=\gamma\lfloor n-1$:
  (a) If $x\in S^\lambda_\alpha$, $y\in S^\lambda_\beta$, $z\in S^\lambda_\gamma$ and $\theta_{(x,y)}\in(\frac{k\pi}{3}-\varepsilon_\lambda,\frac{k\pi}{3}+\varepsilon_\lambda)$, for some $k\in\{0,1,...,5\}$, then $\theta_{(x,z)}\in(\frac{m\pi}{3}-\varepsilon_\lambda,\frac{m\pi}{3}+\varepsilon_\lambda)$ for $m=(k+1)\text{mod}6$ or $m=(k-1)\text{mod}6$.
  (b) If $x,z\in S^\lambda_\alpha$, $y\in S^\lambda_\beta$ and $\theta_{(x,y)}\in(\frac{k\pi}{3}-\varepsilon_\lambda,\frac{k\pi}{3}+\varepsilon_\lambda)$ then $\theta_{(z,y)}\in(\frac{k\pi}{3}-\varepsilon_\lambda,\frac{k\pi}{3}+\varepsilon_\lambda)$ as well.

Now we can define C^∞ functions Ω_λ on S^1 satisfying
(i) $\Omega_\lambda(z)=(-1)^k$ for $\theta_{(z,0)}\in(\frac{k\pi}{3}-\varepsilon_\lambda,\frac{k\pi}{3}+\varepsilon_\lambda)$, $k\in\{0,1,...,5\}$,
(ii) $\Omega_\lambda(-z)=-\Omega_\lambda(z)$ for every $z\in S^1$.

Observe that the second condition also implies
$$\int_{S^1}\Omega_\lambda(z)d\sigma z=0$$
where σ is the normalized surface measure on S^1.

The functions h_λ: Fix some $\lambda\in(0,1/3)$, and choose any function $h_\lambda:(0,\infty)\to\mathbb{R}$ with the following properties,
(i) h_λ is C^∞,
(ii) h_λ is increasing,
(iii) $h_\lambda\lfloor[(\frac{1}{\lambda}-2)\lambda^k,\lambda^{k-1}]=\lambda^{(k-1)d_\lambda}$ for every $k\in\mathbb{N}$.

It follows that for $r\in(0,1]$, $h_\lambda(r)\approx r^{d_\lambda}$. In fact
$$r^{d_\lambda}/C_\lambda\leq h_\lambda(r)\leq C_\lambda r^{d_\lambda}\quad\text{for }0<r\leq1$$
where $C_\lambda=\lambda^{-d_\lambda}$.

Hence we are able, using the above families, to define appropriate kernels
$$K_\lambda:E_\lambda\times E_\lambda\backslash\{(x,y):x=y\}\to\mathbb{R}$$
as
$$K_\lambda(x,y)=\frac{\Omega_\lambda((x-y)/|x-y|)}{h_\lambda(|x-y|)}.$$

For the kernels K_λ there exists some constant C such that for all $x, y, z \in E_\lambda$, $x \neq y$, satisfying $|x-z| < (1-2\lambda)\,|x-y|$,

$$|K_\lambda(x,y)| \leq \frac{C}{|x-y|^{d_\lambda}}, \tag{2.1}$$

$$K_\lambda(x,y) - K_\lambda(z,y) = 0 \tag{2.2}$$

Condition (2.1) follows immediately from the definition of K_λ. To prove (2.2), let $k \in \mathbb{N}^* = \{0, 1, ..\}$ be the largest natural number such that $x, y \in S_\alpha^\lambda$ for some $\alpha \in I^k$. Therefore

$$x \in s_i^\lambda(S_\alpha^\lambda) \text{ and } y \in s_j^\lambda(S_\alpha^\lambda)$$

for some $i, j \in I$, $i \neq j$. This implies that,

$$(\frac{1}{\lambda} - 2)\lambda^{k+1} \leq |y-x| \leq \lambda^k. \tag{2.3}$$

Since $|x-z| < (1-2\lambda)\,|x-y|$ we get

$$|x-z| < (1-2\lambda)\lambda^k.$$

As

$$d(s_i^\lambda(S_\alpha^\lambda), s_q^\lambda(S_\alpha^\lambda)) = (1-2\lambda)\lambda^k \text{ for } q \in I,\ q \neq i,$$

and

$$S_\alpha^\lambda = \bigcup_{p \in I} s_p^\lambda(S_\alpha^\lambda),$$

we deduce that $z \in s_i^\lambda(S_\alpha^\lambda)$ and

$$(\frac{1}{\lambda} - 2)\lambda^{k+1} \leq |y-z| \leq \lambda^k. \tag{2.4}$$

Therefore as $x, z \in s_i^\lambda(S_\alpha^\lambda)$ and $y \in s_j^\lambda(S_\alpha^\lambda)$

$$\theta(x,y), \theta(z,y) \in \left(m\frac{\pi}{3} - \varepsilon_\lambda, m\frac{\pi}{3} + \varepsilon_\lambda\right) \tag{2.5}$$

for some $m \in \{0, 1, .., 5\}$. From (2.3), (2.4), (2.5) and the definition of h_λ and Ω_λ we deduce that

$$h_\lambda(|x-y|) = h_\lambda(|z-y|) = \lambda^{kd_\lambda},$$

and

$$\Omega_\lambda\left(\frac{x-y}{|x-y|}\right) = \Omega_\lambda\left(\frac{z-y}{|z-y|}\right).$$

Hence

$$K_\lambda(x,y) - K_\lambda(z,y) = 0$$

and by antisymmetry

$$K_\lambda(y,x) - K_\lambda(y,z) = 0.$$

It follows that the kernels K_λ are CZ standard, in fact condition (2.2) is much stronger than the ones appearing in the usual definitions of CZ standard kernels (see e.g. [Ch], [D1] and [J]).

As stated before, we want to show that the kernels define singular integral operators that are bounded in $L^2(\mu_\lambda)$, and examine their convergence properties.

3. L^2 BOUNDEDNESS

Theorem 3.1. For all $\lambda \in (0,1/3)$ the maximal singular integral operators T_λ^*,

$$T_\lambda^*(f)(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} K_\lambda(x,y) f(y)\, d\mu_\lambda y \right|,$$

are bounded in $L^2(\mu_\lambda)$.

Proof. The idea is to use the $T(1)$ theorem of David and Journé in the context of [D2]. Start by defining T_λ^n, for $n \geq 1$, as

$$T_\lambda^n(f)(x) = \int_{|x-y|>\lambda^n} K_\lambda(x,y) f(y)\, d\mu_\lambda y.$$

We want to show that $T_\lambda^n(\mathbf{1})$ =0 for all $n \in \mathbb{N}$, by induction.

For $n=1$: Let $x \in S_i^\lambda \cap E_\lambda$ for some $i \in I$. If $j,k \in I \setminus \{i\}$, $j \neq k$, we get

$$\begin{aligned} T_\lambda^1(\mathbf{1})(x) &= \int_{|x-y|>\lambda} K_\lambda(x,y) d\mu_\lambda y \\ &= \int_{S_j^\lambda \cup S_k^\lambda} K_\lambda(x,y) d\mu_\lambda y \\ &= \int_{S_j^\lambda} \frac{\Omega_\lambda((x-y)/|x-y|)}{h_\lambda(|x-y|)} d\mu_\lambda y + \int_{S_k^\lambda} \frac{\Omega_\lambda((x-y)/|x-y|)}{h_\lambda(|x-y|)} d\mu_\lambda y. \end{aligned}$$

Furthermore there exists some $m \in \{0,1,..,5\}$ such that for $y \in S_j^\lambda$,

$$\Omega_\lambda\left(\frac{x-y}{|x-y|}\right) = (-1)^m$$

and for $y \in S_k^\lambda$,

$$\Omega_\lambda\left(\frac{x-y}{|x-y|}\right) = (-1)^{m+1}.$$

Hence

$$T_\lambda^1(\mathbf{1})(x) = (-1)^m \int_{S_j^\lambda} \frac{1}{h_\lambda(|x-y|)} d\mu_\lambda y + (-1)^{m+1} \int_{S_k^\lambda} \frac{1}{h_\lambda(|x-y|)} d\mu_\lambda y$$

But for $y \in S_j^\lambda \cup S_k^\lambda$ we have that $1-2\lambda \leq |x-y| \leq 1$ and consequently $h_\lambda(|x-y|) = 1$. Thus

$$T_\lambda^1(\mathbf{1})(x) = (-1)^m \mu_\lambda(S_j^\lambda) + (-1)^{m+1} \mu_\lambda(S_k^\lambda) = 0.$$

Suppose that $T_\lambda^n(\mathbf{1}) = 0$ and let some $x \in E_\lambda$. We want to show that $T_\lambda^{n+1}(\mathbf{1})(x) = 0$. Let $x \in S_\alpha^\lambda$ for some $\alpha = (i_1, i_2, .., i_n, i_{n+1}) \in I^{n+1}$. If $\beta = (i_1, i_2, ..i_n, j)$ and $\gamma = (i_1, i_2, ..i_n, k)$ for $j, k \in I \setminus \{i_{n+1}\}$, $j \neq k$,

$$\begin{aligned} T_\lambda^{n+1}(\mathbf{1})(x) &= \int_{|x-y|>\lambda^{n+1}} K_\lambda(x,y) d\mu_\lambda y \\ &= \int_{|x-y|>\lambda^n} K_\lambda(x,y) d\mu_\lambda y + \int_{S_\beta^\lambda} K_\lambda(x,y) d\mu_\lambda y + \int_{S_\gamma^\lambda} K_\lambda(x,y) d\mu_\lambda y \\ &= \int_{S_\beta^\lambda} \frac{\Omega_\lambda((x-y)/\left|x-y\right|)}{h_\lambda(|x-y|)} d\mu_\lambda y + \int_{S_\gamma^\lambda} \frac{\Omega_\lambda((x-y)/\left|x-y\right|)}{h_\lambda(|x-y|)} d\mu_\lambda y, \end{aligned}$$

since by the induction hypothesis

$$T_\lambda^n(\mathbf{1})(x) = \int_{|x-y|>\lambda^n} K_\lambda(x,y) d\mu_\lambda y = 0.$$

Using exactly the same argument as in the case for $n = 1$

$$\int_{S_\beta^\lambda} \frac{\Omega_\lambda((x-y)/\left|x-y\right|^{-1})}{h_\lambda(|x-y|)} d\mu_\lambda y + \int_{S_\gamma^\lambda} \frac{\Omega_\lambda((x-y)/\left|x-y\right|^{-1})}{h_\lambda(|x-y|)} d\mu_\lambda y = 0.$$

Therefore $T_\lambda^{n+1}(\mathbf{1})(x) = 0$, completing the induction. As $T_\lambda^n(\mathbf{1}) = 0$ for all $n \in \mathbb{N}$ the same holds for their transposes.

Due to the structure of the spaces $(E_\lambda, \mu_\lambda, \rho)$ the proof of the $T(1)$ theorem in this setting is essentially the same with the one appearing in [D1]. As commented in [D1] and in [D2], in order to be able to use the $T(1)$ theorem we need some suitable decomposition of dyadic type, as in [Ch] and [D1], to replace the usual dyadic cubes in $\mathbb{R}^n$. In our setting the required such family $\mathcal{R}$ consists of all the triangles appearing in every step of the iteration process, i.e.

$$\mathcal{R}_k^\lambda = \{S_\alpha^\lambda : \alpha \in I^k\} \text{ for } k \in \mathbb{N}^*.$$

and

$$\mathcal{R} = \{\mathcal{R}_k^\lambda : k \in \mathbb{N}^*\}.$$

In the assumptions of the original David-Journé $T(1)$ theorem the operators should also satisfy an extra condition the so called weak boundedness. This condition is only used in the proof, as it appears in [D1], to show that there exists some absolute constant C, such that for all dyadic cubes Q

$$\left| \int_Q T(\mathbf{1}_Q)(x) dx \right| \leq C \left|Q\right|.$$

But since the operators T_λ^n are canonically associated with antisymmetric kernels the weak boundedness comes for free, see e.g. [Ch].

Applying the $T(1)$ theorem we derive that every element of the sequence $\{T_\lambda^n\}_{n \in \mathbb{N}}$ is bounded in $L^2(\mu_\lambda)$ with bounds not depending on n. This fact enables us to extract some linear $L^2(\mu_\lambda)$-bounded operator T as a weak limit of some subsequence of $\{T_\lambda^n\}_{n\in\mathbb{N}}$. Finally using the version of Cotlar's inequality, as it is stated in ([D1], p.59), we get that there exists some constant C such that for all $f \in L^2(\mu_\lambda)$

$$T_\lambda^*(f)(x) \leq C(M_\lambda(Tf)(x) + M_\lambda(|f|^{\sqrt{2}}(x))^{\sqrt{2}}),$$

where M_λ is the Hardy-Littlewood maximal operator related to the measure μ_λ. Therefore we conclude that T_λ^* is bounded in $L^2(\mu_\lambda)$. □

4. Divergence of Principal Values

Theorem 4.1. Let $\lambda \in (0, 1/3)$. For μ_λ almost every point in E_λ the principal values of the singular integral operator T_λ do not exist.

Proof. Let $\lambda \in (0, 1/3)$, we want to show that for μ_λ a.e $x \in E_\lambda$ the limit

$$\lim_{\varepsilon\to 0}\left|\int_{\mathbb{R}^2\setminus B(x,\varepsilon)} K_\lambda(x,y)d\mu_\lambda y\right|$$

does not exist. To every $z \in E_\lambda$ assign naturally the code $(z_i)_{i\in\mathbb{N}} \in I^\infty$ such that $\{z\} = \bigcap_{i\geq 1} S^\lambda_{(z_1,..,z_i)}$ and consider the set

$$D_\lambda = \{z \in E_\lambda : z_i \neq z_{i+1} \text{ for infinitely many } i's\}\ .$$

The set D_λ had full μ_λ measure as its complement $E_\lambda \backslash D_\lambda$ is countable. In fact the set $E_\lambda\backslash D_\lambda$ consists of the vertices of every triangle S^λ_α, $\alpha \in I^*$.

Notice that there exist some $C_\lambda > 1$ and some $m_\lambda \in \mathbb{N}$ such that for every $z \in D_\lambda$ and every $i \in \mathbb{N}^*$, satisfying $z_i \neq z_{i+1}$, there exist $\beta_i(z) \in I^{i-1+m_\lambda}$ and positive numbers $R_i(z)$ with the properties,

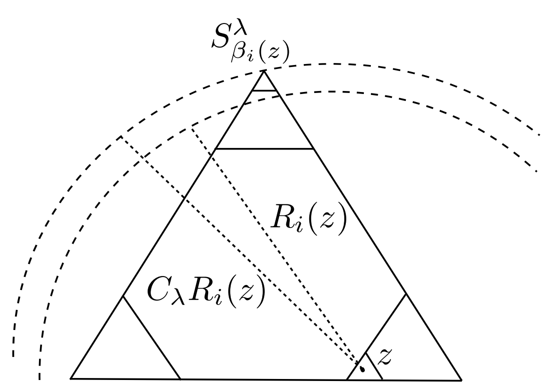

Figure A

(i) $\beta_i(z) = (z_1, ..., z_{i-1}, \overbrace{y(z), .., y(z)}^{m_\lambda \text{ times}})$ where $y(z) \in I \backslash \{z_i, z_{i+1}\}$,
(ii) $R_i(z) \approx \lambda^i$,
(iii) $(B(z, C_\lambda R_i(z)) \backslash B(z, R_i(z))) \cap E_\lambda = S^\lambda_{\beta_i(z)}$.

See also Figure A. This geometric property of the sets E_λ forces the principal values of T_λ to diverge.

To see this, let some $x \in D_\lambda$ and denote $J_x = \{i \in \mathbb{N}^* : x_i \neq x_{i+1}\}$. For all $i \in J_x$,

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^2 \backslash B(x, R_i(x))} K_\lambda(x,y) d\mu_\lambda y - \int_{\mathbb{R}^2 \backslash B(x, C_\lambda R_i(x))} K_\lambda(x,y) d\mu_\lambda y \right| \\
&= \left| \int_{B(x, C_\lambda R_i(x)) \backslash B(x, R_i(x))} \frac{\Omega_\lambda((x-y)\,|x-y|^{-1})}{h_\lambda(|x-y|)} d\mu_\lambda y \right| \\
&= \left| \int_{S^\lambda_{\beta_i(x)}} \frac{\Omega_\lambda((x-y)\,|x-y|^{-1})}{h_\lambda(|x-y|)} d\mu_\lambda y \right|
\end{aligned}
$$

For all $x \in S^\lambda_{a_i(x)}$ and $y \in S^\lambda_{\beta_i(x)}$, where $\alpha_i(x) = (x_1, ..., x_i, x_{i+1})$,

$$(1 - 2\lambda)\lambda^{i-1} \leq |x - y| \leq \lambda^{i-1}$$

and

$$\Omega_\lambda \left(\frac{x-y}{|x-y|} \right) = (-1)^{\varepsilon_i}$$

where $\varepsilon_i = 1$ or $\varepsilon_i = -1$. Hence

$$
\begin{aligned}
\left| \int_{S^\lambda_{\beta_i(x)}} \frac{\Omega_\lambda((x-y)\,|x-y|^{-1})}{h_\lambda(|x-y|)} d\mu_\lambda y \right| &= \int_{S^\lambda_{\beta_i(x)}} \frac{1}{h_\lambda(|x-y|)} d\mu_\lambda y \\
&= \frac{\mu_\lambda(S^\lambda_{\beta_i(x)})}{(\lambda^{i-1})^{d_\lambda}} \\
&= \frac{(\lambda^{i-1+m_\lambda})^{d_\lambda}}{(\lambda^{i-1})^{d_\lambda}} = \lambda^{m_\lambda d_\lambda}.
\end{aligned}
$$

As $R_i(x) \approx \lambda^i \to 0$ we conclude that the principal values of T_λ do not exist μ_λ a.e. □

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