Singular integrals on self-similar subsets of metric groups

Vasilis Chousionis and Pertti Mattila

Abstract In this paper we study singular integrals on small (that is, measure zero and lower than full dimensional) subsets of metric groups. The main examples of the groups we have in mind are Euclidean spaces and Heisenberg groups. We shall pay particular attention to the behaviour of singular integral operators on self-similar subsets.

1 Introduction

The general question we are interested in here is: how is the L^2-boundedness of singular integral operators related to geometric properties of the underlying sets and measures? A little more precisely, in some space, say d-dimensional space in terms of Hausdorff dimension, we study singular integral operators on s-dimensional subsets with $s < d$. The spaces we are mainly interested in, are Euclidean spaces and Heisenberg groups but we shall say something also in more general metric groups. Such questions in Euclidean spaces have been studied systematically for more than 20 years, the book [9] of David and Semmes is a good source for background information. This survey focuses mostly to our recent progress in Heisenberg groups in [5] and [6]. The general setting is the following:

We assume that (G,d) is a complete separable metric group with the following properties:

(i) The left translations $\tau_q : G \to G$,

Vasilis Chousionis
Departament of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana IL 61801
e-mail: vchous@uiuc.edu

Pertti Mattila
Department of Mathematics and Statistics, P.O. Box 68, FI-00014 University of Helsinki, Finland
e-mail: pertti.mattila@helsinki.fi

$$\tau_q(x) = q \cdot x, x \in G,$$

are isometries for all $q \in G$.

(ii) There exist dilations $\delta_r : G \to G, r > 0$, which are continuous group homomorphisms for which,

a. $\delta_1 =$ identity,
b. $d(\delta_r(x), \delta_r(y)) = rd(x,y)$ for $x, y \in G, r > 0$,
c. $\delta_{rs} = \delta_r \circ \delta_s$.

It follows that for all $r > 0$, δ_r is a group isomorphism with $\delta_r^{-1} = \delta_{\frac{1}{r}}$.

Euclidean spaces, Heisenberg groups and the more general Carnot groups are the main examples of such groups.

Let μ be a finite Borel measure on G and let $K : G \times G \setminus \{(x,y) : x = y\} \to \mathbb{R}$ be a Borel measurable kernel which is bounded away from the diagonal, i.e., K is bounded in $\{(x,y) : d(x,y) > \delta\}$ for all $\delta > 0$. The truncated singular integral operators associated to μ and K are defined for $f \in L^1(\mu)$ and $\varepsilon > 0$ as,

$$T_\varepsilon(f)(y) = \int_{G \setminus B(x,\varepsilon)} K(x,y) f(y) d\mu y,$$

and the maximal singular integral operator is defined as usual,

$$T_K^*(f)(x) = \sup_{\varepsilon > 0} |T_\varepsilon(f)(x)|.$$

For a vector-valued kernel $K = (K_1, \dots, K_l)$ we define

$$T_K^*(f)(x) = \max_{1 \le j \le l} \{T_{K_j}^*(f)(x)\}.$$

Then T_K^* bounded in $L^2(\mu)$ means that

$$\int T_K^*(f)^2 d\mu \le C \int |f|^2 d\mu \text{ for all } f \in L^2(\mu).$$

We are particularly interested in the following class of kernels.

Definition 1. For $s > 0$ the s-homogeneous kernels are of the form,

$$K_\Omega(x,y) = \frac{\Omega(x^{-1} \cdot y)}{d(x,y)^s}, \; x, y \in G \setminus \{(x,y) : x = y\},$$

where $\Omega : G \to \mathbb{R}$ is a continuous and homogeneous function of degree zero, that is,

$$\Omega(\delta_r(x)) = \Omega(x) \text{ for all } x \in G, r > 0.$$

We shall discuss results saying that such maximal singular integral operators are often unbounded on fractal type sets. We shall mostly restrict to s-dimensional

Ahlfors-David regular, briefly s-regular, Borel measures μ, which means that for some positive and finite constant C,

$$r^s/C \leq \mu(B(x,r)) \leq Cr^s \text{ for all } x \in \operatorname{spt}\mu, 0 < r < \mathrm{d}(\operatorname{spt}\mu).$$

Here $B(x,r)$ is the closed ball with centre x and radius r, and $\mathrm{d}(E)$ denotes the diameter of E. A closed set E is called s-regular if the s-dimensional Hausdorff measure $\mathscr{H}^s\lfloor E$ restricted to E is s-regular.

First we shall review briefly some of the Euclidean results. Recent surveys are [24] and [15].

2 The one-dimensional case

We start with the following result from [MMV] for 1-dimensional sets. It characterizes geometrically the 1-regular measures on which the singular integral operator related to the 1-dimensional Riesz kernel

$$R_1(x) = x/|x|^2, x \in \mathbb{R}^n,$$

is bounded in $L^2(\mu)$. Note that in the complex plane this kernel is essentially the Cauchy kernel $1/z = \bar{z}/|z|^2$.

Theorem 1. *Let μ be a 1-regular measure in $\mathbb{R}^n$. The following two conditions are equivalent.*

(i) $T^*_{R_1}$ is bounded in $L^2(\mu)$
(ii) $\operatorname{spt}\mu \subset \Gamma$ where Γ is a curve with $\mathscr{H}^1(\Gamma \cap B(x,r)) \leq Cr$ for all $x \in \mathbb{R}^n$ and for all $r > 0$.

The key for the proof was the following identity found by Melnikov in [M] for $z_1, z_2, z_3 \in \mathbb{C}$:

$$c(z_1,z_2,z_3)^2 = \sum_{\sigma} \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}}, \tag{1}$$

where σ runs through all six permutations of 1,2 and 3, and $c(z_1,z_2,z_3)$ is the reciprocal of the radius of the circle passing through z_1, z_2 and z_3. It is called the Menger curvature of this triple. It vanishes exactly when the three points lie on the same line. In general it measures how far they are from being collinear. Melnikov and Verdera used this identity to give a new proof for the boundedness of the Cauchy singular integral operator on Lipschitz graphs in [19]. Integrating the above identity with respect to all three variables and using Fubini's theorem, one can prove Theorem 1 by proving that the conditions (i) and (ii) are both equivalent to

$$\int_B \int_B \int_B c(x,y,z)^2 d\mu x d\mu y d\mu z \leq Cd(B)$$

for all balls $B \subset \mathbb{R}^n$ and for all $r > 0$.

The identity (1) connects the sum over permutations, which is a kind of symmetrization over the three variables, to a nice geometric object. But already the fact that this sum is non-negative is unexpected and useful. The proof of the identity is an exercise.

In the plane, Theorem 1 remains valid if the kernel R_1 is replaced by any of its coordinate parts $x_1/|x|$ or $x_2/|x|, x = (x_1, x_2) \in \mathbb{R}^2$, because the symmetrization method described earlier works in this case as well. Recently, in [4], Theorem 1 was extended to all kernels $k_n(x) = x_1^{2n-1}/|x|^{2n}$, $n \in \mathbb{N}$. It should be noted that the proof in [4] also depends on some good symmetrization properties of the kernels k_n.

Based on earlier work of many people Theorem 1 gives the following corollary:

Corollary 1. *Let E be a compact 1-regular subset of the complex plane. The following three conditions are equivalent.*

(i) E is removable for bounded analytic functions.
(ii) E is removable for Lipschitz harmonic functions.
(iii) E is purely unrectifiable.

Here the pure unrectifiablity of E means that E meets every rectifiable curve in zero length. The removability of E for bounded analytic functions means that if E is contained in an open set U, any bounded analytic function in $U \setminus E$ can be extended analytically to U. The removability for Lipschitz harmonic functions is analogous, but since Lipschitz functions on $U \setminus E$ can be uniquely extended as Lipschitz functions, (ii) means that any Lipschitz function in U which is harmonic in $U \setminus E$ is harmonic in U.

David showed later in [7] that instead of AD-regularity it is enough to assume that E has finite 1-dimensional Hausdorff measure. Still later Tolsa gave in [23] a characterization of removability for all compact subsets of the complex plane in terms of Menger curvature. A consequence of this is that (i) and (ii) in the above corollary are equivalent for any compact set E. An amusing feature is that nobody knows how to prove this without going through the Menger curvature characterization. For a survey, see [24] or [20]. Tolsa's result is

Theorem 2. *Let E be a compact subset of the complex plane. The following three conditions are equivalent.*

(i) E is not removable for bounded analytic functions.
(ii) E is not removable for Lipschitz harmonic functions.
(iii) There is a finite Borel measure μ supported in E such that $\mu(E) > 0$, $\mu(B) \leq d(B)$ for all discs B and

$$\int\int\int c(x,y,z)^2 d\mu x d\mu y d\mu z < \infty.$$

3 The higher dimensional case

The higher dimensional analogues of the above results are unknown. Let R_m be the vector-valued m-dimensional Riesz kernel;

$$R_m(x) = x/|x|^{m+1}, x \in \mathbb{R}^n.$$

Let μ be an m-regular measure and E an m-regular set in $\mathbb{R}^n$. The natural questions are: when m is an integer, is it true that

(a) $T^*_{R_m}$ is bounded in $L^2(\mu)$ if and only if $\operatorname{spt}\mu$ is uniformly rectifiable,
(b) when $m = n-1$, E is removable for Lipschitz harmonic functions if and only if E is purely unrectifiable?

The reason that the Riesz kernel $|x|^{-n}x$ appears in connection of removable sets of Lipschitz harmonic functions is that it is essentially the gradient of the fundamental solution of the Laplacian.

The m-dimensional pure unrectifiability can be defined, for example, as the property that the set intersects every m-dimensional C^1 surface in a set of zero m-dimensional measure. The uniform rectifiability is a quantitative concept of rectifiability due to David and Semmes, see [9]. For 1-dimensional sets it means exactly the condition (ii) of Theorem 2.1. It is known that the “if”-part in (a) and the “only if”-part in (b) are true. Some partial results for the converse can be found in [17], [14] and [12]; they are discussed also in the book [13]. The main problem for the converse is to prove that boundedness such as in (a) implies some sort of rectifiability. One characterization of the rectifiability of E is approximation of E with m-dimensional planes almost everywhere at all small scales. The partial results referred to above are in the spirit that the boundedness implies such approximation almost everywhere at some, but maybe not all, small scales. Such partial results hold also in Heisenberg groups and we shall below formulate them more precisely there.

One can also consider the Riesz kernels when m in not an integer. Vihtilä showed in [26] that then $T^*_{R_m}$ is never bounded in $L^2(\mu)$ for m-regular measures μ.

4 Self-similar sets and singular integrals

We shall now return to the general setting of Introduction. Let $\mathscr{S} = \{S_1,\dots,S_N\}, N \geq 2$, be an iterated function system (IFS) of similarities of the form

$$S_i = \tau_{q_i} \circ \delta_{r_i} \tag{2}$$

where $q_i \in G, r_i \in (0,1)$ and $i = 1,\dots,N$. The self-similar set C with respect to $\mathscr{S}$ is the unique non-empty compact set such that

$$C = \bigcup_{i=1}^{N} S_i(C).$$

If this system satisfies the strong separation condition, that is, the sets $S_i(C)$ are pairwise disjoint for $i = 1,\dots,N$, it follows by a general metric space result of Schief in [21] (which holds also under the open set condition) that

$$0 < \mathscr{H}^s(C) < \infty \text{ for } \sum_{i=1}^{N} r_i^s = 1,$$

and the Hausdorff measure $\mathscr{H}^s \lfloor C$ is d-regular.

The following result was proved in [6]:

Theorem 3. *Let $\mathscr{S} = \{S_1,\dots,S_N\}$ be an iterated function system in G satisfying the strong separation condition, let C be the corresponding s-dimensional self-similar set, and let K_Ω be an s-homogeneous kernel. If there exists a fixed point x for some $S_{i_1} \circ \cdots \circ S_{i_k}$; $S_{i_1} \circ \cdots \circ S_{i_k}(x) = x$, such that*

$$\int_{C \setminus S_{i_1} \circ \cdots \circ S_{i_k}(C)} K_\Omega(x,y)\, d\mathscr{H}^s y \neq 0,$$

then the maximal operator $T^*_{K_\Omega}$ is unbounded in $L^2(\mathscr{H}^s \lfloor C)$, moreover $\|T^*_{K_\Omega}(1)\|_{L^\infty(\mathscr{H}^s \lfloor C)} = \infty$.

Remark 1. Since such fixed points are dense in C, we have infinitely many points in a dense set and it suffices to check the condition at any one of them. Even when the ambient space is Euclidean, Theorem 3 provides new information about the behavior of general homogeneous singular integrals on self-similar sets. For any kernel $K_\Omega(x) = \frac{\Omega(x/|x|)}{|x|^s}, x \in \mathbb{R}^n \setminus \{0\}, s \in (0,n)$, where Ω is continuous, one can easily find Sierpiński-type s-dimensional self-similar sets C_s for which one can check using Theorem 3 that the corresponding operator $T^*_{K_\Omega}$ is unbounded. For example it follows that the operator associated to the kernel $z^3/|z|^4, z \in \mathbb{C} \setminus \{0\}$, is unbounded on many simple 1-dimensional self-similar sets. In the case of the Sierpiński gasket this is immediate while in the case of the $1/4$-Cantor set it requires more computational work and it was checked after compiling a computer program. In [11], Huovinen considered such kernels in the plane and he proved that the a.e. existence of principal values of operators associated to any kernel $\frac{z^{2n-1}}{|z|^{2n}}$, for $n \geq 1$ implies rectifiability.

5 Self-similar sets in Heisenberg groups

For an introduction to Heisenberg groups and some of the facts mentioned below, see for example [2] or [1]. Below we state the basic facts needed in this survey.

The Heisenberg group $\mathbb{H}^n$, identified with $\mathbb{R}^{2n+1}$, is a non-abelian group where the group operation is given by

$$p \cdot q = (p_1 + q_1, \dots, p_{2n} + q_{2n}, p_{2n+1} + q_{2n+1} + A(p,q)),$$

where

$$A(p,q) = -2\sum_{i=1}^{n}(p_i q_{i+n} - p_{i+n} q_i).$$

We will denote points $p \in \mathbb{H}^n$ by $p = (p', p_{2n+1}), p' \in \mathbb{R}^{2n}, p_{2n+1} \in \mathbb{R}$. For any $q \in \mathbb{H}^n$ and $r > 0$, let again $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$ be the left translation

$$\tau_q(p) = q \cdot p,$$

and define the dilation $\delta_r : \mathbb{H}^n \to \mathbb{H}^n$ by

$$\delta_r(p) = (rp_1, \dots, rp_{2n}, r^2 p_{2n+1}).$$

A natural metric d on $\mathbb{H}^n$ is defined by

$$d(p,q) = \|p^{-1} \cdot q\|$$

where

$$\|p\| = (\|(p_1, \dots, p_{2n})\|^4_{\mathbb{R}^{2n}} + p^2_{2n+1})^{\frac{1}{4}}.$$

The metric is left invariant, that is $d(q \cdot p_1, q \cdot p_2) = d(p_1, p_2)$, and the dilations satisfy $d(\delta_r(p_1), \delta_r(p_2)) = rd(p_1, p_2)$. All the conditions of the general setting of Introduction are satisfied.

A subgroup G of $\mathbb{H}^n$ is called homogeneous if it is closed and invariant under the dilations; $\delta_r(G) = G$ for all $r > 0$. Every homogeneous subgroup G is a linear subspace of $\mathbb{R}^{2n+1}$. We call G a k-subgroup if its linear dimension is k. The homogeneous subgroups fall into two categories, vertical and horizontal: the vertical homogeneous k-subgroups are the linear subspaces of $\mathbb{R}^{2n+1}$ of the form $V \times \mathbb{T}$ where V is a $(k-1)$-dimensional linear subspace of $\mathbb{R}^n$ and $\mathbb{T}$ is the t-, that is, p_{2n+1}-axis. Their Hausdorff dimension is $k+1$. The horizontal homogeneous k-subgroups are those k-dimensional linear subspaces of $\mathbb{R}^{2n}$ on which A vanishes identically. Their Hausdorff dimension is k. The Haar measure on a k-subgroup is just the k-dimensional Lebesgue measure on it. We denote the set of these measures by $\mathscr{H}(n,k)$.

In this section we consider certain families of self-similar sets in $\mathbb{H}^n$ and we discuss their relations with Riesz-type transforms.

Definition 2. Let $Q = [0,1]^{2n} \subset \mathbb{R}^{2n}$ and $r \in (0, \frac{1}{2})$. Let $z_j \in \mathbb{R}^{2n}, j = 1, \dots, 2^{2n}$, be distinct points such that $z_{j,i} \in \{0, 1-r\}$ for all $j = 1, \dots, 2^{2n}$ and $i = 1, \dots, 2n$. We consider the following 2^{2n+2} similitudes depending on the parameter r,

$$S_j = \tau_{(z_j,0)}\delta_r, \text{ for } j = 1,\dots,2^{2n},$$
$$S_j = \tau_{(z_{\lfloor j\rfloor_{2^{2n}}},\frac{1}{4})}\delta_r, \text{ for } j = 2^{2n}+1,\dots,2\cdot 2^{2n},$$
$$S_j = \tau_{(z_{\lfloor j\rfloor_{2\cdot 2^{2n}}},\frac{1}{2})}\delta_r, \text{ for } j = 2\cdot 2^{2n}+1,\dots,3\cdot 2^{2n},$$
$$S_j = \tau_{(z_{\lfloor j\rfloor_{3\cdot 2^{2n}}},\frac{3}{4})}\delta_r, \text{ for } j = 3\cdot 2^{2n}+1,\dots,2^{2n+2},$$

where $\lfloor j\rfloor_m := j \mod m$ and $1 \le \lfloor j\rfloor_m \le m$.

Theorem 4. *Let $r \in (0,\frac{1}{2})$ and $\mathscr{S}_r = \{S_1,\dots,S_{2^{2n+2}}\}$ where the S_j's are the similitudes of Definition 2. Let K_r be the self-similar set defined by*

$$K_r = \bigcup_{j=1}^{2^{2n+2}} S_j(K_r).$$

Then the the sets $S_j(K_r)$ are disjoint for $j = 1,\dots,2^{2n+2}$, and

$$0 < \mathscr{H}^s(K_r) < \infty \text{ with } s = \frac{(2n+2)\log(2)}{\log(\frac{1}{r})}.$$

We give a sketch of the proof. It is similar to the one given by Strichartz in [22] in the case $r = 1/2$. He obtains then a fractal tiling of $\mathbb{H}^n$. It is enough to find some set $R \supset K$ such that for all $j = 1,..,2^{2n+2}$,

(i) $S_j(R) \subset R$ and
(ii) the sets $S_j(R)$ are disjoint.

This is established by finding a continuous function $\varphi : Q \to \mathbb{R}$ such that the set

$$R = \{q \in \mathbb{H}^n : q' \in Q \text{ and } \varphi(q') \le q_{2n+1} \le \varphi(q') + 1\}$$

satisfies (i) and (ii).

This will follow immediately if we find some continuous $\varphi : Q \to \mathbb{R}$ which satisfies for all $j = 1,\dots,2^{2n}$,

$$\tau_{(z_j,0)}\delta_r(R) = \{q \in \mathbb{H}^n : q' \in Q_j \text{ and } \varphi(q') \le q_{2n+1} \le \varphi(q') + r^2\}, \tag{3}$$

where $Q_j = \tau_{(z_j,0)}(\delta_r(Q))$. Since

$$\tau_{(z_j,0)}\delta_r(R) = \{p \in \mathbb{H}^n : p' \in Q_j \text{ and } r^2\varphi(\frac{p'-z_j}{r}) - 2\sum_{i=1}^{n}(z_{j,i}p_{i+n} - z_{j,i+n}p_i) \le p_{2n+1}$$
$$\le r^2\varphi(\frac{p'-z_j}{r}) - 2\sum_{i=1}^{n}(z_{j,i}p_{i+n} - z_{j,i+n}p_i) + r^2\},$$

proving (3) amounts to showing that

$$\varphi(w) = r^2\varphi(\frac{w-z_j}{r}) - 2\sum_{i=1}^{n}(z_{j,i}w_{i+n} - z_{j,i+n}w_i) \text{ for } w \in Q_j, j = 1, \dots, 2^{2n}. \quad (4)$$

Such a function φ is found with an application of the Banach fixed point theorem to a contraction T satisfying

$$T(f)(w) = r^2 f(\frac{w-z_j}{r}) - 2\sum_{i=1}^{n}(z_{j,i}w_{i+n} - z_{j,i+n}w_i) \text{ for } w \in Q_j.$$

6 Riesz-type kernels in Heisenberg groups

Definition 3. The s-Riesz kernels in $\mathbb{H}^n$, $s \in (0, 2n+2)$, are defined as

$$R_s(p) = (R_{s,1}(p), \dots, R_{s,2n+1}(p))$$

where

$$R_{s,i}(p) = \frac{p_i}{\|p\|^{s+1}} \text{ for } i = 1, \dots, 2n$$

and

$$R_{s,2n+1}(p) = \frac{p_{2n+1}}{\|p\|^{s+2}}.$$

Notice that these kernels are antisymmetric,

$$R_s(p^{-1}) = (R_s(p))^{-1},$$

and s-homogeneous,

$$R_s(\delta_r(p)) = \frac{1}{r^s}(R_s(p)).$$

Let μ be a finite Borel measure in $\mathbb{H}^n$. The image $f_\#\mu$ under a map $f : \mathbb{H}^n \to \mathbb{H}^n$ is the measure on $\mathbb{H}^n$ defined by

$$f_\#\mu(A) = \mu\big(f^{-1}(A)\big) \text{ for all } A \subset \mathbb{H}^n.$$

For $a \in \mathbb{H}^n$ and $r > 0$, $T_{a,r} : \mathbb{H}^n \to \mathbb{H}^n$ is defined for all $p \in \mathbb{H}^n$ by

$$T_{a,r}(p) = \delta_{1/r}(a^{-1} \cdot p).$$

Definition 4. We say that ν is a *tangent measure* of μ at $a \in \mathbb{H}^n$ if ν is a Radon measure on $\mathbb{H}^n$ with $\nu(\mathbb{H}^n) > 0$ and there are positive numbers c_i and r_i, $i = 1, 2, \dots$, such that $r_i \to 0$ and

$$c_i T_{a,r_i\#}\mu \to \nu \text{ weakly as } i \to \infty.$$

We denote by $\mathrm{Tan}(\mu, a)$ the set of all tangent measures of μ at a.

The numbers c_i are normalization constants which are needed to keep ν non-trivial and locally finite. Often one can use $c_i = \mu(B(a, r_i))^{-1}$.

The following result was proved in [5] (recall that $\mathscr{H}(n,k)$ denotes the set of the Haar measures of the k-subgroups):

Theorem 5. *Let $s \in (0,2n+2)$ and let μ be an s-regular measure in $\mathbb{H}^n$. If $T^*_{R_s}$ is bounded in $L^2(\mu)$, then*

(i) s is an integer in $[1,2n+1]$,
(ii) for μ-a.e. $a \in \mathbb{H}^n$, the set of tangent measures of μ at a, $\mathrm{Tan}(\mu,a)$*, contains measures in $\mathscr{H}(n,s)$.*

One can show that the s-dimensional self-similar sets of Theorem 4 don't have tangent measures in $\mathscr{H}(n,s)$; they are too spread at all scales for that. This leads to

Corollary 2. *The maximal operators $T^*_{R_s}$ are unbounded in $L^2(\mathscr{H}^s\lfloor C)$ for the s-dimensional self-similar sets of Theorem 4.*

Theorem 5 corresponds to what is known in $\mathbb{R}^n$ for s-regular sets and Riesz kernels in this respect (in other respects much more is known by results of Tolsa, Volberg and others, see e.g., [25] and [10]). The disadvantage here is that the kernels are not natural in the same way as Riesz kernels in $\mathbb{R}^n$; they don't seem to relate to any function classes. Analogues of harmonic functions lead to other kernels which we look at now.

7 $\Delta_{\mathbb{H}}$-removability and singular integrals

The Lie algebra of left invariant vector fields in $\mathbb{H}^n$ is generated by

$$X_i := \partial_i + 2x_{i+n}\partial_{2n+1}, \;\; Y_i := \partial_{i+n} - 2x_i\partial_{2n+1}, \;\; T := \partial_{2n+1},$$

for $i = 1,\dots,n$. In fact, these vectorfields generate the whole group and metric structure of $\mathbb{H}^n$.

If f is a real function defined on an open set of $\mathbb{H}^n$ its $\mathbb{H}$-gradient is given by

$$\nabla_{\mathbb{H}} f = (X_1 f,\dots,X_n f, Y_1 f,\dots,Y_n f).$$

The $\mathbb{H}$-divergence of a function $\phi = (\phi_1,\dots,\phi_{2n}) : \mathbb{H}^n \to \mathbb{R}^{2n}$ is defined as

$$\operatorname{div}_{\mathbb{H}} \phi = \sum_{i=1}^{n} (X_i\phi_i + Y_i\phi_{i+n}).$$

The sub-Laplacian in $\mathbb{H}^n$ is given by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{n} (X_i^2 + Y_i^2)$$

or equivalently

$$\Delta_{\mathbb{H}} = \operatorname{div}_{\mathbb{H}} \nabla_{\mathbb{H}}.$$

Definition 5. Let $U \subset \mathbb{H}^n$ be an open set. A real valued function f on U is called $\Delta_{\mathbb{H}}$-harmonic, or simply harmonic, on U if $\Delta_{\mathbb{H}} f = 0$ on U.

We shall consider removable sets for Lipschitz solutions of the sub-Laplacian:

Definition 6. A compact set $C \subset \mathbb{H}^n$ will be called removable, or $\Delta_{\mathbb{H}}$-removable for Lipschitz $\Delta_{\mathbb{H}}$-harmonic functions, if for every open set U with $C \subset U$ and every Lipschitz function $f : U \to \mathbb{R}$,

$$\Delta_{\mathbb{H}} f = 0 \text{ in } U \setminus C \text{ implies } \Delta_{\mathbb{H}} f = 0 \text{ in } U.$$

Fundamental solutions for sub-Laplacians in homogeneous Carnot groups are defined in accordance with the classical Euclidean setting. In particular in the case of the sub-Laplacian in $\mathbb{H}^n$:

Definition 7 (Fundamental solutions). A function $\Gamma : \mathbb{R}^{2n+1} \setminus \{0\} \to \mathbb{R}$ is a fundamental solution for $\Delta_{\mathbb{H}}$ if:

(i) $\Gamma \in C^\infty(\mathbb{R}^{2n+1} \setminus \{0\})$,
(ii) $\Gamma \in L^1_{\text{loc}}(\mathbb{R}^{2n+1})$ and $\lim_{\|p\| \to \infty} \Gamma(p) \to 0$,
(iii) for all $\varphi \in C_0^\infty(\mathbb{R}^{2n+1})$,

$$\int_{\mathbb{R}^{2n+1}} \Gamma(p) \Delta_{\mathbb{H}} \varphi(p) dp = -\varphi(0).$$

It also follows easily that for every $p \in \mathbb{H}^n$,

$$\Gamma * \Delta_{\mathbb{H}} \varphi(p) = -\varphi(p) \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^{2n+1}). \tag{5}$$

Convolutions are defined as usual by

$$f * g(p) = \int f(q^{-1} \cdot p) g(q) dq$$

for $f, g \in L^1$ and $p \in \mathbb{H}^n$.

The fundamental solution Γ of $\Delta_{\mathbb{H}}$ is given by

$$\Gamma(p) = C_d \|p\|^{2-d} \text{ for } p \in \mathbb{H}^n \setminus \{0\}$$

where $d = 2n+2$ is the Hausdorff dimension of $\mathbb{H}^n$.

Let $K = \nabla_{\mathbb{H}} \Gamma$, then $K = (K_1, \dots, K_{2n}) : \mathbb{H}^n \to \mathbb{R}^{2n}$ where

$$K_i(p) = c_d \frac{p_i |p'|^2 + p_{i+n} p_{2n+1}}{\|p\|^{d+2}} \text{ and } K_{i+n}(p) = c_d \frac{p_{i+n} |p'|^2 - p_i p_{2n+1}}{\|p\|^{d+2}}, \tag{6}$$

for $i = 1, \dots, n$, $p \in \mathbb{H}^n \setminus \{0\}$ and $c_d = (2-d)C_d$. We will also use the following notation,

$$\Omega_i(p) = c_d \frac{(p_i |p'|^2 + p_{i+n} p_{2n+1})}{\|p\|^3} \text{ and } \Omega_{i+n}(p) = c_d \frac{(p_{i+n} |p'|^2 - p_i p_{2n+1})}{\|p\|^3}, \tag{7}$$

for $i=1,\dots,n$ and $p\in\mathbb{H}^n\setminus\{0\}$. Hence,

$$K_i(p)=\frac{\Omega_i(p)}{\|p\|^{d-1}} \text{ and } K(p)=\frac{\Omega(p)}{\|p\|^{d-1}}, \tag{8}$$

for $i=1,\dots,2n, \Omega=(\Omega_1,\dots,\Omega_{2n})$ and $p\in\mathbb{H}^n\setminus\{0\}$. The functions Ω_i are homogeneous and hence, recalling Definition 1, the kernels K_i are $(d-1)$-homogeneous.

The following proposition asserts that K is a standard kernel.

Proposition 1. *For all $i=1,\dots,2n$,*

(i) $|K_i(p)|\lesssim\|p\|^{1-d}$ *for* $p\in\mathbb{H}^n\setminus\{0\}$,
(ii) $|\nabla_{\mathbb{H}}K_i(p)|\lesssim\|p\|^{-d}$ *for* $p\in\mathbb{H}^n\setminus\{0\}$,
(iii) $|K_i(p^{-1}\cdot q_1)-K_i(p^{-1}\cdot q_2)|\lesssim\max\left\{\dfrac{d(q_1,q_2)}{d(p,q_1)^d},\dfrac{d(q_1,q_2)}{d(p,q_2)^d}\right\}$ *for* $q_1,q_2\neq p\in\mathbb{H}^n$.

The following theorem, which makes use of Proposition 1, was proved in [6]. With d replaced by n, it is also valid for Lipschitz harmonic functions in $\mathbb{R}^n$, as it was shown in [17].

Theorem 6. *Let C be a compact subset of $\mathbb{H}^n$.*

(i) If $\mathscr{H}^{d-1}(C)=0$, C is removable.
(ii) If $\dim C>d-1$, C is not removable.

8 $\Delta_{\mathbb{H}}$-removable self-similar Cantor sets in $\mathbb{H}^n$

In this section we consider a modified class of the self-similar Cantor sets C in $\mathbb{H}^n$ which were introduced in Section 3. Notice that there is one piece $S_0(C_{r,N})$ of $C_{r,N}$ below, which is well separated from the others. This is in order to make the condition of Theorem 3 easily checkable. It is very probable that also the more symmetric self-similar sets of Section 3 would satisfy that condition, but the calculation would become much more complicated.

Let $Q=[0,1]^{2n}\subset\mathbb{R}^{2n}, r>0, N\in 2\mathbb{N}$, be such that $r<\frac{1}{N}<\frac{1}{2}$. Let $z_j\in\mathbb{R}^{2n}, j=1,...,N^{2n}$, be distinct points such that $z_{j,i}\in\{\frac{l}{N}:l=0,1,\cdots,N-1\}$ for all $j=1,\cdots,N^{2n}$ and $i=1,..,2n$.

The similarities $\mathscr{S}_{r,N}=\{S_0,\dots,S_{\frac{1}{2}N^{2n+2}}\}$, depending on the parameters r and N, are defined as follows,

$$S_0=\delta_r,$$

$$S_j=\tau_{(z_{\lfloor j\rfloor_{N^{2n}}},\frac{1}{2}+\frac{i}{N^2})}\circ\delta_r, \text{ for } i=0,\cdots,\frac{N^2}{2}-1 \text{ and } j=iN^{2n}+1,\cdots,(i+1)N^{2n}.$$

where $\lfloor j\rfloor_m := j \mod m$.

Let $C_{r,N}$ be the self-similar set defined by

$$C_{r,N} = \bigcup_{j=0}^{\frac{1}{2}N^{2n+2}} S_j(C_{r,N}).$$

Then

$$0 < \mathscr{H}^s(C_{r,N}) < \infty \text{ with } s = \frac{\log(\frac{1}{2}N^{2n+2}+1)}{\log(\frac{1}{r})}.$$

Denote by C_{d-1} the set C_{r_{d-1},N_0} for which

$$0 < \mathscr{H}^{2n+1}(C_{r_{d-1},N_0}) < \infty.$$

Theorem 7. *The Cantor set C_{d-1} satisfies $0 < \mathscr{H}^{d-1}(C_{d-1}) < \infty$ and is removable.*

The proof of Theorem 7 can be found in [6] and to prove it one verifies the condition of the general Theorem 3.

9 Concluding comments

As discussed above, the question for what kind of 1-regular measures the singular integral operators based on the 1-dimensional Riesz kernel are L^2-bounded is solved. So are the corresponding removability questions, both even much more generally than for regular measures and sets. For other integral dimensional Riesz kernels in $\mathbb{R}^n$ and Riesz-type kernels in $\mathbb{H}^n$ we have partial results for general regular measures and sets. For other kernels, such as the gradient of the fundamental solution of the sub-Laplacian, we only know results for some special self-similar sets. A natural direction would be to proceed further with self-similar sets, studying more systematically their properties and defining conditions in relation with kernels and L^2-boundedness. The L^2-boundedness on some particular self-similar sets for kernels adapted to them was shown in [8], by David, and in [3].

Acknowledgements P.M and V.C were supported by the Academy of Finland.

References

1. Bonfiglioli, A., Lanconelli, E., and Uguzzoni, F.: Stratified Lie Groups and Potential Theory for Their Sub-Laplacians, Springer Monographs in Mathematics 2007.
2. Capogna, L., Danielli, D., Pauls, S. D., Tyson, J. T.: An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem, Birkhäuser 2007.
3. Chousionis, V.: Singular integrals on Sierpinski gaskets, Publ. Mat. 53 (2009), **1**, 245–256.
4. Chousionis, V., Mateu, J., Prat, L., Tolsa, X.: Calderón-Zygmund kernels and rectifiability in the plane, Adv. Math.**231** (1) (2012). 535–568.
5. Chousionis, V., Mattila, P.: Singular integrals on Ahlfors-David subsets of the Heisenberg group, J. Geom. Anal. **21** (2011), no. 1 , 56–77.

6. Chousionis, V., Mattila, P.: Singular integrals on self-similar sets and removability for Lipschitz harmonic functions in Heisenberg groups, to appear in J. Reine. Angew. Math.
7. David, G.: Unrectifiable 1-sets have vanishing analytic capacity, Rev. Math. Iberoam. **14** (1998) 269–479.
8. David, G.: Des intégrales singulières bornées sur un ensemble de Cantor, C. R. Acad. Sci. Paris Sr. I Math. **332** (2001), 5, 391–396.
9. David, G., Semmes, S.: Analysis of and on Uniformly Rectifiable Sets, Mathematical Surveys and Monographs, **38**. American Mathematical Society, Providence, RI, (1993).
10. Eiderman, V. Nazarov, F., Volberg, A.:Vector-valued Riesz potentials: Cartan-type estimates and related capacities, Proc. London Math. Soc. (2010) **101** (3): 727–758.
11. Huovinen, P.: Singular integrals and rectifiability of measures in the plane. Ann. Acad. Sci. Fenn. Math. Diss. **109** (1997).
12. Lorent, A.: A generalised conical density theorem for unrectifiable sets, Ann. Acad. Sci. Fenn. Math. 28 (2003), no. 2, 415–431.
13. Mattila, P.: Geometry of Sets and Measures in Euclidean Spaces, Cambridge University Press, (1995).
14. Mattila, P.: Singular integrals and rectifiability, Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000). Publ. Mat. 2002, Vol. Extra, 199–208.
15. Mattila, P.: Removability, singular integrals and rectifiability, Rev. Roumaine Math. Pures Appl. 54 (2009), **5-6**, 483–491.
16. Mattila, P., Melnikov, M. S., Verdera, J.: The Cauchy integral, analytic capacity, and uniform rectifiability, Ann. of Math. (2) **144** (1996), no. 1, 127–136.
17. Mattila, P., Paramonov P. V.: On geometric properties of harmonic Lip1-capacity, Pacific J. Math., **171:2** (1995), 469–490.
18. Melnikov, M. S.: Analytic capacity: discrete approach and curvature of a measure, Sbornik: Mathematics **186(6)** (1995), 827–846.
19. Melnikov, M. S., Verdera, J.: A geometric proof of the L^2 boundedness of the Cauchy integral on Lipschitz graphs, Internat. Math. Res. Notices (1995), 325–331.
20. Pajot, H.: Analytic capacity, rectifiability, Menger curvature and the Cauchy integral, Lecture Notes in Mathematics, Vol. 1799. Springer-Verlag, Berlin.
21. Schief, A.: Self-similar sets in complete metric spaces, Proc. Amer. Math. Soc. **124** (1996), 481–490.
22. Strichartz, R.: Self-similarity on nilpotent Lie groups, Contemp. Math. **140**, (1992) 123–157.
23. Tolsa, X.: Painlevé's problem and the semiadditivity of analytic capacity, Acta Math. **190:1** (2003), 105–149.
24. Tolsa, X.: Analytic capacity, rectifiability, and the Cauchy integral, Contemp. International Congress of Mathematicians. Vol. II, 1505–1527, Eur. Math. Soc., Zürich, 2006.
25. Tolsa, X.:Calderon-Zygmund capacities and Wolff potentials on Cantor sets. J. Geom. Anal. **21** (1) (2011), 195–223.
26. Vihtilä, M.: The boundedness of Riesz s-transforms of measures in $\mathbb{R}^n$, Proc. Amer. Math. Soc. **124**124 (1996), 481–490.