RIGOROUS HAUSDORFF DIMENSION ESTIMATES FOR CONFORMAL FRACTALS

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ABSTRACT. We develop a versatile framework which allows us to rigorously estimate the Hausdorff dimension of maximal conformal graph directed Markov systems in $\mathbb{R}^n$ for $n \geq 2$. Our method is based on piecewise linear approximations of the eigenfunctions of the Perron-Frobenius operator via a finite element framework for discretization and iterative mesh schemes. One key element in our approach is obtaining bounds for the derivatives of these eigenfunctions, which, besides being essential for the implementation of our method, are of independent interest.

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1. Introduction

Understanding and determining the Hausdorff dimension of various and diverse attractors has played a crucial role in advancing the fields of fractal geometry and dynamical systems. In particular, one of the most influential results in iterated function systems, due to Hutchinson [25], asserts that if $\mathscr{S} = \{\phi_i\}_{i=1}^k$ is a set of similitudes which satisfies the open set condition, and J is the unique compact set such that $J = \cup_{i=1}^k \phi_i(J)$ (frequently called the *limit set* or the *attractor* of $\mathscr{S}$), then $\dim_{\mathscr{H}}(J)$ is the parameter $t \in [0,\infty)$ so that

$$\sum_{i=1}^{k} r_i^t = 1, \tag{1.1}$$

where $r_i \in (0,1)$ are the contraction ratios of the maps ϕ_i.

The dimension theory of *conformal iterated function systems* (CIFS) is much more complex. In [34] Mauldin and the third named author employed thermodynamic formalism to determine the Hausdorff dimension of limit sets of CIFSs. According to [34], given a finite or countable collection of uniformly contracting conformal maps which satisfies some natural assumptions then the Hausdorff dimension of its limit set coincides with the zero of a corresponding (topological) pressure function, see Section 2 for more details. We note that that this approach traces back to the the fundamental work of Rufus Bowen [3], and frequently the zero of the previously mentioned pressure function is called the Bowen's parameter. Using Hutchinson's formula (1.1) one can determine the Hausdorff dimension of self similar sets with very high precision. However, due to the complexity of the pressure function, obtaining rigorous and effective estimates for the Hausdorff dimension of self-conformal sets is significantly subtler.

Consider for example the set of irrational numbers whose continued fraction expansion can only contain digits from a prescribed set $E \subset \mathbb{N}$, i.e.

$$J_E = \left\{[e] : e \in E^{\mathbb{N}}\right\} \quad \text{where} \quad [e] = [e_1, e_2, \dots] = \cfrac{1}{e_1 + \cfrac{1}{e_2 + \dots}}.$$

Quite conveniently, the set J_E is the limit set of the CIFS $\mathscr{CF}_E = \{\phi_e : [0,1] \to [0,1]\}_{e \in E}$, where

$$\phi_e(x) = \frac{1}{e+x}.$$

Estimating $\dim_{\mathscr{H}}(J_E)$ for $E \subset \mathbb{N}$ is of particular historical and contemporary interest. The problem first appeared in Jarnik's work [26] during the late 1920s in relation to *Diophantine approximation* and *badly-approximable numbers*. Specifically, Jarnik obtained dimension estimates when $E = \{1,2\}$. Jarnik's result was subsequently improved and extended by many authors [6, 5, 10, 11, 12, 17, 21, 23, 22, 20, 27, 28, 29, 19, 40]. Notably, Pollicott and Vytnova in [40], were able to rigorously estimate $\dim_{\mathscr{H}}(J_{1,2})$ with an accuracy of 200 digits. They used the zeta function—an approach introduced in this topic by Pollicott and his collaborators in previous studies—along with their "bisection method" to deliver very precise estimates for $\dim_{\mathscr{H}}(J_E)$ when the alphabet E is quite specific (for example when E is an initial segment of $\mathbb{N}$ or specific arithmetic progressions). Additionally, rigorous bounds for $\dim_{\mathscr{H}}(J_E)$ were needed in a seminal work by Kontorovich and Bourgain [2] and follow up work of Huang [24] to prove an almost everywhere version of Zaremba's Conjecture. More precisely, lower bounds for

$\dim_{\mathcal{H}}(J_{\{1,2,\ldots,50\}})$ and $\dim_{\mathcal{H}}(J_{\{1,2,\ldots,5\}})$ were respectively employed in [2] and [24]. These bounds were justified rigorously in [30] and they also follow from [16].

Falk and Nussbaum [16, 18, 17], developed a quite versatile (although frequently less accurate) method in order to provide rigorous estimates for CIFSs arising from continued fraction algorithms, both real and complex. In [8] the three first-named authors further refined the Falk-Nussbaum method in order to rigorously estimate $\dim_{\mathcal{H}}(J_E)$ for a wide variety of subsets $E \subset \mathbb{N}$, such as the primes, various powers, arithmetic progressions, etc. These estimates played a crucial in the study of the dimension spectrum of continued fractions with restricted digits in [8], and they were also recently used in [13].

So far we have only discussed rigorous Hausdorff dimension estimates for one very specific family of CIFSs in the real line. As it happens, there exist very few rigorous dimension estimates for other CIFSs. Falk and Nussbaum [18] obtained rigorous dimension estimates for complex continued fractions and Vytnova and Wormell [43] recently obtained very sharp dimension estimates for the Apollonian gasket (which as discovered in [35] can be viewed as an infinite CIFS). These approaches are fundamentally based on the specifics of the aforementioned systems. Our goal in this paper is to develop a versatile method that can provide rigorous and effective Hausdorff dimension estimates for a very broad family of conformal fractals.

We will focus our attention on dimension estimates of limit sets in the general framework of *conformal graph directed Markov systems* (CGDMS). For the moment, we will only describe CGDMSs briefly and we will discuss them in more detail in Section 2. A CGDMS in $\mathbb{R}^n$ is structured around a directed multigraph (E, V) with a countable set of edges E and a finite set of vertices V, and an incidence matrix $A: E \times E \to \{0,1\}$. Each vertex $v \in V$ corresponds to a pair of sets (X_v, W_v), $X_v, W_v \subset \mathbb{R}^n$ such that X_v is compact and connected, W_v is open and connected and $X_v \subset W_v$. For each each edge $e \in E$ there exists a contracting map $\phi_e : X_{t(e)} \to X_{i(e)}$ which extends to C^1 conformal diffeomorphism from $W_{t(e)}$ into $W_{i(e)}$. The incidence matrix $A: E \times E \to \{0,1\}$ determines if a pair of these maps is allowed to be composed. A CGDMS is called *maximal* when $t(a) = i(b)$ if and only if $A_{a,b} = 1$; i.e. all possible compositions are admissible.

We will always assume that CGDMSs satisfy the *Open Set Condition (OSC)* and the *finite irreducibility condition*. Assuming these two conditions, we have at our disposal a very rich and robust dimension theory of CGDMSs developed by Mauldin and the third-named author in [36], see also [9, 39, 42, 31] for related recent advances.

We will show that:

The Hausdorff dimension of limit sets of maximal and finitely irreducible CGDMSs in $\mathbb{R}^n, n \geq 2$, *which satisfy either the WCC or the SOSC is effectively and rigorously computable.*

Limit sets of maximal and finitely irreducible CGDMSs encompass a diverse range of geometric objects, including limit sets of Kleinian groups, complex hyperbolic Schottky groups, Apollonian circle packings, as well as self-conformal and self-similar sets. This diversity justifies our focus on studying dimension estimates within the unified framework of CGDMSs.

Our approach relies on piecewise linear approximations of the eigenfunctions of the following *Perron-Frobenius operator*. Given any maximal and finitely irreducible CGDMS $\mathcal{S}$ we define the Perron-Frobenius operator

$$F_t : C(X) \to C(X), \qquad F_t(g)(x) = \sum_{e \in E} \|D\phi_e(x)\|^t g(\phi_e(x)) \chi_{X_{t(e)}}(x),$$

where $X = \cup_{v\in V} X_v$ and t is any parameter such that $P(t)$, the topological pressure of the system evaluated at t, is finite.

In Section 3 we prove that there exists a unique continuous function $\rho_t : X \to [0,\infty)$ so that

$$F_t(\rho_t) = e^{P(t)}\rho_t. \tag{1.2}$$

Moreover, we show that the eigenfunctions ρ_t are uniformly bounded above and below (with bounds depending on t) and they are the uniform limits of the sequences $\{e^{-nP(t)}F_t^n(\mathbf{1})\}_{n=1}^{\infty}$. We also prove that the eigenfunctions ρ_t are the Radon-Nikodym derivatives $\frac{dm_t}{d\mu_t}$, where m_t is the t-conformal measure of the system and μ_t is the push forward of the unique shift-invariant Gibbs state. Some of the results from Section 3 were earlier proved in [36, Section 6.1] for the case of CIFSs. We stress that the open set condition is not required for any of our results in Section 3.

Since the Hausdorff dimension of the limit set of a CGDMS is the zero of its pressure function $P(t)$, it follows from (1.2) that it coincides with the parameter t^*, for which the Perron-Frobenius operator F_{t^*} has 1 as the leading eigenvalue. So, instead of trying to compute directly the zero of $P(t)$, one can try to estimate t^*. Especially if the corresponding eigenfunction ρ_{t^*} is smooth with derivatives that can be estimated, then this alternative approach has proven to be very effective and has led to several rigorous computational methods for estimating the Hausdorff dimension of the limit set.

One such method is based on the following fact: if for some positive function $g > 0$, $F_{\bar{t}}g < g$, then $P(\bar{t}) < 0$ and if $F_{\underline{t}}g > g$, then $P(\underline{t}) > 0$. As a result, we get $\underline{t} \le t^* \le \bar{t}$, and if the interval $[\underline{t}, \bar{t}]$ is small, one obtains a rigorous and effective estimate for the Hausdorff dimension of the limit set. Thus, the main task in this method is to construct such functions g. In the recent work [40], Pollicott and Vytnova constructed the desired functions g as global polynomials. Once the basis is chosen, the problem of computing the parameters $\underline{t}$ and $\bar{t}$ reduces to a finite dimensional linear algebra problem. For certain problems this approach yields very impressive results with many digits of accuracy; see for example the aforementioned paper [40], where highly accurate estimates are obtained for several one dimensional continued fractions susbsystems, and the very recent paper of Vytnova and Wormell [40, 43] where the Hausdorff dimension of the Apollonian gasket is estimated with high precision. We note however that this approach is heavily problem dependent and it is not straightforward to extend it to higher dimensional problems.

Inspired by the work of Falk and Nussbaum [17, 18, 16], we develop a universal method, which can be applied in a straightforward manner to any maximal and finitely irreducible CGDMS in $\mathbb{R}^n$, $n \ge 2$, although presently, and due to computer power limitations, is less precise than the method described in the previous paragraph. In this approach, instead of dealing with the finite dimensional problem of restricting the action of the Perron-Frobenius operator F_t to global polynomials, we focus our attention to the action of F_t on piecewise linear approximations of the eigenfunction ρ_t on some mesh domain $X^h \supseteq X$. Provided that h is small and good estimates for the second derivatives of ρ_t are available, we have accurate piecewise linear approximations of ρ_t and our method yields rigorous Hausdorff dimension estimates with several digits of accuracy even for limit sets in $\mathbb{R}^n$ for $n \ge 2$.

As mentioned earlier, our strategy depends on certain derivative bounds for the eigenfunctions ρ_t of the Perron-Frobenius operator F_t. Falk and Nussbaum obtained such bounds for second order derivatives in the case of CIFSs defined via real and complex continued fraction algorithms using some very technical arguments (especially in the case of complex continued fractions). In Section 4 (Theorem 4.1) we prove that the eigenfuntions ρ_t admit real analytic extensions and they satisfy the desired inequalities for derivatives of all orders. More precisely if $\mathscr{S}$ is a maximal and finitely irreducible CGDMS in $\mathbb{R}^n$, $n \geq 2$, then for any multi-index α:

(1) There exists a computable constant $C_1(t) > 0$ such that if $\mathscr{S}$ consists of Möbius maps:

$$|D^\alpha \rho_t(x)| \leq \alpha!\, n^{1/2} \mathrm{dist}(X, \partial W)^{-|\alpha|} C_1(t)\, \rho_t(x), \quad \forall x \in X.$$

(2) There exists a computable constant $C_2(t) > 0$ such that if $n = 2$:

$$|D^\alpha \rho_t(x)| \leq \alpha! \,\mathrm{dist}(X, \partial W)^{-|\alpha|} C_2(t)\, \rho_t(x), \quad \forall x \in X.$$

Besides being key ingredients in our methods, we consider that these derivative bounds have independent value and they might also find applications in other related problems. The proof of Theorem 4.1, which is quite short and streamlined, employs complexification and some basic tools from the theory of several complex variables. We also stress that the open set condition is not required for Theorem 4.1, i.e. for (1) and (2).

In Sections 5 and 6 we discuss a sampler of CGDMSs where our method can be applied. Due to length considerations we decided not to include an exhaustive list of applications, but we focused on examples which highlight the versatility of our method. We gather our estimates in Table 1.

We pay particular attention to CIFSs which are defined by continued fraction algorithms. We rigorously estimate the Hausdorff dimension of limit sets of CIFSs defined by complex continued continued fractions, earlier considered in [18], and for the first time, we also provide estimates for the complex continued fraction system whose alphabet is the set of Gaussian primes. We also introduce a CIFS modeled on higher dimensional continued fraction algorithms and we provide the first dimension estimates for the limit set of the three-dimensional continued fraction system. To the best of our knowledge this is the first example of a genuine 3-dimensional CIFS (meaning that the generating conformal maps are defined in $\mathbb{R}^3$, they are not similarities, and the limit set is not contained in any lower dimensional affine subspace of $\mathbb{R}^3$) where a rigorous numerical method is applied in order to estimate the Hausdorff dimension of its limit set.

We also discuss how our method can be applied to limit sets of systems defined by quadratic perturbations of linear maps. We included this example in order to highlight the fact that our method can be also applied to systems which do not consist of Möbius maps. All other known numerical methods for the estimation of the Hausdorff dimension of conformal fractals have focused on systems consisting of very specific Möbius maps.

Since our method encompasses the general framework of CGDMSs, and not only CIFSs, we also include a toy example of a system defined by a Schottky group (one of the most well known families of fractals which can be viewed as limit sets of CGDMSs) and we estimate its Hausdorff dimension.

Finally, we also provide rigorous estimates for the Hausdorff dimension of the Apollonian gasket, and for the Hausdorff dimension of several limit sets of its subsystems. Although there exist several non-rigorous estimates for the Hausdorff dimension of the Apollonian gasket [37, 1], until this year there was only one rigorous estimate, due to Boyd [4]. As mentioned earlier, Vytnova and Wormell [43] recently obtained rigorous and very accurate (up to 128 digits) estimates for the Hausdorff dimension of the Apollonian gasket. While our method applied to the gasket yields estimates that are notably less accurate compared to those achieved by Vytnova and Wormell, it offers the advantages of ease of implementation and high flexibility. These attributes allow us to derive rigorous and effective estimates for the Hausdorff dimensions of various subsystems of the Apollonian gasket. This is crucial for an upcoming project aiming to identify the gasket's dimension spectrum, where we need rapid and reliable estimates for a broad range of its subsystems.

We summarize our numerical findings in the following table.

TABLE 1. Hausdorff dimension estimates for various examples.

Example	Hausdorff dimension
$2D$ Continued fractions with 4 generators	$1.149576 \pm 5.5e-06$
$2D$ Continued fractions	$1.853 \pm 4.2e-03$
$2D$ Continued fractions on Gaussian primes	$1.510 \pm 4.0e-03$
$3D$ Continued fractions with 5 generators	$1.452 \pm 9.7e-03$
$3D$ Continued fractions	$2.57 \pm 1.7e-02$
A quadratic abc-example	$0.6327142857142865 \pm 5.0e-16$
An example of a Schottky group	$0.7753714285 \pm 1.5e-10$
12 map Apollonian subsystem	$1.0285714285713 \pm 1.1e-13$
Apollonian gasket	$1.30565 \pm 5e-05$
Apollonian gasket without a generator	$1.2196 \pm 2e-04$
Apollonian gasket without a spiral	$1.2351 \pm 5.5e-04$

Table 1 illustrates the generality of our method by providing several rather distinct examples, for which the Hausdorff dimensions are computed with various order of accuracy. The accuracy of the computations depends mainly on the size of the alphabet and the size of the discrete problem (see Section 5 for more details). Naturally, the largest and the most computationally intensive problem is 3D Continued fractions on an infinite lattice while our Schottky group example is the smallest. Our main objective in this paper is to elaborate that Hausdorff dimensions of a very broad family of conformal fractals are effectively computable. We did not pursue the avenue of giving the best results possible, which we plan to do in future works where we will explore the computational boundaries of our method.

2. PRELIMINARIES

In this section we introduce all the necessary background and definitions about conformal graph directed Markov systems and their thermodynamic formalism.

Definition 2.1. A *graph directed Markov system* (GDMS)

$$\mathscr{S} = \left\{V, E, A, t, i, \{X_v\}_{v\in V}, \{\phi_e\}_{e\in E}\right\} \tag{2.1}$$

consists of

(1) a directed multigraph (E,V) with a countable set of edges E, which we will call the *alphabet* of $\mathscr{S}$, and a finite set of vertices V,
(2) an incidence matrix $A : E \times E \to \{0,1\}$,
(3) two functions $i, t : E \to V$ such that $t(a) = i(b)$ whenever $A_{ab} = 1$,
(4) a family of non-empty compact metric spaces $\{X_v\}_{v\in V}$,

(5) a family of injective contractions

$$\{\phi_e : X_{t(e)} \to X_{i(e)}\}_{e\in E}$$

such that every ϕ_e, $e \in E$, has Lipschitz constant no larger than s for some $s \in (0,1)$.

When it is clear from context we will use the simpler notation $\mathcal{S} = \{\phi_e\}_{e\in E}$ for a GDMS. We will always assume that the alphabet E is not a singleton and for every $v \in V$ there exist $e, e' \in E$ such that $t(e) = v$ and $i(e') = v$. GDMSs with finite alphabets will be called *finite*.

Remark 2.2. When V is a singleton and for every $e_1, e_2 \in E$, $A_{e_1 e_2} = 1$ if and only if $t(e_1) = i(e_2)$, the GDMS is called an *iterated function system* (IFS).

We will use the following standard notation from symbolic dynamics. For every $\omega \in E^* := \bigcup_{n=0}^{\infty} E^n$, we denote by $|\omega|$ the unique integer $n \geq 0$ such that $\omega \in E^n$, and we call $|\omega|$ the *length* of ω. We also set $E^0 = \{\emptyset\}$. For $n \in \mathbb{N}$ and $\omega \in E^{\mathbb{N}}$, we let

$$\omega|_n := \omega_1 \dots \omega_n \in E^n.$$

If $\tau \in E^*$ and $\omega \in E^* \cup E^{\mathbb{N}}$, then

$$\tau\omega := (\tau_1, \dots, \tau_{|\tau|}, \omega_1, \dots).$$

For $\omega, \tau \in E^{\mathbb{N}}$, the longest initial block common to both ω and τ will be denoted by $\omega \wedge \tau \in E^{\mathbb{N}} \cup E^*$. The *shift map*

$$\sigma : E^{\mathbb{N}} \to E^{\mathbb{N}}$$

is given by the formula

$$\sigma\left((\omega_n)_{n=1}^{\infty}\right) = \left((\omega_{n+1})_{n=1}^{\infty}\right).$$

For a matrix $A : E \times E \to \{0,1\}$ we let

$$E_A^{\mathbb{N}} := \{\omega \in E^{\mathbb{N}} : \ A_{\omega_i \omega_{i+1}} = 1 \text{ for all } \ i \in \mathbb{N}\},$$

and we call its elements A-*admissible* (infinite) words. We also set

$$E_A^n := \{w \in E^n : \ A_{\omega_i \omega_{i+1}} = 1 \text{ for all } 1 \leq i \leq n-1\}, \quad n \in \mathbb{N},$$

and

$$E_A^* := \bigcup_{n=0}^{\infty} E_A^n.$$

The elements of E_A^* are called A-admissible (finite) words. Slightly abusing notation, if $\omega \in E_A^*$ we let $t(\omega) = t(\omega_{|\omega|})$ and $i(\omega) = i(\omega_1)$. For every $\omega \in E_A^*$, we let

$$[\omega] := \{\tau \in E_A^{\mathbb{N}} : \ \tau_{||\omega|} = \omega\}.$$

Given $v \in V$ we denote

$$E_A^n(v) = \{\omega \in E_A^n : t(\omega) = v\}$$

and

$$E_A^*(v) = \cup_{n\in\mathbb{N}} E_A^n(v).$$

For each $a \in E$, we let

$$E_a^{\infty} := \{\omega \in E_A^{\mathbb{N}} : A_{a\omega_1 = 1}\}$$

Definition 2.3. A matrix $A : E \times E \to \{0,1\}$ will be called *finitely irreducible* if there exists a finite set $\Lambda \subset E_A^*$ such that for all $i, j \in E$ there exists $\omega \in \Lambda$ for which $i\omega j \in E_A^*$. If the associated matrix of a GDMS is finitely irreducible, we will call the GDMS finitely irreducible as well.

We will be interested in maximal GDMSs.

Definition 2.4. A GDMS $\mathscr{S}$ with an incidence matrix A is called *maximal* if it satisfies the following condition:

$$A_{ab} = 1 \text{ if and only if } t(a) = i(b).$$

This notion has an easy colloquial description — a GDMS is maximal when one can compose maps whose range and domain coincide.

Let $\mathscr{S} = \{V, E, A, t, i, \{X_v\}_{v\in V}, \{\phi_e\}_{e\in E}\}$ be a GDMS. For $\omega \in E_A^*$ we define the map coded by ω:

$$\phi_\omega = \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n} : X_{t(\omega_n)} \to X_{i(\omega_1)} \qquad \text{if } \omega \in E_A^n. \tag{2.2}$$

For $\omega \in E_A^{\mathbb{N}}$, the sequence of non-empty compact sets $\{\phi_{\omega|_n}(X_{t(\omega_n)})\}_{n=1}^\infty$ is decreasing (in the sense of inclusion) and therefore their intersection is nonempty. Moreover,

$$\operatorname{diam}(\phi_{\omega|_n}(X_{t(\omega_n)})) \leq s^n \operatorname{diam}(X_{t(\omega_n)}) \leq s^n \max\{\operatorname{diam}(X_v) : v \in V\}$$

for every $n \in \mathbb{N}$, hence

$$\pi(\omega) := \bigcap_{n\in\mathbb{N}} \phi_{\omega|_n}(X_{t(\omega_n)})$$

is a singleton. Thus we can now define the coding map

$$\pi : E_A^{\mathbb{N}} \to \bigoplus_{v\in V} X_v := X, \tag{2.3}$$

the latter being a disjoint union of the sets X_v, $v \in V$. The set

$$J = J_{\mathscr{S}} := \pi(E_A^{\mathbb{N}})$$

will be called the *limit set* (or *attractor*) of the GDMS $\mathscr{S}$.

For $\alpha > 0$, we define the metrics d_α on $E_A^{\mathbb{N}}$ by setting

$$d_\alpha(\omega, \tau) = \mathrm{e}^{-\alpha|\omega\wedge\tau|}. \tag{2.4}$$

We record that all the metrics d_α induce the same topology. Moreover, see [9, Proposition 4.2], the coding map $\pi : E_A^{\mathbb{N}} \to \bigoplus_{v\in V} X_v$ is Hölder continuous, when $E_A^{\mathbb{N}}$ is equipped with any of the metrics d_α as in (2.4) and $\bigoplus_{v\in V} X_v$ is equipped with the direct sum metric.

Let U be an open and connected subset of $\mathbb{R}^n$. A C^1 diffeomorphism $\phi : U \to \mathbb{R}^n$ will be called *conformal* if its derivative at every point of U is a similarity map. We will denote the derivative of ϕ evaluated at the point z by $D\phi(z) : \mathbb{R}^n \to \mathbb{R}^n$ and we denote its operator norm by $\|D\phi(z)\|$. It is well known by *Liouville's theorem*, see [41, Theorem 19.2.1], that for

- $n = 1$ the map ϕ is conformal if and only if it is a C^1-diffeomorphism,
- $n = 2$ the map ϕ is conformal if and only if it is either holomorphic or antiholomorphic,
- $n \geq 3$ the map ϕ is conformal if and only if it is a Möbius transformation.

We can now define conformal GDMSs. [1]

Definition 2.5. A graph directed Markov system $\mathscr{S} = \{V, E, A, t, i, \{X_v\}_{v\in V}, \{\phi_e\}_{e\in E}\}$ is called *conformal* (CGDMS) if the following conditions are satisfied.

(i) The metric spaces X_v, $v \in V$, are compact and connected subsets of a fixed Euclidean space $\mathbb{R}^n$ and $X_v = \overline{\operatorname{Int}(X_v)}$ for all $v \in V$.

[1] There are several variants for a definition of GDMS, see e.g. [41, 31]. The definition we are using is slightly more restrictive however it is the more convenient for our applications.

(ii) (*Open Set Condition* or *OSC*). For all $a, b \in E$, $a \neq b$,

$$\phi_a(\mathrm{Int}(X_{t(a)})) \cap \phi_b(\mathrm{Int}(X_{t(b)})) = \emptyset.$$

(iii) For every vertex $v \in V$ there exist open and connected sets $W_v \supset X_v$ such that for every $\omega \in E^*$, the map ϕ_ω extends to a C^1 conformal diffeomorphism of $W_{t(\omega)}$ into $W_{i(\omega)}$.

(iv) (*Bounded Distortion Property* or *BDP*) For each $v \in V$ there exist compact and connected sets S_v such that $X_v \subset \mathrm{Int}(S_v) \subset S_v \subset W_v$ so that $\phi_e(S_{t(e)}) \subset S_{i(e)}$ for all $e \in E$ and

$$\left| \frac{\|D\phi_e(p)\|}{\|D\phi_e(q)\|} - 1 \right| \leq L|p-q|^\alpha \text{ for all } e \in E \text{ and } p, q \in S_{t(e)},$$

where $\alpha > 0$ and $L \geq 1$ are two constants depending only on $\mathscr{S}, S_v$ and W_v.

We will use the abbreviation CIFS for conformal IFS.

Remark 2.6. If $n \geq 2$ the definition of a conformal GDMS can be significantly simplified. First, condition (iii) can be replaced by the following weaker condition:

(iii)' For every vertex $v \in V$ there exists an open connected set $W_v \supset X_v$ such that for every $e \in E$, the map ϕ_e extends to a C^1 conformal diffeomorphism of $W_{t(e)}$ into $W_{i(e)}$.

Moreover, Condition (iv) is superfluous since Condition (iii)' $\Longrightarrow$ Condition (iv)(with $\alpha = 1$), see e.g. [36, 31].

We record that the Bounded Distortion Property(BDP) implies that there exists some constant depending only on $\mathscr{S}$ such that

$$K^{-1} \leq \frac{\|D\phi_\omega(p)\|}{\|D\phi_\omega(q)\|} \leq K \tag{2.5}$$

for every $\omega \in E_A^*$ and every pair of points $p, q \in S_{t(\omega)}$.

For $\omega \in E_A^*$ we set

$$\|D\phi_\omega\|_\infty := \|D\phi_\omega\|_{X_{t(\omega)}}.$$

Note that (2.5) and the Leibniz rule easily imply that if $\omega \in E_A^*$ and $\omega = \tau\upsilon$ for some $\tau, \upsilon \in E_A^*$, then

$$K^{-1}\|D\phi_\tau\|_\infty \|D\phi_\upsilon\|_\infty \leq \|D\phi_\omega\|_\infty \leq \|D\phi_\tau\|_\infty \|D\phi_\upsilon\|_\infty. \tag{2.6}$$

Moreover, there exists a constant M, depending only on $\mathscr{S}$, such that for every $\omega \in E_A^*$, and every $p, q \in S_{t(\omega)}$,

$$d(\phi_\omega(p), \phi_\omega(q)) \leq MK\|D\phi_\omega\|_\infty d(p,q), \tag{2.7}$$

where d is the Euclidean metric on $\mathbb{R}^n$. In particular for every $\omega \in E_A^*$

$$\mathrm{diam}(\phi_\omega(X_{t(\omega)})) \leq MK\|D\phi_\omega\|_\infty \mathrm{diam}(X_{t(\omega)}). \tag{2.8}$$

2.1. **Thermodynamic formalism.** We will now recall some well known facts from the thermodynamic formalism of GDMSs. Let $\mathscr{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS. For $t \geq 0$ and $n \in \mathbb{N}$ let

$$Z_n(\mathscr{S}, t) := Z_n(t) := \sum_{\omega \in E_A^n} \|D\phi_\omega\|_\infty^t. \tag{2.9}$$

Note that (2.6) implies that

$$Z_{m+n}(t) \leq Z_m(t) Z_n(t), \tag{2.10}$$

and consequently the sequence $\{\log Z_n(t)\}_{n=1}^{\infty}$ is subadditive. Therefore, the limit

$$P_{\mathscr{S}}(t) := P(t) := \lim_{n\to\infty} \frac{\log Z_n(t)}{n} = \inf_{n\in\mathbb{N}} \frac{\log Z_n(t)}{n}$$

exists and it is called the *topological pressure* of the system $\mathscr{S}$ evaluated at the parameter t. We also define two special parameters related to topological pressure;

$$\theta(\mathscr{S}) := \theta = \inf\{t \geq 0 : P(t) < +\infty\} \quad \text{and} \quad h(\mathscr{S}) := h = \inf\{t \geq 0 : P(t) \leq 0\}.$$

The parameter $h(\mathscr{S})$ is known as *Bowen's parameter*.

It is well known that $t \mapsto P(t)$ is decreasing on $[0, +\infty)$ with $\lim_{t\to+\infty} P(t) = -\infty$, and it is convex and continuous on $\overline{\{t \geq 0 : P(t) < \infty\}}$, see e.g. [41, 19.4.6]. Moreover

$$\theta(\mathscr{S}) := \theta = \inf\{t \geq 0 : P(t) < \infty\} = \inf\{t \geq 0 : Z_1(t) < \infty\}, \tag{2.11}$$

and for $t \geq 0$

$$P(t) < +\infty \text{ if and only if } Z_1(t) < +\infty. \tag{2.12}$$

The proofs of these facts can be found in [9, Proposition 7.5] and [7, Lemma 3.10].

Thermodynamic formalism, and topological pressure in particular, plays a fundamental role in the dimension theory of conformal dynamical systems:

Theorem 2.7. *If $\mathscr{S}$ is a finitely irreducible conformal GDMS, then*

$$h(\mathscr{S}) = \dim_{\mathscr{H}}(J_{\mathscr{S}}) = \sup\{\dim_{\mathscr{H}}(J_F) : F \subset E \textit{ finite}\}.$$

For the proof see [9, Theorem 7.19] or [36, Theorem 4.2.13].

We close this section with a discussion regarding conformal measures and Perron-Frobenius operators. If $\mathscr{S} = \{\phi_e\}_{e\in E}$ is a finitely irreducible conformal GDMS we define

$$\operatorname{Fin}(\mathscr{S}) := \{t > 0 : Z_1(t) < +\infty\} = \left\{ t > 0 : \sum_{e\in E} ||D\phi_e||_\infty^t < +\infty \right\}.$$

Gibbs measures are of crucial importance in thermodynamic formalism of countable alphabet symbolic dynamics.

Definition 2.8. Let $\mathscr{S}$ be a finitely irreducible conformal GDMS and let $t \in \operatorname{Fin}(\mathscr{S})$. A Borel probability measure μ on $E_A^{\mathbb{N}}$ is called t-Gibbs state for $\mathscr{S}$ (or a Gibbs state for the potential $\omega \to t\log\|D\phi_{\omega_1}(\pi(\sigma(\omega)))\|$) if and only if there exist some constant $C_{\mu,t} \geq 1$ such that

$$C_{\mu,t}^{-1} e^{-P(t)n} \|D\phi_{\omega|_n}(\pi(\sigma^n(\omega)))\| \leq \mu([\omega|_n]) \leq C_{\mu,t} e^{-P(t)n} \|D\phi_{\omega|_n}(\pi(\sigma^n(\omega)))\|, \tag{2.13}$$

for all $\omega \in E_A^{\mathbb{N}}$ and $n \in \mathbb{N}$.

For $t \in \operatorname{Fin}(\mathscr{S})$ the *Perron-Frobenius operator* with respect to $\mathscr{S}$ and t is defined as

$$\mathscr{L}_t g(\omega) = \sum_{i:\, A_{i\omega_1}=1} g(i\omega) \|D\phi_i(\pi(\omega))\|^t \quad \text{for } g \in C_b(E_A^{\mathbb{N}}) \text{ and } \omega \in E_A^{\mathbb{N}}, \tag{2.14}$$

where $C_b(E_A^{\mathbb{N}})$ is the Banach space of real-valued bounded continuous functions on $E_A^{\mathbb{N}}$. It is well known that $\mathscr{L}_t : C_b(E_A^{\mathbb{N}}) \to C_b(E_A^{\mathbb{N}})$. Moreover, by a straightforward inductive calculation:

$$\mathscr{L}_t^n g(\omega) = \sum_{\tau\in E_A^n:\, A_{\tau_n\omega_1}=1} g(\tau\omega) \|D\phi_\tau(\pi(\omega))\|^t \quad \text{for } g \in C_b(E_A^{\mathbb{N}}) \text{ and } \omega \in E_A^{\mathbb{N}}. \tag{2.15}$$

We will also denote by $\mathscr{L}_t^* : C_b^*(E_A^{\mathbb{N}}) \to C_b^*(E_A^{\mathbb{N}})$ the dual operator of $\mathscr{L}_t$. The proof of the following theorem can be found in [9, Theorem 7.4].

Theorem 2.9. *Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS and let $t \in$ Fin($\mathcal{S}$).*

(1) *There exists a unique eigenmeasure $\tilde{m}_t$ of the conjugate Perron-Frobenius operator $\mathcal{L}_t^*$ and the corresponding eigenvalue is $e^{P(t)}$.*
(2) *The eigenmeasure $\tilde{m}_t$ is a t-Gibbs state.*
(3) *There exists a unique shift-invariant t-Gibbs state $\tilde{\mu}_t$ which is ergodic and globally equivalent to $\tilde{m}_t$.*

For all $t \in \mathrm{Fin}(\mathcal{S})$ we will denote

$$m_t := \tilde{m}_t \circ \pi^{-1} \quad \text{and} \quad \mu_t := \tilde{\mu}_t \circ \pi^{-1}. \tag{2.16}$$

Note that the measures m_t, μ_t are probability measures supported on $J_{\mathcal{S}}$. The measures m_t will be called t-conformal and in the case when $t = h = h(\mathcal{S})$, the measure m_h is simply called the *conformal measure* of $\mathcal{S}$.

We will conclude this section with a bound for $\mathcal{L}_t^n(\mathbf{1})$ which will be of paramount importance in Sections 3 and 4.

Proposition 2.10. Let $\mathcal{S} = \{\phi_e\}_{e\in E}$ be a finitely irreducible conformal GDMS and let $t \in \mathrm{Fin}(\mathcal{S})$. There exists a constant $M_t \geq 1$ such that

$$M_t^{-1} e^{P(t)n} \leq \mathcal{L}_t^n(\mathbf{1})(\omega) \leq M_t e^{P(t)n}, \tag{2.17}$$

for all $\omega \in E_A^N$ and $n \in \mathbb{N}$.

Proof. The upper bound follows from [41, Lemma 18.1.1]. We will now present the proof for the lower bound. We remark that a much more general statement, which establishes lower bounds for Perron-Frobenius operators with respect to general potentials, will appear in the forthcoming book [14].

We will first show that for all $a \in E$ and $n \in \mathbb{N}$:

$$\sum_{\omega\in E_A^n : A_{\omega_n a}=1} \sup\{\|D\phi_\omega(\pi(\tau))\| : \tau \in [a]\} \geq C_t^{-2} e^{P(t)n}. \tag{2.18}$$

By Theorem 2.9 (3) and Definition 2.8 we know that there exists some $C_t > 0$ such that

$$C_t^{-1} e^{-P(t)n} \|D\phi_{\omega|_n}(\pi(\sigma^n(\omega)))\| \leq \mu_t([\omega|_n]) \leq C_t e^{-P(t)n} \|D\phi_{\omega|_n}(\pi(\sigma^n(\omega)))\|, \tag{2.19}$$

for all $\omega \in E_A^{\mathbb{N}}$ and $n \in \mathbb{N}$. Note that (2.19) and the chain rule imply that

$$\begin{aligned} \mu_t([\alpha\beta]) \leq C_t \exp\big(-(|\alpha| + |\beta|)P(t)\big) &\sup\{\|D\phi_\alpha(\pi(\tau))\| : \tau \in [\beta]\} \\ &\cdot \sup\{\|D\phi_\beta(\pi(\rho))\| : \beta\rho \in E_A^{\mathbb{N}}\}, \end{aligned} \tag{2.20}$$

for any $\alpha, \beta \in E_A^*$ such that $\alpha\beta \in E_A^*$.

Let $a \in E$. We then see that:

$$\begin{aligned} &C_t^{-1} e^{-P(t)} \sup\{\|D\phi_a(\pi(\tau))\| : a\tau \in E_A^{\mathbb{N}}\} \\ &\overset{(2.19)}{\leq} \mu_t([a]) = \mu_t(\sigma^{-n}([a])) \\ &= \sum_{\omega\in E_A^n : A_{\omega_n a}=1} \mu_t([\omega a]) \\ &\overset{(2.20)}{\leq} C_t e^{-(n+1)P(t)} \sup\{\|D\phi_a(\pi(\tau))\| : a\tau \in E_A^{\mathbb{N}}\} \sum_{\omega\in E_A^n : A_{\omega_n a}=1} \sup\{\|D\phi_\omega(\pi(\tau))\| : \tau \in [a]\}. \end{aligned}$$

Thus (2.18) follows.

We can now prove the lower bound in (2.17). Let $\tau \in E_A^{\mathbb{N}}$ and $n \in \mathbb{N}$. If $\omega \in E_A^n$ and $\omega\tau \in E_A^{\mathbb{N}}$ then by the bounded distrotion property:

$$\|D\phi_\omega(\pi(\tau))\| \overset{(2.5)}{\geq} K^{-1} \sup\{\|D\phi_\omega(\pi(\rho))\| : \rho \in [\tau_1]\}. \tag{2.21}$$

Therefore,

$$\begin{aligned}\mathscr{L}_t^n(\mathbf{1})(\tau) &= \sum_{\omega \in E_A^n : \omega\tau \in E_A^{\mathbb{N}}} \|D\phi_\omega(\pi(\tau))\|^t \\ &\overset{(2.21)}{\geq} K^{-t} \sum_{\omega \in E_A^n : \omega\tau \in E_A^{\mathbb{N}}} \sup\{\|D\phi_\omega(\pi(\rho))\| : \rho \in [\tau_1]\} \overset{(2.18)}{\geq} K^{-t} C_t^{-2} e^{P(t)n}.\end{aligned}$$

The proof is complete. □

3. The Radon-Nikodym derivative $\rho_t = \frac{d\mu_t}{dm_t}$ for maximal CGDMS

In this section we study another Perron-Frobenius operator for CGDMSs. This operator is defined on $C(X, \mathbb{C})$ and it is strongly related to the Perron-Frobenius operator that we encountered in Section 2. A detailed study of its eigenfunctions are of paramount importance for our method. We also note that the restriction of these eigenfunctions to the limit set of the system coincide with the Radon-Nikodym derivative $d\mu_t/dm_t$.

Our treatment generalizes earlier results from [36, Section 6.1], which only dealt with CIFSs, to maximal CGDMSs. We also show discuss extensions of the Perron-Frobenius operator to $C(S)$. In what follows

$$\mathscr{S} = \{V, E, A, t, i, \{X_v\}_{v \in V}, \{\phi_e\}_{e \in E}\}$$

will denote a maximal CGDMS. We stress that the results in this sections do not require any separation condition; in particular the open set condition is not needed.

Recall that $X = \bigoplus_{v \in V} X_v$ and similarly define $S := \bigoplus_{v \in V} S_v$. We will assume that these unions are disjoint. This is not an essential restriction because, as it was described in [9, Remark 4.20], given any GDMS we can use formal lifts to obtain a new GDMS with essentially the same limit set but whose corresponding compact sets are disjoint.

In the rest of the section we will focus on the spaces of complex valued continuous functions $C(X)$ and $C(S)$. We will denote by $\mathbf{1}$ and $\mathbf{0}$ the constant functions (defined on X or S, depending on context) with values 1 and 0 respectively. We start by introducing a Perron-Frobenius operator on $C(X)$. For $t \in \mathrm{Fin}(\mathscr{S})$, $g \in C(X)$, let

$$F_t(g)(x) = \sum_{e \in E_A} \|D\phi_e(x)\|^t g(\phi_e(x)) \chi_{X_{t(e)}}(x). \tag{3.1}$$

The following proposition shows that F_t maps $C(X)$ to itself.

Proposition 3.1. Suppose that $\mathscr{S}$ is a finitely irreducible, maximal CGDMS of and $t \in \mathrm{Fin}(\mathscr{S})$. For F_t defined as above,

$$F_t : C(X) \to C(X),$$

and $F_t : (C(X), \|\cdot\|_\infty) \to (C(X), \|\cdot\|_\infty)$ is a bounded linear operator.

Proof. Since $t \in \mathrm{Fin}(\mathscr{S})$, we have that

$$\sum_{e \in E} \|D\phi_e\|_\infty^t = C_F < \infty.$$

Let $g \in C(X)$. By the compactness of X and the continuity of g,

$$F_t(g)(x) = \sum_{e \in E_A} \|D\phi_e(x)\|^t g(\phi_e(x)) \chi_{X_{t(e)}}(x) \leq \|g\|_\infty \sum_{e \in E} \|D\phi_e\|_\infty^t < \infty, \tag{3.2}$$

for all $x \in X$.

Let $x \in X$ and $\epsilon > 0$. If $|V| > 1$, set $\delta_1 = \min\{\mathrm{dist}(X_{v_1}, X_{v_2})\}$, and let $\delta_1 = 1$ otherwise. By [9, Lemma 4.16] or [41, Lemma 19.3.4] there exists $\delta_2 > 0$ so that whenever $|x-y| < \delta_2$,

$$\left|t(\log\|D\phi_e(x)\| - \log\|D\phi_e(y)\|)\right| < \min\left\{1, \frac{\epsilon}{6C_F}\right\}, \tag{3.3}$$

for all $e \in E$. Moreover, using the uniform continuity of g and contractivity of ϕ_e, we can find $\delta_3 > 0$ so that

$$|g(\phi_e(x)) - g(\phi_e(y))| < \frac{\epsilon}{2C_F} \tag{3.4}$$

for all all $e \in E$ and all $x, y \in X_{t(e)}$ satisfying $|x-y| < \delta_3$.

If $x, y \in X_v$ then,

$$\begin{aligned}\left|F_t(g)(x) - F_t(g)(y)\right| &= \left|\sum_{e\in E,\ t(e)=v} \|D\phi_e(x)\|^t g(\phi_e(x)) - \|D\phi_e(y)\|^t g(\phi_e(y))\right| \\ &\le \sum_{e\in E,\ t(e)=v} \left|\|D\phi_e(x)\|^t - \|D\phi_e(y)\|^t\right| \|g\|_\infty + |g(\phi_e(x)) - g(\phi_e(y))| \|D\phi_e\|_\infty^t.\end{aligned} \tag{3.5}$$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and let $y \in X$ such that $|y-x| < \delta$. We analyze each part of (3.5) separately. For the first part, notice that since $|x-y| < \delta_2$

$$\begin{aligned}\left|\|D\phi_e(x)\|^t - \|D\phi_e(y)\|^t\right| &= \|D\phi_e(x)\|^t \left|1 - \left(\frac{\|D\phi_e(y)\|}{\|D\phi_e(x)\|}\right)^t\right| \\ &= \|D\phi_e(x)\|^t \left|1 - e^{t(\log\|D\phi_e(y)\| - \log\|D\phi_e(x)\|)}\right| \\ &\le \|D\phi_e(x)\|^t 3\left|t(\log\|D\phi_e(x)\| - \log\|D\phi_e(y)\|)\right| \\ &\overset{(3.3)}{\le} \|D\phi_e(x)\|^t \frac{\epsilon}{2C_F},\end{aligned} \tag{3.6}$$

where we also used that $|e^s - 1| \le 3s$ when $|s| < 1$. Hence,

$$\sum_{e\in E,\ t(e)=v} \left|\|D\phi_e(x)\|^t - \|D\phi_e(y)\|^t\right| \le \frac{\epsilon}{2C_F} \sum_{e\in E,\ t(e)=v} \|D\phi_e(x)\|^t \le \frac{\epsilon}{2C_F} \sum_{e\in E} \|D\phi_e\|_\infty^t \le \frac{\epsilon}{2}. \tag{3.7}$$

For the second part of the sum, note that since $|x-y| < \delta_3$:

$$\sum_{e\in E,\ t(e)=v} |g(\phi_e(x)) - g(\phi_e(y))| \|D\phi_e\|_\infty^t \overset{(3.4)}{\le} \frac{\epsilon}{2C_F} \sum_{e\in E,\ t(e)=v} \|D\phi_e\|_\infty^t \le \frac{\epsilon}{2C_F} \sum_{e\in E} \|D\phi_e\|_\infty^t \le \frac{\epsilon}{2}. \tag{3.8}$$

Hence, (3.5), (3.7) and (3.8) imply that $F_t(g)$ is continuous at x. Since x was arbitrary we deduce that $F_t(g) \in C(X)$. The fact that $F_t(C(X), \|\cdot\|_\infty) \to (C(X), \|\cdot\|_\infty)$ is a bounded linear operator follows by (3.2). □

We will now find an explicit formula for F_t^n. Starting with a single composition,

$$\begin{aligned}F_t^2(g)(x) &= F_t\left(\sum_{e_1\in E_A} \|D\phi_{e_1}(\cdot)\|^t g(\phi_{e_1}(\cdot))\chi_{X_{t(e_1)}}(\cdot)\right)(x) \\ &= \sum_{e_2\in E_A} \|D\phi_{e_2}(x)\|^t \sum_{e_1\in E_A} \|D\phi_{e_1}(\phi_{e_2}(x))\|^t g(\phi_{e_1}(\phi_{e_2}(x)))\chi_{X_{t(e_1)}}(\phi_{e_2}(x))\chi_{X_{t(e_2)}}(x).\end{aligned}$$

To simplify this equation, notice first that when $X_{t(e_1)} \ne X_{i(e_2)}$, $\chi_{X_{t(e_1)}}(\phi_{e_2}(x)) = 0$, disallowing compositions for which $X_{t(e_1)} \ne X_{i(e_2)}$. Moreover, whenever $\phi_{e_2}(x) \in X_{i(e_1)}$ we must have that $X_{t(e_1)} = X_{i(e_2)}$, so the characteristic function $\chi_{X_{t(e_1)}}(\phi_{e_2}(x))$ can be absorbed into the expression $e_1 e_2 \in E_A^*$. Hence the second iterate of F_t is given by

$$F_t^2(g)(x) = \sum_{e_1 e_2 \in E_A^2} \left\|D\phi_{e_1}(\phi_{e_2}(x))\right\|^t \left\|D\phi_{e_2}(x)\right\|^t g(\phi_{e_1} \circ \phi_{e_2}(x))\chi_{X_{t(e_2)}}(x).$$

Such reasoning easily generalizes to the n-th iterate of the operator, so

$$F_t^n(g)(x) = \sum_{w \in E_A^n} \left\|D\phi_\omega(x)\right\|^t g(\phi_\omega(x))\chi_{X_{t(w)}}(x). \tag{3.9}$$

Note that if $x \in X_v$ then

$$F_t^n(g)(x) = \sum_{w \in E_A^n(v)} \left\|D\phi_\omega(x)\right\|^t g(\phi_\omega(x)). \tag{3.10}$$

The connection between the Perron-Frobenius operator F_t and the symbolic Perron-Frobenius operator defined in Section 2 can be easily obtained. For every $g \in C(X)$ and $n \in \mathbb{N}$:

$$\mathscr{L}_t^n(g \circ \pi) = F_t^n(g) \circ \pi. \tag{3.11}$$

See [31, p. 425] for the straightforward calculation leading to (3.11).

Remark 3.2. We note that the main reason why we restrict ourselves to maximal systems is the fact that the iterates of the Perron-Frobenius operator F_t^n, see (3.10), are not well defined if the GDMS is not maximal.

We will now show that the iterates $F_t^{(n)}(\mathbf{1})$ are uniformly bounded above and below with bounds depending on t and n.

Proposition 3.3. Let $\mathscr{S}$ be a finitely irreducible, maximal CGDMS. If $t \in \mathrm{Fin}(\mathscr{S})$ then for all $x \in X$ and $n \in \mathbb{N}$:

$$M_t^{-1} K^{-t} e^{nP(t)} \le F_t^{(n)}(\mathbf{1})(x) \le M_t K^t e^{nP(t)}, \tag{3.12}$$

where M_t is as in Proposition 2.10.

Proof. Let $x \in X$.Then $x \in X_v$ for some $v \in V$. Let $\tau \in E_A^{\mathbb{N}}$ such that $i(\tau) = i(\tau_1) = v$. Then, by Theorem 2.10

$$\begin{aligned} F_t^n(\mathbf{1})(x) &= \sum_{\omega \in E_A^n} \left\|D\phi_\omega(x)\right\|^t \chi_{X_{t(\omega)}(x)} = \sum_{\omega \in E_A^n : t(\omega_n) = v} \left\|D\phi_\omega(x)\right\|^t \\ &\overset{(2.5)}{\le} K^t \sum_{\omega \in E_A^n : A_{\omega_n \tau_1} = 1} \left\|D\phi_\omega(\pi(\tau))\right\|^t = K^t \mathscr{L}_t^n(\mathbf{1})(\tau) \overset{(2.17)}{\le} K^t M_t e^{nP(t)}. \end{aligned}$$

The lower bound follows by a similar argument. □

In order to simplify notation we will also use the following normalized version of F_t. For $t \in \mathrm{Fin}(\mathscr{S})$ we let

$$\tilde{F}_t(g)(x) := \lambda_t^{-1} F_t(g)(x) = \lambda_t^{-1} \sum_{e \in E_A} \left\|D\phi_e(x)\right\|^t g(\phi_e(x))\chi_{X_{t(e)}}(x),$$

where $\lambda_t = e^{P(t)}$ is the spectral radius of F_t. Recalling (3.9), we obtain a formula for $\tilde{F}_t^n$ given by

$$\tilde{F}_t^n(g)(x) := \lambda_t^{-n} F_t^{(n)}(g)(x) = \lambda_t^{-n} \sum_{w \in E_A^n} \left\|D\phi_\omega(x))\right\|^t g(\phi_\omega(x))\chi_{X_{t(w)}}(x).$$

Clearly, (3.11) implies that for every $g \in C(X)$ and $n \in \mathbb{N}$:

$$\widetilde{\mathscr{L}_t}^n(g \circ \pi) = \tilde{F}_t^n(g) \circ \pi, \tag{3.13}$$

where $\widetilde{\mathscr{L}_t} = \lambda_t^{-1}\mathscr{L}_t$. Moreover, Theorem 2.9 (1) implies that

$$\begin{aligned} m_t(\tilde{F}_t(g)) &= \tilde{m}_t \circ \pi^{-1}(\tilde{F}_t(g)) = \int_{J_{\mathscr{S}}} \tilde{F}_t(g)\, d(\tilde{m}_t \circ \pi^{-1}) \\ &= \int_{E_A^{\mathbb{N}}} \tilde{F}_t(g) \circ \pi\, d\tilde{m}_t = \int_{E_A^{\mathbb{N}}} \tilde{F}_t(g) \circ \pi\, d\tilde{m}_t \\ &= \int_{E_A^{\mathbb{N}}} \widetilde{\mathscr{L}_t}(g \circ \pi)\, d\tilde{m}_t = \int_{E_A^{\mathbb{N}}} g \circ \pi\, d\tilde{m}_t = \int_{J_{\mathscr{S}}} g\, d(\tilde{m}_t \circ \pi^{-1}) = m_t(g), \end{aligned} \tag{3.14}$$

for all $g \in C(X)$.

As it turns out, the operator $\tilde{F}_t : C(X) \to C(X)$ is *almost periodic*. We first recall the definition of almost periodicity.

Definition 3.4 (Almost Periodicity)**.** Suppose that L is a bounded operator on a Banach space B, with $L : B \to B$. Then L is called *almost periodic* if, for every $x \in B$, the orbit $(L^n(x))_{n=0}^{\infty}$ is relatively compact in B.

We will now prove that $\tilde{F}_t$ is almost periodic.

Proposition 3.5 ($\tilde{F}_t$ is Almost-Periodic)**.** Let $\mathscr{S}$ is a finitely irreducible, maximal CGDMS. If $t \in \mathrm{Fin}(\mathscr{S})$ then the operator $\tilde{F}_t : C(X) \to C(X)$ is almost periodic.

Proof. Fix a function $g \in C(X)$ and an $\epsilon \in (0,1)$. By the compactness of X we know that g is uniformly continuous, and so there is a $\delta_1 > 0$ so that

$$|g(x) - g(y)| < \epsilon \text{ whenever } |x - y| < \delta_1. \tag{3.15}$$

Since the family of functions $\{\log \|D\phi_\omega(\cdot)\|\}_{\omega \in E_A^*}$ is equicontinuous, see e.g. [9, Lemma 4.16] or [41, Lemma 19.3.4], we can choose $\delta_2 > 0$ so that

$$\left|\log \|D\phi_\omega(x)\| - \log \|D\phi_\omega(y)\|\right| < \min\left\{\frac{1}{2t}, \epsilon\right\} \tag{3.16}$$

for all $\omega \in E_A^*$ and all $x, y \in X_{t(\omega)}$ satisfying $|x - y| < \delta_2$. We also let $\delta_3 = \min\{\mathrm{dist}(X_v, X_r) : v, r \in V, v \neq r\}$. The quantity δ_3 is positive since we assume that the sets X_v, $v \in V$, are disjoint. Hence, taking $x, y \in X$ such that $|x - y| < \delta = \min\{\delta_1, \delta_2, \delta_3\}$ we know that $x, y \in$

X_v for some $v \in V$ and

$$\begin{aligned}
&\left|\tilde{F}_t^n(x) - \tilde{F}_t^n(y)\right| = \\
&= \left| \frac{1}{\lambda_t^n} \sum_{\omega \in E_A^n} \left\|D\phi_\omega(x)\right\|^t g(\phi_\omega(x)) \chi_{X_{t(\omega)}}(x) - \frac{1}{\lambda_t^n} \sum_{\omega \in E_A^n} \left\|D\phi_\omega(y)\right\|^t g(\phi_\omega(y)) \chi_{X_{t(\omega)}}(y) \right| \\
&= \frac{1}{\lambda_t^n} \left| \sum_{\omega \in E_A^n(v)} \left\|D\phi_\omega(x)\right\|^t g(\phi_\omega(x)) - \left\|D\phi_\omega(y)\right\|^t g(\phi_\omega(y)) \right| \\
&\le \frac{1}{\lambda_t^n} \sum_{\omega \in E_A^n(v)} \left(g(\phi_\omega(x)) \left| \left\|D\phi_\omega(x)\right\|^t - \left\|D\phi_\omega(y)\right\|^t \right| + \left\|D\phi_\omega(y)\right\|^t \left| g(\phi_\omega(x)) - g(\phi_\omega(y)) \right| \right) \\
&\le \frac{1}{\lambda_t^n} \sum_{\omega \in E_A^n(v)} |g(\phi_\omega(x))| \left| \left\|D\phi_\omega(x)\right\|^t - \left\|D\phi_\omega(y)\right\|^t \right| \\
&\quad + \frac{1}{\lambda_t^n} \sum_{\omega \in E_A^n(v)} \left\|D\phi_\omega(y)\right\|^t \left| g(\phi_\omega(x)) - g(\phi_\omega(y)) \right|.
\end{aligned}$$

We start by analyzing the latter term. Using the uniform continuity of g and our choice of x and y, we see that

(3.17)
$$\frac{1}{\lambda_t^n} \sum_{\omega \in E_A^n(v)} \left\|D\phi_\omega(y)\right\|^t \left| g(\phi_\omega(x)) - g(\phi_\omega(y)) \right| \overset{(3.15)}{<} \frac{\epsilon}{\lambda_t^n} \sum_{\omega \in E_A^n(v)} \left\|D\phi_\omega(y)\right\|^t \overset{(3.12)}{\le} \epsilon K^t M_t.$$

Arguing exactly as in (3.6), we also get that

$$\left| \left\|D\phi_\omega(x)\right\|^t - \left\|D\phi_\omega(y)\right\|^t \right| \le 3\epsilon t \left\|D\phi_\omega(x)\right\|^t. \tag{3.18}$$

Therefore,

$$\begin{aligned}
&\frac{1}{\lambda_t^n} \sum_{w \in E_A^n(v)} |g(\phi_\omega(x))| \left| \left\|D\phi_\omega(x)\right\|^t - \left\|D\phi_\omega(y)\right\|^t \right| \\
&\qquad \overset{(3.18)}{\le} 3\epsilon t \left\|g\right\|_\infty \sum_{\omega \in E_A^n(v)} \left\|D\phi_\omega(x)\right\|^t \overset{(3.12)}{\le} 3\epsilon t \left\|g\right\|_\infty K^t M_t.
\end{aligned} \tag{3.19}$$

Combining these bounds, we find that for every $n \in \mathbb{N}$ if $x, y \in X$ and $|x - y| < \delta$ then

$$\left|\tilde{F}_t^n(g)(x) - \tilde{F}_t^n(g)(y)\right| < \epsilon(3t\left\|g\right\|_\infty + 1) K^t M_t.$$

Hence, $\{\tilde{F}_t^n(g)\}_{n=1}^\infty$ is equicontinuous. Since it is also uniformly bounded by Proposition 3.3, so the Arzela-Ascoli theorem implies that $\{\tilde{F}_t^n(g)\}_{n=1}^\infty$ has a convergent subsequence. Therefore $\{\tilde{F}_t^n(g)\}_{n=1}^\infty$ is relatively compact. Ergo by definition, $\tilde{F}_t$ is almost periodic. □

We are now ready to prove the main result in this section.

Theorem 3.6. *Let $\mathcal{S}$ be a finitely irreducible, maximal CGDMS and let $t \in Fin(\mathcal{S})$. There exists a unique continuous function $\rho_t : X \to [0, \infty)$ so that*

$$\tilde{F}_t \rho_t = \rho_t, \text{ and } \int \rho_t \, dm_t = 1. \tag{3.20}$$

Moreover:

(1) $K^{-t} M_t^{-1} \le \rho_t \le K^t M_t$,
(2) $\{\tilde{F}_t^n(\mathbf{1})\}_{n=1}^\infty$ *converges uniformly to* ρ_t *on* X,
(3) $\rho_t|_{J_{\mathcal{S}}} = \frac{d\mu_t}{dm_t}$.

Proof. By Proposition 3.3 the sequence $\{\frac{1}{n}\sum_{j=0}^{n-1}\tilde{F}_t^j(\mathbf{1})\}_{n=1}^{\infty}$ is uniformly bounded between $K^{-t}M_t^{-1}$ and K^tM_t. Moreover, recalling the proof of Proposition 3.5, we know that

$$\left\{\frac{1}{n}\sum_{j=0}^{n-1}\tilde{F}_t^j(\mathbf{1})\right\}_{n=1}^{\infty}$$

is equicontinuous. Here, we make the convention that $\tilde{F}_t^0(\mathbf{1}) = \mathbf{1}$. Hence, the Arzela-Ascoli Theorem implies that there exists some subsequence

$$\{f_k\}_{k=1}^{\infty} := \left\{\frac{1}{n_k}\sum_{j=0}^{n_k-1}\tilde{F}_t^j(\mathbf{1})\right\}_{k=1}^{\infty}$$

which converges to a function $\rho_t \in C(X)$. Clearly, ρ_t satisfies (1). We will now show that $\tilde{F}_t(\rho_t) = \rho_t$. Since F_t is a bounded linear operator

$$F_t(f_k) \to F_t(\rho_t).$$

On the other hand if $x \in X$ then by Proposition 3.3

$$\tilde{F}_t(f_k)(x) = \frac{1}{n_k}\sum_{j=1}^{n_k}\tilde{F}_t^j(\mathbf{1})(x) = \frac{1}{n_k}\sum_{j=0}^{n_k-1}\tilde{F}_t^j(\mathbf{1})(x) + \frac{\tilde{F}_t^{n_k}(\mathbf{1})(x)}{n_k} - \frac{1}{n_k} \to \rho_t(x).$$

Thus $\tilde{F}_t(\rho_t) = \rho_t$ in X. Note also that for all $n \in \mathbb{N}$,

$$\int \tilde{F}_t^n(\mathbf{1})dm_t \overset{(3.14)}{=} \int \mathbf{1}dm_t = 1.$$

Therefore, $\int f_k dm_t = 1$ for all $k \in \mathbb{N}$. By Lebesgue's dominated convergence (since $K^{-t}M_t \le f_k \le K^tM_t$) we then deduce that

$$\int \rho_t dm_t = 1.$$

The next step in the proof is to show that if $\rho : X \to [0,\infty)$ is a continuous function such that $\rho(X) \subset [a,b]$ for some $a, b > 0$ and

$$\tilde{F}_t\rho = \rho \text{ and } \int \rho dm_t = 1,$$

then $\rho|_{J_{\mathscr{S}}} = \frac{d\mu_t}{dm_t}$. In particular, this will imply that ρ_t satisfies (3). By (3.13) we see that $\rho\circ\pi$ is a fixed point of the normalized symbolic transfer operator $\widetilde{\mathscr{L}_t} = \lambda_t^{-1}\mathscr{L}_t$. Specifically, $\widetilde{\mathscr{L}_t}(\rho\circ\pi) = \rho\circ\pi$. An application of [41, Corollary 17.7.6] implies that

$$(\rho\circ\pi)\circ\sigma^{-1} = (\rho\circ\pi) \in \mathrm{AI}(\tilde{m}_t)$$

where

$$\mathrm{AI}(\tilde{m}_t) = \left\{g \in L_1(\tilde{m}_t) : g(\tilde{m}_t\circ\sigma^{-1}) = g\tilde{m}_t, \int_{E_A^{\mathbb{N}}} g\,d\tilde{m}_t = 1, \text{ and } g \ge 0\right\}.$$

Now consider the measure

$$\tilde{\nu}_t(A) = \int_A \rho\circ\pi(\omega)d\tilde{m}_t(\omega).$$

Therefore,

$$\int_A \rho\circ\pi(\omega)d\tilde{m}_t(\omega) = \int_A \rho\circ\pi(\omega)d(\tilde{m}_t\circ\sigma^{-1})(\omega) = \int_{\sigma(A)} \rho\circ\pi(\sigma(\omega))d\tilde{m}_t(\omega),$$

and $\tilde{\nu}_t$ is shift-invariant. Note also that $\tilde{\nu}_t$ is a t-Gibbs because ρ is bounded away from 0 and infinity. Since $\tilde{\nu_t}$ is both shift-invariant and a Gibbs State, [41, Corollary 17.7.5] implies that $\tilde{\nu}_t = \tilde{\mu}_t$. Therefore, $\rho\circ\pi = \frac{d\tilde{\mu}_t}{\tilde{m}_t}$ and consequently $\rho|_{J_{\mathscr{S}}} = \frac{d\mu_t}{dm_t}$.

We will now show that ρ_t is the unique continuous function bounded away from zero and infinity which satisfies (3.20). Suppose that ρ_1 and ρ_2 both satisfy (3.20) and moreover $\rho_i(X) \subset [a,b], i = 1,2$ for some $a, b > 0$. By the previous step $\rho_1|_{\overline{J_{\mathscr{S}}}} = \rho_2|_{\overline{J_{\mathscr{S}}}}$. We denote this common restriction by $\hat{\rho}_t$. For an $\epsilon > 0$ and choose $\eta > 0$ so small that both $|\rho_1(x) - \rho_1(y)| < \epsilon$ and $|\rho_2(x) - \rho_2(y)| < \epsilon$ whenever $x, y \in X$ and $|x - y| < \eta$. We can assume that η is so small so that if $|x - y| < \eta$, then both $x, y \in X_v$ for a single $v \in V$. Since the maps $\{\phi_e\}_{e \in E}$ are contractive with Lipschitz constants bounded by $s < 1$,

$$\mathrm{diam}(\phi_\omega(X_{t(\omega)})) \le \mathrm{diam}(X_{t(\omega)}) s^{|\omega|}.$$

Fix $n \ge 1$ so large so that $\max_{v \in V}\{\mathrm{diam}(X_v)\} s^n < \eta$. Let $z \in X$ and let $v \in V$ such that $z \in X_v$. Let $\omega \in E_A^n(v)$ and take $x \in J_{\mathscr{S}} \cap \phi_\omega(X_{t(\omega)})$,

$$|\rho_2(\phi_\omega(z)) - \rho_1(\phi_\omega(z))| \le |\rho_2(\phi_\omega(z)) - \hat{\rho}(x)| + |\hat{\rho}(x) - \rho_1(\phi_\omega(z))| < 2\epsilon.$$

Therefore,

$$\begin{aligned} |\rho_2(z) - \rho_1(z)| &= \left|\tilde{F}_t^n \rho_2(z) - \tilde{F}_t^n \rho_1(z)\right| = |\tilde{F}_t^n(\rho_2 - \rho_1)(z)| \\ &\le \frac{1}{\lambda_t^n} \sum_{\omega \in E_A^n(v)} |\rho_2(\phi_\omega(z)) - \rho_1(\phi_\omega(z))| \cdot \left\|D\phi_\omega(z)\right\|^t \\ &< \frac{2\epsilon}{\lambda_t^n} \sum_{\omega \in E_A^n(v)} \left\|D\phi_\omega\right\|_\infty^t \overset{(3.12)}{\le} 2\epsilon K^t M_t. \end{aligned}$$

Taking $\epsilon \to 0$, we see that $\rho_1(z) = \rho_2(z)$.

Recall that by Proposition 3.5 the Perron–Frobenius operator $\tilde{F}_t : C(X) \to C(X)$ is almost periodic. By a well known result of Lyubich [33] we can then deduce that

$$C(X) = E_0 \oplus \overline{\mathrm{span} E_u} \tag{3.21}$$

where

$$E_0 = \{f \in C(X) : \|\tilde{F}_t^n(f)\|_\infty \to 0\}$$

and

$$E_u = \{f \in C(X) : \tilde{F}_t(f) = \lambda f \text{ for some } \lambda \in \mathbb{C} \text{ with } |\lambda| = 1\}.$$

Replicating the argument from [36, pg 147] we obtain that:

$$\overline{\mathrm{span} E_u} = \{c\rho_t : c \in \mathbb{C}\}. \tag{3.22}$$

Note that (3.21) and (3.22) imply that if a function $\rho \in C(X)$ satisfies $\tilde{F}_t(\rho) = \rho$ then $\rho = c\rho_t$ for some $c \in \mathbb{C}$. This follows because if $\rho \in C(X)$ satisfies $\tilde{F}_t(\rho) = \rho$ then there exist a unique $\rho_0 \in E_0$ and a unique $c \in \mathbb{C}$ such that $\rho = \rho_0 + c\rho_t$. Therefore,

$$\rho = \tilde{F}_t^n(\rho) = \tilde{F}_t^n(\rho_0) + c\tilde{F}_t^n(\rho_t) \overset{(3.20)}{=} \tilde{F}_t^n(\rho_0) + c\rho_t.$$

Letting $n \to 0$, we conclude that $\rho = c\rho_t$.

We showed earlier that ρ_t is the unique function in $C(X)$ which is bounded away from zero and infinity and it satisfies (3.20). However, if $\rho \in C(X)$ satisfies (3.20) then by the previous paragraph $\rho = c\rho_t$. Since,

$$\int \rho \, dm_t = \int \rho_t \, dm_t = 1$$

we deduce that $c = 1$. Hence, we have proved the uniqueness of ρ_t and the first part of the theorem is complete.

Note that if $f \in E_0$ then $\int f \, dm_t = 0$. To see this, let $f \in E_0$. By (3.14)

$$\int f \, dm_t = \int \tilde{F}_t^n(f) \, dm_t \text{ for all } n \in \mathbb{N}.$$

Since $\tilde{F}_t^n(f) \to 0$ uniformly and m_t is a finite measure, Lebesgue's dominated convergences implies that

$$\int f\,dm_t = \lim_{n\to\infty} \int \tilde{F}_t^n(f)\,dm_t = 0.$$

By (3.21), the function $\mathbf{1}$ can be represented in a unique way as

$$\mathbf{1} = \rho_0 + c\rho_t, \tag{3.23}$$

where $\rho_0 \in E_0$ and $c \in \mathbb{C}$. Since $\int \rho_0\,dm_t = 0$ and $\int \rho_t\,dm_t = \int \mathbf{1}\,dm_t = 1$ we deduce that $c = 1$. Therefore,

$$\|\tilde{F}_t^n(\mathbf{1} - \rho_t)\|_\infty = \|\tilde{F}_t^n(\mathbf{1}) - \rho_t\|_\infty \to 0.$$

The proof is complete. □

We will also need extensions of the eigenfunctions ρ_t on neighborhoods of X. They will be used in Section 4 in order to show that the functions ρ_t admit real analytic extensions on S, and for technical reasons they will also be useful in the implementation of our method in Section 5. First we need to define an extension of the Perron-Frobenius operator in $C(S)$. We assume that the sets S_v are disjoint. For $t \in \text{Fin}(\mathcal{S})$ and $g \in C(S)$, we let

$$G_t(g)(x) = \sum_{e\in E} \left\|D\phi_e(x)\right\|^t g(\phi_e(x))\chi_{S_{t(e)}}(x). \tag{3.24}$$

We also consider the normalized operators

$$\tilde{G}_t(g)(x) := \lambda_t^{-1} G_t(g)(x) = \lambda_t^{-1} \sum_{e\in E_A} \left\|D\phi_e(x)\right\|^t g(\phi_e(x))\chi_{S_{t(e)}}(x),$$

where $\lambda_t = e^{P(t)}$.

Since (2.5) and [41, Lemma 19.3.4] hold in S, replicating the the proof of Proposition 3.1 and only replacing X by S we see that $G_t : C(S) \to C(S)$ and $G_t : (C(S), \|\cdot\|_\infty) \to (C(S), \|\cdot\|_\infty)$ is a bounded linear operator. Similarly we obtain analogues of Propositions 3.3 and Proposition 3.5.

Proposition 3.7. Let $\mathcal{S}$ be a finitely irreducible, maximal CGDMS and let $t \in \text{Fin}(\mathcal{S})$. Then for all $x \in S$ and $n \in \mathbb{N}$:

$$M_t^{-1}K^{-t}e^{nP(t)} \le G_t^{(n)}(\mathbf{1})(x) \le K^t M_t^{-1} e^{nP(t)}. \tag{3.25}$$

Proposition 3.8. Let $\mathcal{S}$ be a finitely irreducible, maximal CGDMS. If $t \in \text{Fin}(\mathcal{S})$ then the operator $\tilde{G}_t : C(S) \to C(S)$ is almost periodic.

We can now state and prove the extension theorem that we will use in the following.

Theorem 3.9. *Let $\mathcal{S}$ be a finitely irreducible, maximal CGDMS. If $t \in \text{Fin}(\mathcal{S})$ then there exists a unique continuous function $\tilde{\rho}_t : S \to [0,\infty)$ so that:*

$$\tilde{G}_t\tilde{\rho}_t = \tilde{\rho}_t, \text{ and } \int \tilde{\rho}_t\,dm_t = 1. \tag{3.26}$$

(1) $\tilde{G}_t\tilde{\rho}_t = \tilde{\rho}_t$
(2) $M_t^{-1}K^{-2t} \le \tilde{\rho}_t \le M_t K^{2t}$,
(3) $\tilde{\rho}_t|_X = \rho_t$, *where ρ_t is as in Theorem 3.6,*
(4) $\{\tilde{G}_t^n(\mathbf{1})\}_{n=1}^\infty$ *converges uniformly to $\tilde{\rho}_t$ on S.*

Proof. The proof is identical to the proof of Theorem 4. Everything goes through without issues because (2.5) and [41, Lemma 19.3.4] hold in S, and the open set condition (which is not satisfied by the system $\{\phi_e : S_{t(e)} \to S_{i(e)}\}$) was never used in the proof of Theorem 4 or in any other result in this section. We only comment on (3). If $x \in X$ then

$$
\begin{aligned}
\tilde{\rho}_t(x) &\overset{(1)}{=} \tilde{G}_t(\tilde{\rho}_t)(x) \\
&= \lambda_t^{-1} \sum_{e \in E_A} \left\| D\phi_e(x) \right\|^t \tilde{\rho}_t(\phi_e(x)) \chi_{S_{t(e)}}(x) \\
&= \lambda_t^{-1} \sum_{e \in E_A} \left\| D\phi_e(x) \right\|^t \tilde{\rho}_t(\phi_e(x)) \chi_{X_{t(e)}}(x) \\
&= \tilde{F}_t(\tilde{\rho}_t)(x).
\end{aligned}
\tag{3.27}
$$

By the proof of uniqueness in Theorem 3.6 we know that $\rho_t : X \to [0,\infty)$ is the unique continuous function such that $\tilde{F}_t(\rho_t) = \rho_t$. Therefore, (3.27) implies that $\rho_t = \tilde{\rho}_t$ in X, and thus (3) has been proven. □

We conclude this section with a small, visual prelude to our numerical results following from this theory. Our numerical method uses approximations on ρ_t to estimate the Hausdorff dimension of GDMS attractors. The corresponding approximate eigenfunctions for both the full Apollonian IFS $\mathscr{A}$ and a truncation to its first 12 maps $\mathscr{A}|_{12}$ are shown below.

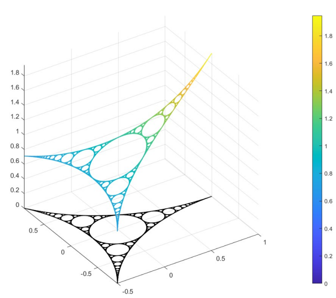

FIGURE 1. An approximation of ρ_t for $\mathscr{A}$.

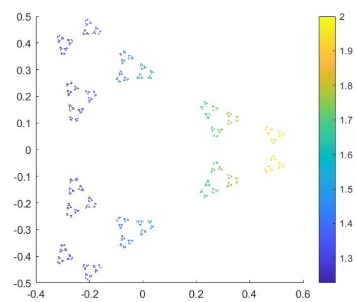

FIGURE 2. An approximation of ρ_t for $\mathscr{A}|_{12}$

4. DERIVATIVE BOUNDS FOR ρ_t

In this section we will prove derivative bounds for the eigenfunctions of the Perron-Frobenius operator F_t on maximal CGDMSs. These bounds will play a crucial role in our numerical method. We stress that, as in Section 3, the open set condition is not needed for any of the results in this section.

We start by introducing some standard notation. A *multi-index* α is an n-tuple of non-negative integers α_i. The *length* of α is

$$|\alpha| := \sum_{i=1}^{n} \alpha_i,$$

and we also denote

$$\alpha! = \alpha_1! \cdot \alpha_2! \cdots \alpha_n!.$$

For a weakly $|\alpha|$-differentiable function u, we define the operator D^α by

$$D^\alpha u = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} (u).$$

As in Section 3,

$$\mathscr{S} = \left\{V, E, A, t, i, \{X_v\}_{v\in V}, \{\phi_e\}_{e\in E}\right\}$$

will denote a maximal CGDMS and we will again assume that the sets X_v are disjoint. Moreover, we will let

$$\eta_{\mathscr{S}} = \min_{v\in V} \operatorname{dist}(X_v, \partial W_v).$$

Theorem 4.1. *Let $\mathscr{S} = \{\phi_e\}_{e\in E}$ be a a finitely irreducible, maximal CGDMS in $\mathbb{R}^n$, $n \geq 2$. Let $t \in Fin(\mathscr{S})$, let ρ_t be as in Theorem 3.6, and let α be any multi-index.*

(1) *The eigenfunctions ρ_t admit real analytic extensions on* $\operatorname{Int}(S) = \cup_{v\in V} \operatorname{Int}(S_v)$.

(2) *If $\mathscr{S}$ consists of Möbius maps then for any u, s such that* $0 < u < s < \sqrt{2} - 1$,

$$|D^\alpha \rho_t(x)| \leq \alpha! \left(\frac{n^{1/2}}{u\eta_{\mathscr{S}}}\right)^{|\alpha|} c(s)^t \rho_t(x), \quad \forall x \in X, \tag{4.1}$$

where $c(s) = (1 - s(2+s))^{-1}$.

(3) *If $n = 2$, then*

$$|D^\alpha \rho_t(x)| \leq \alpha! \left(\frac{ML}{s\eta_{\mathscr{S}}}\right)^{|\alpha|} \exp\left(tC_r \left(\frac{L}{L-2}\right)^2\right) \rho_t(x), \quad \forall x \in X, \tag{4.2}$$

where r, s, M, L can be any numbers such that $r \in (0,1)$, $s \in (0,r)$, $M > 1$, $L > 2$ and

$$C_r = \log\left(\frac{(1+r\eta)^3}{(1-r\eta)^5}\right).$$

Proof. We will denote translation by $a \in \mathbb{R}^n$ by $\tau_a(x) = x + a$, $x \in \mathbb{R}^n$. The definition of the Möbius group implies that for all $\omega \in E_A^*$, the map ϕ_ω has the form

$$\phi_\omega = \tau_{b_\omega} \circ \lambda_\omega L_\omega \circ \iota^{\varepsilon_\omega} \circ \tau_{-a_\omega},$$

where $a_\omega, b_\omega \in \mathbb{R}^n$, $\lambda_\omega > 0$, L_ω is an orthogonal transformation, $\varepsilon_\omega \in \{0,1\}$, $\iota^0 = \mathrm{Id}$ and

$$\iota^1(z) = \iota(z) = \begin{cases} \frac{1}{z}, & z \in \mathbb{C} \\ \frac{z}{|z|^2}, & z \in \mathbb{R}^n, n \geq 3. \end{cases}$$

Thus,

$$\left\|D\phi_\omega(z)\right\| = \begin{cases} \frac{\lambda_\omega}{|z-a_\omega|^2} & \text{if } \varepsilon_\omega = 1, \\ \lambda_\omega & \text{if } \varepsilon_\omega = 0 \quad (\text{ i.e. } \iota^{\varepsilon_\omega} = \mathrm{Id}). \end{cases}$$

When $\iota^{\varepsilon_\omega}$ is not the identity we have that $a_\omega \notin W_{t(\omega)}$.

We will first prove statement (2). We fix $v \in V$ and $x \in X_v$. For any $\omega \in E_A^*(v)$ we define a function $\rho_\omega : \mathbb{C}^n \to \mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ given by

$$\rho_\omega(z) = \begin{cases} \frac{|x-a_\omega|^2}{\sum_{j=1}^n \left(z_j - (a_\omega)_j\right)^2} & \text{if } \varepsilon_\omega = 1 \\ 1, & \text{if } \varepsilon_\omega = 0, \end{cases}$$

where, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{C}^n$.

For simplicity of notation we let $\eta := \eta_{\mathscr{S}}$. Let $0 < u < s < \sqrt{2} - 1$ and set $r = s\eta$. We will first show that if $\omega \in E_A^n(v)$ then

$$|\rho_\omega(z)| \leq c(s), \text{ for all } z \in B_{\mathbb{C}^n}(x, r) := \{z \in \mathbb{C}^n : \|z - x\| < r\}. \tag{4.3}$$

Note that if $\rho_\omega(z) = 1$, we have nothing to prove. Therefore we may assume that

$$\rho_\omega(z) = \frac{|x - a_\omega|^2}{\sum_{j=1}^n \left(z_j - (a_\omega)_j\right)^2}.$$

Let $z \in B_{\mathbb{C}^n}(x, r)$. Then:

$$\sum_{j=1}^n (z_j - (a_\omega)_j)^2 = \sum_{j=1}^n (z_j - x_j + x_j - (a_\omega)_j)^2 = \sum_{j=1}^n (z_j - x_j)^2 + \sum_{j=1}^n (x_j - (a_\omega)_j)^2 + 2\sum_{j=1}^n (z_j - x_j)(x_j - (a_\omega)_j),$$

and consequently

$$\begin{aligned} \left\| \sum_{j=1}^n (z_j - (a_\omega)_j)^2 \right\| &\geq \left\| \sum_{j=1}^n (x_j - (a_\omega)_j)^2 \right\| - \sum_{j=1}^n |z_j - x_j|^2 - 2\sum_{j=1}^n |z_j - x_j||x_j - (a_\omega)_j| \\ &\overset{x \in \mathbb{R}^n}{=} |x - a_\omega|^2 - \|z - x\|^2 - 2\sum_{j=1}^n |z_j - x_j||x_j - (a_\omega)_j|. \end{aligned} \tag{4.4}$$

Since $z \in B_{\mathbb{C}^n}(x, s\eta)$ and $a_\omega \notin W_{t(\omega)}$

$$\frac{\|z - x\|}{|x - a_\omega|} \leq \frac{s\eta}{\eta} = s. \tag{4.5}$$

Using the Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_{j=1}^n |z_j - x_j||x_j - (a_\omega)_j| &\leq \left(\sum_{j=1}^n |z_j - x_j|^2 \right)^{1/2} \left(\sum_{j=1}^n |x_j - (a_\omega)_j|^2 \right)^{1/2} \\ &= \|z - x\||x - a_\omega| \overset{(4.5)}{\leq} s|x - a_\omega|^2. \end{aligned} \tag{4.6}$$

Thus,

$$\begin{aligned} \left\| \sum_{j=1}^n (z_j - (a_\omega)_j)^2 \right\| &\overset{(4.4)\wedge(4.5)\wedge(4.6)}{\geq} |x - a_\omega|^2 - s^2|x - a_\omega|^2 - 2s|x - a_\omega|^2 \\ &= (1 - s(2 + s))|x - a_\omega|^2. \end{aligned} \tag{4.7}$$

Therefore,

$$|\rho_\omega(z)| = \frac{|x - a_\omega|^2}{\left\| \sum_{j=1}^n \left(z_j - (a_\omega)_j\right)^2 \right\|} \overset{(4.7)}{\leq} c(s).$$

Since $B_{\mathbb{C}^n}(x, r)$ is simply connected, the analytic function

$$z \longmapsto \rho_\omega(z), \quad z \in B_{\mathbb{C}^n}(x, r),$$

has an analytic logarithm, see e.g. [32, Lemma 6.1.10]. Thus,

$$z \longmapsto \rho_\omega(z)^t$$

is analytic for $z \in B_{\mathbb{C}^n}(x, r)$. We then let

$$b_m(z) = \sum_{\omega \in E_A^m(v)} e^{-P(t)m} \rho_\omega(z)^t \left\| D\phi_\omega(x) \right\|^t, \quad z \in B_{\mathbb{C}^n}(x, r).$$

Using Proposition 3.3 we see that for all $m \in \mathbb{N}$ and $z \in B_{\mathbb{C}^n}(x,r)$,

$$\begin{aligned}|b_m(z)| &\le e^{-P(t)m} \sum_{\omega\in E_A^m(v)} |\rho_\omega(z)|^t \left\|D\phi_\omega(x)\right\|^t \overset{(4.3)}{\le} e^{-P(t)m} c(s)^t \sum_{\omega\in E_A^m(v)} \left\|D\phi_\omega(x)\right\|^t \\ &\le c(s)^t e^{-P(t)m} F_t^m(\mathbf{1})(x) \overset{(3.12)}{\le} c(s)^t K^t M_t.\end{aligned} \tag{4.8}$$

Since the maps $z \to \rho_\omega(z)^t$ are analytic in $B_{\mathbb{C}^n}(x,r)$, Montel's theorem (see e.g. [38, Proposition 6]) and (4.8) imply that the maps b_m are analytic in $B_{\mathbb{C}^n}(x,r)$. Let $\tilde{s} \in (u,s)$ and set

$$\tilde{r} = \tilde{s}\eta.$$

A second application of Montel's Theorem implies that there is some subsequence $(b_{m_k})_{k=1}^\infty$ and a holomorphic function $b : B_{\mathbb{C}^n}(x,\tilde{r}) \to \mathbb{C}$ such that

$$b_{m_k} \to b \text{ uniformly on } B_{\mathbb{C}^n}(x,\tilde{r}). \tag{4.9}$$

Therefore, Theorem 3.6 (2), (4.8) and (4.9) imply that

$$b(z) \le c(s)^t \rho_t(x) \text{ for all } z \in B_{\mathbb{C}^n}(x,\tilde{r}). \tag{4.10}$$

Note that for $z \in B_{\mathbb{C}^n}(x,r) \cap X_v$:

$$\begin{aligned} b_m(z) &= \sum_{\omega\in E_A^m(v)} e^{-P(t)m} \rho_\omega(z)^t \left\|D\phi_\omega(x)\right\|^t = e^{-P(t)m} \sum_{\omega\in E_A^m(v)} \left(\rho_\omega(z)\frac{\lambda_\omega}{|x-a_\omega|^2}\right)^t \\ &= e^{-P(t)m} \sum_{\omega\in E_A^m(v)} \left(\frac{|x-a_\omega|^2}{\sum_{j=1}^n (z_j-(a_\omega)_j)^2}\frac{\lambda_\omega}{|x-a_\omega|^2}\right)^t \\ &= e^{-P(t)n} \sum_{\omega\in E_A^m(v)} \left(\frac{\lambda_\omega}{\sum_{j=1}^n (z_j-(a_\omega)_j)^2}\right)^t \\ &= e^{-P(t)m} \sum_{\omega\in E_A^m(v)} \left\|D\phi_\omega(z)\right\|^t \\ &= e^{-P(t)m} F_t^m(\mathbf{1})(z). \end{aligned}$$

Thus, combining Theorem 3.6 (2) and (4.9) we deduce that

$$b = \rho_t \text{ in } X \cap B_{\mathbb{C}^n}(x,\tilde{r}). \tag{4.11}$$

Recall that the *polydisk metric in* $\mathbb{C}^n$ is defined as

$$\|z-w\|_P = \max\{|z_i - w_i| : i = 1,\dots n\}, \quad z,w \in \mathbb{C}^n.$$

A *polydisk in* $\mathbb{C}^n$ is a set of the form

$$P(z,r) := \left\{w \in \mathbb{C}^n : \|w-z\|_P < r\right\}, \quad \text{where } z \in \mathbb{C}^n, r > 0.$$

It is easy to check that

$$\|z-w\|_P \le \|z-w\| \le \sqrt{n}\,\|z-w\|_P. \tag{4.12}$$

Therefore,

$$\overline{P}\left(x, \frac{1}{\sqrt{n}}u\eta\right) \subset \overline{B}_{\mathbb{C}_n}(x,u\eta).$$

Recall that b is holomorphic in $B_{\mathbb{C}_n}(x,\tilde{r})$ which is an open neighborhood of $\overline{B}_{\mathbb{C}_n}(x,u\eta)$. Therefore, if α is any multi-index, applying the Cauchy estimates (see e.g. [38, Chapter 1, Proposition 3]), we see that

(4.13)

$$|D^\alpha \rho_t(x)| \overset{(4.11)}{=} |D^\alpha b(x)| \le \alpha! \left(\frac{n^{1/2}}{u\eta}\right)^{|\alpha|} \max_{z\in\partial P(x,\frac{u\eta}{\sqrt{n}})} |b(z)| \overset{(4.10)}{\le} \alpha! \left(\frac{n^{1/2}}{u\eta}\right)^{|\alpha|} c(s)^t \rho_t(x).$$

Since $v\in V$ and $x\in X_v$ were arbitrary, the proof of statement 2 is complete.

We will now prove statement 3. We fix $v\in V$ and we define

$$b_n(z) = e^{-nP(t)} \sum_{\omega\in E_A^n(v)} \left\|D\phi_\omega(z)\right\|^t \tag{4.14}$$

for $z\in W_v$ and $n\in\mathbb{N}$. Note that for $z\in X_v$,

$$b_n(z) = e^{-nP(t)} F_t^n(\mathbf{1})(z). \tag{4.15}$$

Let $\omega\in E_A^*$. Recall that since the maps ϕ_ω are conformal we have that either $\left\|D\phi_\omega(z)\right\| = |\phi'_\omega(z)|$ (when ϕ_ω is holomorphic) or $\left\|D\phi_\omega(z)\right\| = |(\overline{\phi_\omega})'(z)|$ (when ϕ_ω is antiholomorphic). By Proposition 3.3

$$b_n(z) \overset{(3.12)}{\le} K^t M_t, \quad \text{for all } z\in X \text{ and } n\in\mathbb{N}. \tag{4.16}$$

For $\omega\in E_A^*$, define

$$\psi_\omega = \begin{cases} \phi_\omega, & \text{if } \phi_\omega \text{ is holomorphic} \\ \bar{\phi}_\omega, & \text{if } \phi_\omega \text{ is anti-holomorphic.} \end{cases}$$

Thus $\left\|D\phi_\omega(z)\right\| = |\psi'_\omega(z)|$. Fix some $\zeta_v\in X_v$ and, without loss of generality, assume that $\zeta_v = 0$. Given any $\omega\in E_A^*(v)$, define

$$\rho_\omega(z) = \frac{\psi'_\omega(z)}{\psi'_\omega(0)}, \quad z\in W_v.$$

To simplify notation we again let $\eta := \eta_{\mathscr{S}}$. Since $B(0,\eta)$ is simply connected, ρ_ω is analytic and it does not vanish, all of the branches of $\log\rho_\omega$ are well defined on $B(0,\eta)$. After choosing a suitable branch, an application of Köebe's Distortion Theorem [42, Theorem 23.1.6] gives

$$|\rho_\omega(z)| \le \frac{1+r\eta}{(1-r\eta)^3}$$

and

$$|\arg\rho_\omega(z)| \le 2\log\left(\frac{1+r\eta}{1-r\eta}\right)$$

on $\bar{B}(0,r\eta)$ for $r\in(0,1)$. Therefore $\log\rho_\omega = \log|\rho_\omega| + i\arg\rho_\omega$ is an analytic logarithm for ρ_ω and

$$|\log\rho_\omega(z)| \le \log\left(\frac{1+r\eta}{(1-r\eta)^3}\right) + 2\log\left(\frac{1+r\eta}{1-r\eta}\right) := C_r. \tag{4.17}$$

for $z\in\bar{B}(0,r\eta)$ and $r\in(0,1)$. Therefore we can write $\log\rho_\omega$ as a power series

$$\log\rho_\omega = \sum_{m=0}^\infty a_m z^m \text{ in } B(0,r\eta),$$

and by Cauchy estimates we can see that for all $s\le r$,

$$|a_m| \le \frac{C_r}{s^m\eta^m}. \tag{4.18}$$

Hence, if $z = x + iy \in B(0, r\eta)$

$$\begin{aligned}
\mathrm{Re}(\log\rho_\omega(z)) &= \mathrm{Re}\left(\sum_{m=0}^{\infty} a_m (x+iy)^m\right)\\
&= \mathrm{Re}\left(\sum_{m=0}^{\infty} a_m \sum_{k=0}^{m} \binom{m}{k} x^k (iy)^{m-k}\right)\\
&= \mathrm{Re}\left(\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{n+k} \binom{n+k}{k} i^n x^k y^n\right)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathrm{Re}\left(a_{n+k}\binom{n+k}{k} i^n\right) x^k y^n\\
&:= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} c_{k,n} x^k y^n.
\end{aligned}$$

Thus for all $s \le r$

$$|c_{k,n}| \le |a_{n+k}|\binom{n+k}{k} \le |a_{n+k}| 2^{n+k} \overset{(4.18)}{\le} \frac{C_r 2^{n+k}}{(s\eta)^{n+k}}. \tag{4.19}$$

Consider the complex valued function, formally defined on $\mathbb{C}^2$, given by

$$F(z,w) = \sum_{n,k=0}^{\infty} c_{k,n} z^k w^n, \quad z, w \in \mathbb{C}.$$

Note that for $L > 2$, the function F is holomorphic in the polydisk $P\left(0, \frac{s\eta}{L}\right)$. Indeed, $(z,w) \in P\left(0, \frac{s\eta}{L}\right)$:

$$\begin{aligned}
|F(z,w)| &\le \sum_{k,n=0}^{\infty} |c_{k,n}||z|^k |w|^n\\
&\le \sum_{k,n=0}^{\infty} \frac{C_r 2^{n+k}}{(s\eta)^{n+k}} \frac{s^{n+k}}{L^{n+k}} \eta^{n+k}\\
&= C_r \sum_{k,n=0}^{\infty} \left(\frac{2}{L}\right)^{n+k} = C_r \left(\sum_{k=0}^{\infty}\left(\frac{2}{L}\right)^k\right)^2\\
&= C_r \left(\frac{L}{L-2}\right)^2 := C_1(r,L).
\end{aligned} \tag{4.20}$$

In the following we will use the embedding $\iota : \mathbb{C} \to \mathbb{C}^2$,

$$\iota(x+iy) = (x+i0, y+i0)$$

for all $x, y \in \mathbb{R}$. To simplify notation, we let

$$A = \iota(A) \text{ if } A \subset \mathbb{C}.$$

Note also that $B(0,r) = \iota(B(0,r)) \subset P(0,r)$. Hence,

$$F = \mathrm{Re}(\log\rho_\omega) \text{ on } B(0, s\eta/L). \tag{4.21}$$

Let

$$B_n(z,w) = e^{-nP(t)} \sum_{\omega \in E_A^n(v)} \left\|D\phi_\omega(0)\right\|^t e^{tF(z,w)}, \quad z, w \in \mathbb{C}, n \in \mathbb{N}.$$

For $(x,y)\equiv x+iy=\zeta\in B\left(0,\frac{s\eta}{L}\right)$

$$
\begin{aligned}
B_n(\zeta)=B_n(x,y) &\overset{(4.21)}{=} e^{-nP(t)}\sum_{\omega\in E_A^n(v)}\left\|D\phi_\omega(0)\right\|^t e^{t\,\mathrm{Re}(\log\rho_\omega(\zeta))}\\
&= e^{-nP(t)}\sum_{\omega\in E_A^n(v)}\left\|D\phi_\omega(0)\right\|^t e^{\log|\rho_\omega(\zeta)|^t}\\
&= e^{-nP(t)}\sum_{\omega\in E_A^n(v)}\left\|D\phi_\omega(0)\right\|^t \left|\frac{\psi'_\omega(\zeta)}{\psi'_\omega(0)}\right|^t\\
&= e^{-nP(t)}\sum_{\omega\in E_A^n(v)}\left\|D\phi_\omega(\zeta)\right\|^t = b_n(\zeta).
\end{aligned}
\tag{4.22}
$$

Now note that for all $(z,w)\in P(0,s\eta/L)$

$$
\begin{aligned}
|B_n(z,w)| &= \left|e^{-nP(t)}\sum_{\omega\in E_A^n(v)}\left\|D\phi_\omega(0)\right\|^t e^{tF(z,w)}\right|\\
&\le e^{-nP(t)}\sum_{\omega\in E_A^n(v)}\left\|D\phi_\omega(0)\right\|^t e^{\mathrm{Re}(tF(z,w))}\\
&\le e^{-nP(t)}\sum_{\omega\in E_A^n(v)}\left\|D\phi_\omega(0)\right\|^t e^{t|F(z,w)|}\\
&\overset{(4.20)}{\le} e^{tC_1(r,L)}e^{-nP(t)}\sum_{\omega\in E_A^n(v)}\left\|D\phi_\omega(0)\right\|^t\\
&= e^{tC_1(r,L)}b_n(0).
\end{aligned}
\tag{4.23}
$$

Thus,

$$
|B_n(z,w)| \overset{(4.23)\wedge(4.16)}{\le} K^t M_t^{-1} e^{tC_1(r,L)} \quad \text{for } (z,w)\in P(0,s\eta/L).
$$

Since the functions

$$
(z,w)\longmapsto e^{tF(z,w)}
$$

are holomorphic in $P(0,s\eta/L)$ and the partial sums of $B_n(z,w)$ are uniformly bounded, an application of Montel's Theorem implies that the functions

$$
(z,w)\longmapsto B_n(z,w)
$$

are holomorphic in $P(0,s\eta/L)$. Via another application of Montel's Theorem, we can extract a sequence of functions B_{n_k} converging uniformly to a holomorphic function B in $\bar P\left(0,\frac{s\eta}{ML}\right)$ for any $M>1$. Thus, Theorem 3.6 (2) and (4.22) imply that

$$
B=\rho_t \text{ on } B\left(0,\frac{s\eta}{ML}\right)\cap X_v. \tag{4.24}
$$

Moreover, Theorem 3.6 (2) and (4.23) imply that

$$
|B(z,w)|\le e^{tC_1(r,L)}\rho_t(0) \text{ for all } (z,w)\in \bar P\left(0,\frac{s\eta}{ML}\right). \tag{4.25}
$$

By the Cauchy Estimates, if α is any multiindex,

$$
\begin{aligned}
|D^\alpha\rho_t(0)| \overset{(4.24)}{=} |D^\alpha B(0)| &\le \frac{\alpha!}{\left(\frac{s\eta}{ML}\right)^{|\alpha|}}\max_{(z,w)\in\partial P\left(0,\frac{s\eta}{ML}\right)}|B(z,w)|\\
&\overset{(4.25)}{\le} \frac{\alpha!}{(s\eta)^{|\alpha|}}(ML)^{|\alpha|}e^{tC_1(r,L)}\rho_t(0)=\frac{\alpha!}{(s\eta)^{|\alpha|}}(ML)^{|\alpha|}e^{tC_r\left(\frac{L}{L-2}\right)^2}\rho_t(0).
\end{aligned}
$$

The proof of (3) is complete.

We will now prove (1). First observe that using (4.11) and (4.24) we can deduce that for every $x \in X$ there exists an analytic function $R_x : B_{\mathbb{C}^n}(x, 4^{-1}\eta) \to \mathbb{C}$ such that

$$R_x|_{X \cap B_{\mathbb{C}^n}(x,4^{-1}\eta)} = \rho_t.$$

We now set

$$\tilde{\eta} = \min_{v \in V} \operatorname{dist}(S_v, \partial W_v).$$

Using Proposition 3.7, Theorem 3.9 and arguing exactly as in the proofs of (2) and (3) we can deduce that for every $x \in S$ there exists an analytic function $\tilde{R}_x : B_{\mathbb{C}^n}(x, 4^{-1}\tilde{\eta}) \to \mathbb{C}$ such that

$$\tilde{R}_x|_{X \cap B_{\mathbb{C}^n}(x,4^{-1}\tilde{\eta})} = \tilde{\rho}_t.$$

Clearly, $\tilde{\rho}_t$ is real analytic on $\operatorname{Int}(S)$ and (1) follows after we recall Theorem 3.9 (4). The proof is complete. □

We conclude this section with two remarks.

Remark 4.2. Using Proposition 3.7, Theorem 3.9 and replicating the proofs of (2) and (3) we obtain derivative bounds for the extensions $\tilde{\rho}_t$ of the eigenfunctions ρ_t:

(1) If $\mathscr{S}$ consists of Möbius maps then:

$$|D^\alpha \tilde{\rho}_t(x)| \le \alpha! \left(\frac{n^{1/2}}{u\tilde{\eta}}\right)^{|\alpha|} c(s)^t \tilde{\rho}_t(x), \quad \forall x \in S, \tag{4.26}$$

where $0 < u < s < \sqrt{2} - 1$ and $c(s) = (1 - s(2+s))^{-1}$.

(2) If $n = 2$, then

$$|D^\alpha \tilde{\rho}_t(x)| \le \alpha! \left(\frac{ML}{s\tilde{\eta}}\right)^{|\alpha|} \exp\left(t\tilde{C}_r \left(\frac{L}{L-2}\right)^2\right) \tilde{\rho}_t(x), \quad \forall x \in S, \tag{4.27}$$

where r, s, M, L can be any numbers such that $r \in (0,1), s \in (0,r), M > 1, L > 2$ and

$$\tilde{C}_r = \log\left(\frac{(1+r\tilde{\eta})^3}{(1-r\tilde{\eta})^5}\right).$$

Remark 4.3. It is straightforward to check that Theorem 4.1 (2) also holds if $\mathscr{S}$ consists of *extended Möbius maps* in $\mathbb{C}$. Recall that a map $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is an extended Möbius map if f or $\bar{f}$ belong to the Möbius group.

5. Numerical method

In this section, we describe an algorithm that rigorously computes the Hausdorff dimension of limit sets of maximal GDMSs. The method is based on the Falk-Nussbaum approach of approximating the eigenfunctions of the Perron-Frobenius operator [16], and consists of the following steps:

- Discretizing $C(X)$.
- Approximating the Perron-Frobenius operator.
- Computing upper and lower bounds for the Hausdorff dimension of the limit set.

Before we describe the method, we introduce some notation and supplementary results.

5.1. **Notation and the Bramble-Hilbert lemma.** Our numerical estimates apply results from finite element methods. Suppose we are working on an open, bounded *domain* Ω in $\mathbb{R}^n$. Throughout the paper, we will use the usual notation for the Lebesgue (L^p), Sobolev ($W^{m,p}$) and Hölder ($C^{k,\alpha}$) spaces with the corresponding norms and semi-norms. Thus if $u \in W^{m,p}(\Omega)$, the corresponding norm is defined by

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \le m} \left\| D^\alpha u \right\|^2_{L^p(\Omega)} \right)^{1/2},$$

and the *semi-norm* by

$$|u|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| = m} \left\| D^\alpha u \right\|^2_{L^p(\Omega)} \right)^{1/2}.$$

To state the following version of the Bramble-Hilbert lemma, we recall that a domain Ω is *star-shaped* with respect to $x_0 \in \Omega$ if the segment

$$[x_0, x] = \{x_0 t + x(1-t) : t \in [0,1]\} \subset \Omega$$

for all $x \in X$. Let $\mathscr{P}_m$ be the space of piecewise m-degree polynomials on Ω. We will use a version of the Bramble-Hilbert Lemma with a computational constant, found in [15].

Lemma 5.1 (Explicit Bramble-Hilbert)**.** *Suppose Ω is an open bounded set which is star-shaped with respect to every point in a measurable set of positive measure $B \subseteq \Omega$. Let $p \ge q > 1$, suppose that $j < m$, and let $d = diam(\Omega)$. If $f \in W^{p,m}(\Omega)$, then*

$$\inf_{P \in \mathscr{P}_m} |f - P|_{W^{j,q}(\Omega)} \le C_{BH} \frac{d^{m-j+n/q}}{\lambda(B)^{1/p}} |f|_{W^{m,p}(\Omega)} \tag{5.1}$$

where

$$C_{BH} = \#\left\{\alpha : |\alpha| = j\right\} \cdot \frac{m-j}{n^{1/q}} \cdot \frac{p}{p-1} \omega_{n-1}^{1/q} \left(\sum_{|\beta| = m-j} (\beta!)^{-2} \right)^{1/2}.$$

5.2. **Discretizing $C(X)$.** To discretize $C(X)$ we use a finite element approach. Take $\delta > 0$ so that $X(\delta) \subset W$, where

$$X(\delta) = \left\{x \in \mathbb{R}^n : d(x, X) < \delta\right\}.$$

For $h < \delta$ choose a subdomain $X^h \subset \mathbb{R}^n$ such that $X \subset X^h \subset X(\delta)$. We partition (triangulate) X^h into simplices, i.e. $X^h = \cup_\tau \bar{\tau}$. For simplicity we choose a conformal mesh, meaning that two neighboring simplices can intersect only by lower dimensional simplices (faces, edges, or nodes). An example of 2-dimensional conformal triangulation is shown in Figure 3.

Let $h_\tau = \mathrm{diam}(\tau)$ and define $h = \max_\tau h_\tau$. On an element τ of the mesh, we define $\mathbb{P}_1(\tau)$ the space of linear functions on τ. Furthermore, let S_h be the space of piecewise linear functions on X^h

$$S_h = \{v \in C^0(X) \ : \ v\,|_\tau \in \mathbb{P}_1(\tau)\}.$$

By the Bramble-Hilbert Lemma 5.1, for any $v \in W^{2,\infty}$,

$$\inf_{\chi \in S_h} \|v - \chi\|_{L^\infty} \le C_{BH} h^2 |v|_{W^{2,\infty}}, \tag{5.2}$$

for some constant C_{BH} independent of h, which can be explicitly estimated from the Lemma 5.1.

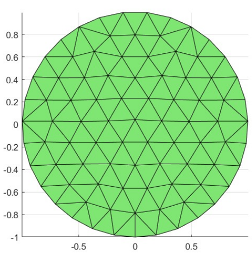

FIGURE 3. An example of a triangular mesh of $\mathbb{D}$.

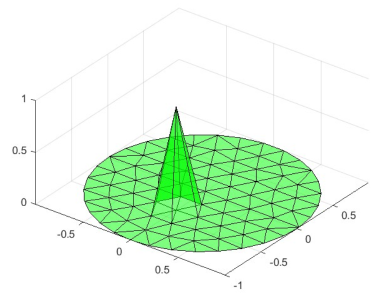

FIGURE 4. A nodal basis function for $\mathbb{P}_1(X^h)$.

Remark 5.2. Instead of triangulation, we could choose any other partition of X^h, for example rectangular elements and use bilinear functions as was done in [18], which is a valid alternative. However, in our opinion the triangulation provides more structure that makes the implementation faster and easier.

To use the finite element space S_h for computations, we need some basis functions. Since any element from S_h is uniquely defined by its values at the nodes of the triangulation $\{x_j\}_{j=1}^N$, we choose basis functions $\{\phi_i(x)\}_{i=1}^N$ satisfying

$$\phi_i(x_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad i, j = 1, 2, \dots, N,$$

and define a nodal (Lagrange) interpolation operator $\mathscr{I}_h : C^0(X) \to S_h$ by

$$\mathscr{I}_h v(x) = \sum_{j=1}^N v(x_j)\phi_j(x).$$

Since the nodal interpolant $\mathscr{I}_h$ is invariant on S_h, i.e. $\mathscr{I}_h q = q$ for any $q \in S_h$, and bounded from $L^\infty \to L^\infty$ with a constant 1, by the triangle inequality, for an arbitrary $q \in S_h$, we have

$$\begin{aligned} \|v - \mathscr{I}_h v\|_{L^\infty} &\le \|v - q\|_{L^\infty} + \|q - \mathscr{I}_h v\|_{L^\infty} \\ &\le \|v - q\|_{L^\infty} + \|\mathscr{I}_h(q - v)\|_{L^\infty} \\ &\le 2\|v - q\|_{L^\infty}. \end{aligned}$$

Thus, we immediately obtain the following corollary.

Corollary 5.3. *For any $v \in W^{2,\infty}(\Omega)$,*

$$\|v - \mathscr{I}_h v\|_{L^\infty(\Omega)} \le 2C_{BH} h^2 |v|_{W^{2,\infty}(\Omega)},$$

where C_{BH} is the same constant as in (5.2).

Provided we have the following continuity and derivative estimates for ρ_t

$$|\rho_t(x) - \rho_t(y)| \le C_1 |x - y| \quad x, y \in X^h \tag{5.3}$$

$$|D^\alpha \rho_t(x)| \le C_2 |\rho_t(x)| \quad x \in X^h, \quad |\alpha| = 2, \tag{5.4}$$

for some computable constants C_1 and C_2, for any $x \in \tau$, we obtain

$$0 \le |\rho_t(x) - \mathscr{I}_h \rho_t(x)| \le 2C_{BH} h_\tau^2 |\rho_t|_{W^{2,\infty}(\tau)} \le 2C_{BH}(C_1 h_\tau + 1) C_2 h_\tau^2 \rho_t(x).$$

Thus we have

$$(1 - \mathrm{err}_\tau)\mathscr{I}_h \rho_t(x) \le \rho_t(x) \le (1 + \mathrm{err}_\tau)\mathscr{I}_h \rho_t(x) \quad \forall x \in \tau,\ \forall \tau, \tag{5.5}$$

where

$$\mathrm{err}_\tau = 2C_{BH}(C_1 h_\tau + 1) C_2 h_\tau^2.$$

Thus, $\mathscr{I}_h \rho_t$ provides upper and lower pointwise bounds for ρ_t and these bounds tend to 1 quadratically as $h \to 0$. From now on we assume that h is sufficiently small, so that

$$\mathrm{err} := \max_\tau \mathrm{err}_\tau < 1.$$

5.3. **Approximating the Perron-Frobenius operator when the alphabet E is finite.** Next we want to approximate the Perron-Frobenius operator $F_t : C(X) \to C(X)$ which was introduced in (3.1). Recall that

$$F_t(g)(x) = \sum_{e \in E_A} \left\| D\phi_e(x) \right\|^t g(\phi_e(x)) \chi_{X_{t(e)}}(x), \quad g \in C(X).$$

Using (5.5), we have

$$\begin{aligned} (1 - \mathrm{err}) \sum_{e \in E_A} \|D\phi_e(x)\|^t \mathscr{I}_h \rho_t(\phi_e(x)) \chi_{X_{t(e)}}(x) &\le F_t \rho_t(x) \\ \le (1 + \mathrm{err}) \sum_{e \in E_A} \|D\phi_e(x)\|^t \mathscr{I}_h \rho_t(\phi_e(x)) \chi_{X_{t(e)}}(x) & \quad \forall x \in X^h. \end{aligned} \tag{5.6}$$

Let $\alpha \in \mathbb{R}^N$ be a vector with entries

$$\alpha_j = \rho_t(x_j) = \mathscr{I}_h \rho_t(x_j) \quad j = 1, 2, \ldots, N,$$

and define two matrices $A_t, B_t \in \mathbb{R}^{N \times N}$ such that

$$\begin{aligned} (A_t \alpha)_j &:= (1 - \mathrm{err}) \sum_{e \in E_A} \|D\phi_e(x_j)\|^t \mathscr{I}_h \rho_t(\phi_e(x_j)) \chi_{X_{t(e)}}(x_j) \\ (B_t \alpha)_j &:= (1 + \mathrm{err}) \sum_{e \in E_A} \|D\phi_e(x_j)\|^t \mathscr{I}_h \rho_t(\phi_e(x_j)) \chi_{X_{t(e)}}(x_j). \end{aligned}$$

One of the technical difficulties of assembling the above matrices is to locate an element τ that contains $\phi_e(x_j)$. At this point, the structure of the triangulation comes very handy as one can use a barycentric point location, which makes the assembly rather efficient. For example if for the node x_j, the image $\phi_e(x_j) \in \tau_i$ for some $1 \le i \le N$, then we have $\phi_e(x_j) = \lambda_1 x_1^i + \cdots + \lambda_{n+1} x_{n+1}^i$, where $x_1^i, \ldots, x_{n+1}^i$ are the vertices of the simplex τ_i and $\lambda_1, \ldots, \lambda_{n+1} \ge 0$, $\lambda_1 + \cdots + \lambda_{n+1} = 1$ are the barycentric coordinates of the point $\phi_e(x_j)$. Thus, we obtain the contribution to the entries of j-th columns of the matrices A and B the rows corresponding to the global indices of the nodes $x_1^i, \ldots, x_{n+1}^i$ weighted by the barycentric coordinates $\lambda_1, \ldots, \lambda_{n+1}$. This step can be vectorized for all $e \in E$, making the assembly very efficient.

5.4. **Computing upper and lower bounds of the Hausdorff dimension.** The matrices A_t, B_t consist of non-negative entries and we can use the following key result for such matrices [18, Lemma 3.2].

Lemma 5.4. *Let M be an $N \times N$ matrix with non-negative entries and w an N-vector with strictly positive components. Then,*

$$\text{if} \quad (Mw)_j \geq \lambda w_j, \quad j = 1, \dots, N, \text{ then } r(M) \geq \lambda,$$
$$\text{if} \quad (Mw)_j \leq \lambda w_j, \quad j = 1, \dots, N, \text{ then } r(M) \leq \lambda,$$

where $r(M)$ denotes the spectral radius of M.

Since

$$(F_t \rho_t)(x_j) = r(F_t)\rho_t(x_j) \quad j = 1, \dots, N,$$

where $r(F_t) = \lambda_t = e^{P(t)}$ denotes the spectral radius of F_t, for all $j = 1, \dots, N$,

$$(A_t \alpha_t)_j \leq F_t \rho_t(x_j) = \lambda_t \rho_t(x_j) = r(F_t)(\alpha_t)_j,$$

and

$$(B_t \alpha_t)_j \geq F_t \rho_t(x_j) = \lambda_t \rho_t(x_j) = r(F_t)(\alpha_t)_j.$$

Therefore Lemma 5.4 implies that

$$r(A_t) \leq r(F_t) = \lambda_t \leq r(B_t).$$

Let $t^* = \dim_{\mathcal{H}}(J_S)$ and recall by Bowen's formula from Section 2 that $r(F_{t^*}) = \lambda_{t^*} = 1$. Thus, our goal is to compute tight upper and lower bounds $\underline{t}, \overline{t}$ such that $t^* \in (\underline{t}, \overline{t})$. Since the map $t \to \lambda_t$ is strictly decreasing, if we find $\underline{t}$ such that $r(A_{\underline{t}}) > 1$, then $r(F_{t^*}) = 1 < r(A_{\underline{t}}) \leq r(F_{\underline{t}})$ and as a result $t^* > \underline{t}$. Similarly, if we find $\overline{t}$ such that $r(B_{\overline{t}}) < 1$, then $r(F_{\overline{t}}) \leq r(B_{\overline{t}}) < 1 = r(F_{t^*})$ and as a result $t^* < \overline{t}$. In conclusion, we would have $\underline{t} < t^* < \overline{t}$, which is a rigorous effective estimate for the Hausdorff dimension of the set J_S.

Thus, given matrices A_t and B_t the problem essentially reduces to nonlinear problem of computing a parameter t that corresponds to a leading eigenvalue 1. Since there is a spectrum gap between the leading eigenvalue 1 and the rest, this problem is well-suited for a power method, which starting from arbitrary vector x_0, generates an iterative sequence $\{x_k\}_{k=0}^{\infty}$ given by

$$x_{k+1} = \frac{A_t x_k}{|A_t x_k|}$$

and the corresponding sequence of numbers $\{\mu_k\}_{k=0}^{\infty}$ is given by

$$\mu_k = \frac{x_k^T A_t x_k}{x_k^T x_k},$$

such that $\mu_k \to 1$ and x_k converges to the corresponding eigenvector at the rate $|\mu_2|^k$. Since $|\mu_2| < 1$, the power method is rather efficient. Furthermore, since the power method only requires matrix-vector multiplication, there is no need to construct the matrices A_t and B_t explicitly, which is an important issue for large problems, for example in 3D.

Using the logarithm, the above nonlinear problem is equivalent to root finding problem. There are many good choices can be used. In our computations, we used a variation of a secant method, since good initial guesses for such problem are available.

5.5. **Case of infinite alphabet.** In the case of the infinite alphabet, we consider the truncated finite alphabet $\tilde{E} \subset E$ and initially define the matrices on the truncated alphabet as,

$$(\tilde{A}_t\alpha)_j = (1-\text{err}) \sum_{e\in\tilde{E}_A} \|D\phi_e(x_j)\|^t \mathscr{I}_h\rho_t(\phi_e(x_j))\chi_{X_{t(e)}}(x_j)$$

$$(\tilde{B}_t\alpha)_j = (1+\text{err}) \sum_{e\in\tilde{E}_A} \|D\phi_e(x_j)\|^t \mathscr{I}_h\rho_t(\phi_e(x_j))\chi_{X_{t(e)}}(x_j).$$

For estimating the lower bound $\underline{t}$, we can use the matrix $\tilde{A}_t$, however for estimating the upper bound $\overline{t}$, we need to modify the matrix $\tilde{B}_t$ to account for the tail

$$(1+\text{err}) \sum_{e\in E_A\setminus\tilde{E}_A} \|D\phi_e(x_j)\|^t \mathscr{I}_h\rho_t(\phi_e(x_j))\chi_{X_{t(e)}}(x_j).$$

Provided that

$$\sum_{e\in E_A\setminus\tilde{E}_A} \|D\phi_e(x)\|^t \mathscr{I}_h\rho_t(\phi_e(x))\chi_{X_{t(e)}}(x)$$

converges uniformly in x, in view of the continuity estimate (5.3), we have that for any $1\le j\le N$

$$(1+\text{err}) \sum_{e\in E_A\setminus\tilde{E}_A} \|D\phi_e(x_j)\|^t \mathscr{I}_h\rho_t(\phi_e(x_j))\chi_{X_{t(e)}}(x_j) \le C_0\rho(x_1).$$

Thus, for each j column of $\tilde{B}_t$ we only need to modify the first row of $\tilde{B}_t$. In the above estimate, the choice of x_1 is arbitrary, we could select any other node (or nodes) as well. The exact estimate of the constant C_0, depends of course on a concrete problem and the size of $\tilde{E}$. In many examples, we can chose the size of the truncated so large that the modified matrix allows us to obtain a sharp upper bound $\overline{t}$.

Remark 5.5. In the case of infinite alphabet, We have two sources of errors, one is due to discretization of the domain X and the other is due to truncation of the alphabet E. The sizes of the matrices $\tilde{A}_t$ and $\tilde{B}_t$ only depend on the discretization parameter h and not on the truncated alphabet $\tilde{E}$. The size of the truncation alphabet affects of course the entries of the matrices $\tilde{A}_t$ and $\tilde{B}_t$ and the time it takes to assemble them. However, as we already mentioned in the section 5.3, this step can be made very efficient and in all our examples given below, we are able to take $\tilde{E}$ so large (corresponding to C_0 be very small) that the dominating error is due to the discretization parameter h only.

5.6. **Mesh Trimming.** In this section we provide a meshing scheme for fractals generated by CGDMSs, that eliminates computationally redundant points in the iterative creation of the mesh and some cases can reduce the number of degrees of freedom by orders of magnitude. We visualize the method using the Apollonian gasket, although the algorithm is presented below works for general CGDMS.

In section 5.3, we showed that elements of the approximating matrices are defined by the expression

$$\sum_{e\in E} \|D\phi_e(x_j)\|^t \mathscr{I}_h\rho_t(\phi_e(x_j)). \tag{5.7}$$

Looking at Figure 5, which shows a general mesh on X^h and the image of all the nodes under the map ϕ_e as blue dots, one can see that the blue dots are not distributed uniformly. In fact most of the elements (triangles) do not contain any such dots. This implies that when we generate the matrices using formula (5.7), the resulting matrices will have many zero columns. Such columns do not play any role in computing the corresponding eigenvectors, and the size of the matrix can be significantly reduced by

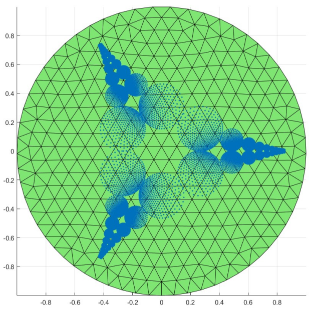

FIGURE 5. The initial mesh X^h with the image set $\mathscr{X}^{h_1}$.

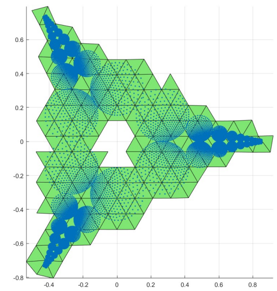

FIGURE 6. A new mesh after deleting redundancies.

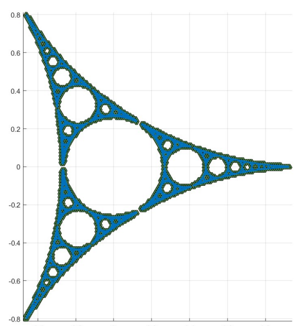

FIGURE 7. The 5th iteration of mesh trimming applied to a truncated Apollonian gasket IFS. The corresponding image points are overlayed in blue.

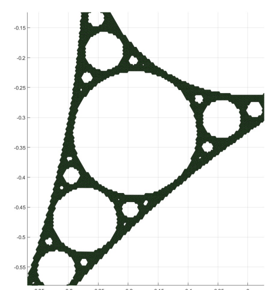

FIGURE 8. A zoomed in version of the aforementioned mesh.

removing those zero columns and the corresponding rows. Alternatively, we could just remove those elements altogether at the beginning, see the in the mesh on Figure 6, which significantly saves time for generating the approximating matrices.

Remark 5.6. The step 3 for unstructured meshes may be computationally difficult, however, if the meshes consist of shape regular simplices this step can be done directly using the edge data structure.

Algorithm 1 Mesh trimming

1. Choose initial mesh size h and generate an initial mesh on X^h with nodes $\{x_j\}_{j=1}^n$.
2. Set $E = \emptyset$.

for j=1:n **do**

3. Compute $\phi_e(x_j)$.
4. Detect all elements τ containing $\phi_e(x_j)$ and add them to the set E.

end for

5. Remove all elements which are not in E.
6. Refine h and repeat until the desired accuracy is achieved.

6. APPLICATIONS

In this section, we illustrate how the method can be applied to various CGDMSs. In particular, we verify that these systems are indeed CGDMSs and highlight some properties of the general families that these systems belong to. In Section 6 we will describe the specific implementation of our numerical method for these examples.

6.1. n**-dimensional continued fractions.** In this section we review n-dimensional continued fractions and some of their dynamical properties. We find their θ-number and prove they are a CIFS.

Definition 6.1 (n-dimensional Continued Fractions IFS)**.** Let $v_{1/2} = (1/2, 0, \ldots, 0)$ and let $|\cdot|$ denote the Euclidean norm. The n-dimensional continued fraction IFS, denoted $\mathcal{CF}_E$, consists of the maps

$$\left\{\phi_e : X \to X |\ e \in \mathbb{N} \times \mathbb{Z}^{n-1},\ \phi_e(x) = \frac{x+e}{|x+e|^2}\right\}, \tag{6.1}$$

where

$$X = \left\{x \in \mathbb{R}^n : |x - v_{1/2}| \leq \frac{1}{2}\right\}.$$

To verify that $\mathcal{CF}_E$ is a CIFS, first note that $X = \overline{\mathrm{Int}(X)}$. We are left with three properties to check. First, the the system has to satisfy the OSC. Second, each ϕ_e must map X to itself to be an IFS. Finally, there must exist an open set $W \supset X$ furnishing a conformal extension for each $e \in E$.

Lemma 6.2. *For any* $e_1, e_2 \in E$ *with* $e_1 \neq e_2$*,*

$$\phi_{e_1}(\mathrm{Int}(X)) \cap \phi_{e_2}(\mathrm{Int}(X)) = \emptyset.$$

Proof. Each ϕ_e in $\mathcal{CF}_E$ is the composition of two distinct maps — a translation τ_e followed by an inversion ι about the unit sphere:

(1) $\tau_e : x \mapsto x + e$, and
(2) $\iota : x \mapsto x/|x|^2$.

Since $|e_1 - e_2| \geq 1 = \mathrm{diam}(X)$, we see that for distinct $e_1, e_2 \in E$

$$\tau_{e_1}(\mathrm{Int}(X)) \cap \tau_{e_2}(\mathrm{Int}(X)) = \emptyset.$$

Applying the injectivity of an inversion,

$$\iota \circ \tau_{e_1}(X) \cap \iota \circ \tau_{e_2}(X) = \emptyset,$$

so the open set condition is satisfied. □

We now provide an analytic proof that each ϕ_e maps X to itself, proving that $\mathscr{C}\mathscr{F}_E$ is an IFS.

Lemma 6.3. *For each $e \in E$, $\phi_e : X \to X$.*

Proof. It suffices to show that for all $x \in X$, $e \in E$

$$\left|\phi_e(x) - v_{1/2}\right| \leq \frac{1}{2}$$

Since $X = B(v_{1/2}, 1/2)$, for all $x \in X$, $x_1 + e_1 \geq 1$,

$$\sqrt{1 + (x_2+e_2)^2 + \ldots + (x_n+e_n)^2} \leq |x+e|.$$

Dividing through by $|x+e|^2$ and squaring both sides gives

$$\left(\frac{1}{|x+e|}\right)^4 + \left(\frac{x_2+e_2}{|x+e|^2}\right)^2 + \ldots + \left(\frac{x_n+e_n}{|x+e|^2}\right)^2 \leq \frac{1}{|x+e|^2}.$$

From here, subtracting terms yields

$$\left[-\frac{1}{|x+e|^2} + \frac{1}{|x+e|^4} + \frac{1}{4}\right] + \left(\frac{x_2+e_2}{|x+e|^2}\right)^2 + \ldots + \left(\frac{x_n+e_n}{|x+e|^2}\right)^2 \leq \frac{1}{4}.$$

Equating

$$\left[-\frac{1}{|x+e|^2} + \frac{1}{|x+e|^4} + \frac{1}{4}\right] = \left(\frac{1}{|x+e|^2} - \frac{1}{2}\right)^2$$

and taking square roots, we see that

$$\left|\phi_e(x) - v_{1/2}\right| \leq \sqrt{\left(\frac{1}{|x+e|^2} - \frac{1}{2}\right)^2 + \left(\frac{x_2+e_2}{|x+e|^2}\right)^2 + \ldots + \left(\frac{x_n+e_n}{|x+e|^2}\right)^2} \leq \frac{1}{2}.$$

□

We are interested in the existence and maximality of conformal extensions of $\mathscr{C}\mathscr{F}_E$. The existence of a conformal extension shows that $\mathscr{C}\mathscr{F}_E$ is a CIFS, while finding maximal extensions is needed for eigenfunction bounds. Introducing some notation, for all $\delta > 0$, let

$$X(\delta) = \left\{x \in \mathbb{R}^n : d(x, X) < \delta\right\}.$$

To show the existence of a uniformly contracting conformal extension we must find a $\delta > 0$ so that

$$\phi_\omega(X(\delta)) \subset X(\delta).$$

Note that in this lemma, we only consider ϕ_ω corresponding to words of finite length greater than one, as it is not true for single letters (specifically, letting $v_1 = (1, 0, \ldots, 0)$, we see that $\left\|D\phi_e(0)\right\| = 1$ whenever $e = v_1$). While this formally corresponds to a different dynamical system, they clearly share the same limit set.

Lemma 6.4. *For any $0 < \delta < 1$,*

$$\phi_w(X(\delta)) \subset X(\delta),$$

where $w \in E^* \setminus E$.

Proof. To show this, note that since $\phi_w(X) \subset X$, it suffices to show that $\phi_{ab}(X(\delta)) \subset X$ for any $a, b \in E$. Consider the set

$$R = \{x \in \mathbb{R}^n : x_1 > 1\}.$$

We wish to show that $\iota(R) = X$. To do so, note that the boundary of ∂R is a half plane, and thus described uniquely by $n+1$ points. If we can show that $\iota(\partial R) = \partial X$, we will be done.

By properties of Möbius transformations, we know that $\iota(\partial R)$ is either a sphere or a $n-1$ hyperplane. Notably, any $n+1$ points determine this image. For the point at infinity, $\iota(\infty) = 0$. Moreover, $\iota(e_1) = e_1$. Now, let p_i, $i = 1, \ldots, n-1$ be the point $e_1 + e_i$. Certainly $p_i \in \partial R$ for each i, as

$$\iota(p_i) = \frac{e_1 + e_i}{|e_1 + e_i|^2} = \frac{1}{2}(e_1 + e_i) \in \partial X,$$

so our claim is proven.

Defining the set

$$R_\delta = \left\{x \in \mathbb{R}^n : x_1 > -\delta\right\} \supset X(\delta),$$

note that the first coordinate of any point in $\phi_b(R_\delta)$ is always positive when $\delta < 1$. Hence for any $x \in R_\delta$ and any $a \in E$, $\pi_1(\phi_b(x) + a) > 1$, so $\phi_{ab}(X(\delta)) \subset \phi_{ab}(R_\delta) \subset X$, verifying our claim. Note also that this inequality is strict, for if $\delta = 1$ then $-e_1 \in X(\delta)$, and $\phi_{e_1}(-e_1)$ is undefined. □

Hence we have shown that n-dimensional continued fractions are a CIFS. We now move onto tail bounds for these systems for continued fraction systems in any dimension.

Lemma 6.5 (Tail Bounds)**.** *Let $R \geq 1$. Then for any $x, y \in X$,*

$$\sum_{e \in E,\ |e| \geq R+2} \frac{1}{|x+e|^{2t}} \rho_t(\phi_e(x)) \leq \frac{\omega_{n-1}}{2} C_{|\alpha|=1}(s,t) \frac{R^{n-2t}}{2t-n} \rho_t(y), \tag{6.2}$$

where ω_{n-1} is the surface area of the $n-1$ sphere of radius R and

$$C_{|\alpha|=1}(s,t) = \min_{0<s<\sqrt{2}-1} \frac{\sqrt{n}}{s}(1 - s(2+s))^{-t}.$$

Proof. Consider $e \in \Omega = \{(e_1, \ldots, e_n) \in \mathbb{N} \times \mathbb{Z}^{n-1} : |e| \geq R+2\}$. From the definition of ϕ_e, we immediately have

$$|\phi_e(x)| \leq \frac{1}{R} \quad \forall x \in X.$$

In addition, by the Mean Value Theorem and the derivative estimate (4.1) with $|\alpha| = 1$, we have

$$\rho_t(x) - \rho_t(y) \leq C_{|\alpha|=1}(s,t)|x-y| \quad \forall x, y \in X,$$

and as a result

$$\sum_{e \in \Omega} \frac{1}{|x+e|^{2t}} \rho_t(\phi_e(x)) \leq \operatorname{diam}(X) C_{|\alpha|=1}(s,t) \rho_t(y) \sum_{e \in \Omega} \frac{1}{|x+e|^{2t}} = C_{|\alpha|=1}(s,t) \rho_t(y) \sum_{e \in \Omega} \frac{1}{|x+e|^{2t}} \tag{6.3}$$

for any $x, y \in X$. To estimate the sum we use the *integral comparison test.* Using that for any $x \in X$ and any $e \in E$,

$$|e-1| \leq |x+e|,$$

we have

$$\sum_{e\in\Omega} \frac{1}{|x+e|^{2t}} \le \sum_{e\in\Omega} \frac{1}{|e-1|^{2t}} \le \frac{1}{2}\int_{|x|\ge R} \frac{dx}{|x|^{2t}}.$$

Using the spherical coordinates $\rho = |x|$, we compute

$$\int_{|x|\ge R} \frac{dx}{|x|^{2t}} = \omega_{n-1}\int_R^\infty \rho^{n-1-2t}\,d\rho = \omega_{n-1}\frac{R^{n-2t}}{2t-n}.$$

Combining, we obtain the result.

□

Remark 6.6. Following the lines of more refined analysis from [17], we could obtain a slightly sharper tail bounds. However, the above bounds are more than sufficient for our purpose, and the dominating error is due to discretization of $C(X)$.

6.2. **Quadratic perturbations of linear maps (***abc-examples).* In this section we discuss a CIFS in the the complex plane which does not consist of Möbius maps. Suppose that $r \in (0,1)$, $X = B(0,r) := \{z \in \mathbb{C} : |z| \le r\}$, and let

$$\phi_e(z) = a_e z + b_e + c_e z^2$$

for $e \in E \subseteq \mathbb{N}$. The corresponding (formal) CIFS is denoted by $\mathcal{S}_{abc} = \{X, I, \{\phi_e : X \to X\}_{e\in E}\}$. An arbitrary set of such maps will not be a CIFS. The maps may not be contractions, have intersecting images, or be non-invertible. Conformality is automatic, so for verification purposes we need to do the following:

(1) Verify the maps ϕ_i are contractions on X.
(2) Find an open, connected set $W \supset X$ for which each ϕ_i extends to a uniformly contracting map taking W into itself.
(3) Verify the OSC holds on X.
(4) Verify the Bounded Distortion Property.

Many of these questions may be verified using computational means, provided the system satisfies appropriate separation properties. An investigation of these algorithms is beyond the scope of the paper, and instead we show how to verify this is a CIFS in one particular case. In particular, consider the CIFS $\mathcal{S}_{abc}$ consisting of the maps

$$\phi_1(z) = .25iz + .1 + .1z^2$$
$$\phi_2(z) = .2iz - .1 - .1i + .05z^2$$
$$\phi_3(z) = .1z + .1 - .1i + .04z^2$$

defined on X with $r = 0.2$. To show this system maps X to itself we use norm estimates. For all $e = 1,2,3$, we have that

$$|\phi_e(z)| \le r|a_e| + |b_e| + r^2|c_e|$$

implying

$$|\phi_1(z)| \le .25r + .1 + .1r^2 = .154 < 0.2$$
$$|\phi_2(z)| \le .2r + \sqrt{.02} + .05r^2 = \frac{\sqrt{2}}{10} + .042 < 0.2$$
$$|\phi_3(z)| \le .1r + \sqrt{.02} + .04r^2 = \frac{\sqrt{2}}{10} + .0216 < 0.2$$

for all $z \in X$. Hence $\phi_e(X) \subset X$ for all $e \in E$. To verify the OSC, simply note that $d(b_{e_i}, b_{e_j}) \geq 0.1$ for all $i \neq j$. Pairing this with the fact that $r|a_e| + r^2|c_e| \leq .054 < 0.1$ for all $e \in E$, it is obvious that $\phi_{e_i}(X) \cap \phi_{e_j}(X) = \emptyset$ for all $i \neq j$. More explicitly, we have that

$$\phi_1(X) \subseteq B(b_1, |a_1|r + |c_1|r^2) = B(0.1, .054)$$
$$\phi_2(X) \subseteq B(b_2, |a_2|r + |c_2|r^2) = B(-0.1 - 0.1i, .042)$$
$$\phi_3(X) \subseteq B(b_3, |a_3|r + |c_3|r^2) = B(0.1 - 0.1i, .0216).$$

Checking case by case, we find that

(1) For ϕ_1 and ϕ_2,
$$|b_1 - b_2| = |0.1 - (-0.1 - 0.1i)| = \frac{\sqrt{5}}{10} \geq .096 = .054 + .042 = r_1 + r_2,$$
so $\phi_1(X)$ and $\phi_2(X)$ are disjoint.

(2) For ϕ_1 and ϕ_3,
$$|b_1 - b_3| = |0.1 - (0.1 - 0.1i)| = \frac{1}{10} \geq .0765 = .054 + .0216 = r_1 + r_3,$$
so $\phi_1(X)$ and $\phi_3(X)$ are disjoint.

(3) For ϕ_2 and ϕ_3,
$$|b_2 - b_3| = |-0.1 - 0.1i - (0.1 - 0.1i)| = \frac{1}{5} \geq .0258 = .042 + .0216 = r_2 + r_3,$$
so $\phi_2(X)$ and $\phi_3(X)$ are disjoint.

Hence our system satisfies the OSC. To find an open set $W \supset X$ satisfying property 3, recall that
$$\eta := \min\{1, \operatorname{dist}(X, \partial W\},$$
we wish to find the supremum of r for which $|D\phi_e(z)| < 1$ whenever $|z| \leq r$. For an arbitrary $r > 0$, taking the supremum norm on $B(0, r)$ yields
$$\left\| D\phi_e \right\|_\infty = |a_e| + 2r|c_e|,$$
we must solve
$$|a_i| + 2r_i|c_i| = 1 \Longrightarrow r_i = \frac{1 - |a_i|}{2|c_i|}.$$
Doing so, we have that
$$r_1 = \frac{1 - 0.25}{2 \cdot 0.1} = 5 \cdot 0.75 = 3.75, \; r_2 = \frac{1 - 0.2}{2 \cdot 0.05} = 0.8 \cdot 10 = 8, \; r_3 = \frac{1 - .1}{2 \cdot .04} = 0.9 \cdot 12.5 = 11.25.$$
Hence $\eta = 1$ for this example.

Moving onto injectivity, it is sufficient to show the existence of a nonzero directional derivative for some direction. In particular, taking derivatives yields
$$-i\phi_1'(z) = .25 - .2iz$$
$$-i\phi_2'(z) = .2 - .1iz$$
$$\phi_3'(z) = .1 + .08z.$$
Since $|z| < 0.2$ we have that
$$\operatorname{Re}(-i\phi_1'(z)) \geq .25 - .04 = .21 > 0$$
$$\operatorname{Re}(-i\phi_2'(z)) \geq .2 - .02 = .18 > 0$$
$$\operatorname{Re}(\phi_3'(z)) \geq .1 - .016 = .084 > 0,$$

so injectivity has been proven. Of course, since the alphabet is finite, the tail bounds are not needed.

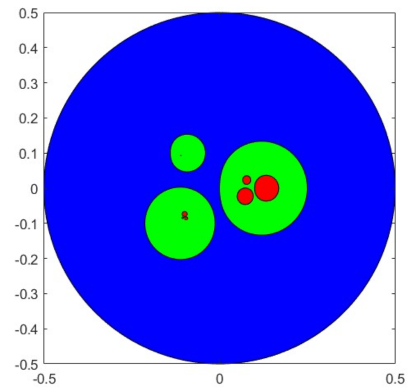

FIGURE 9. The first (green) and second (red) iterations of a system consisting of quadratic perturbations of linear maps.

6.3. **An application to Schottky groups.** Another application of our estimates it to 2D Schottky Groups, specifically classical, nonhyperbolic Schottky Groups generated by Möbius transformations. Suppose that B_j, $j = \pm 1, \pm 2, \ldots, \pm q$, are disjoint closed disks in $\hat{\mathbb{C}}$, and consider Möbius transformations of the form

$$g_j : \hat{\mathbb{C}} \setminus \bar{B}_{-j} \to B_j \text{ defined by } g_j(z) = \frac{a_j z + b_j}{c_j z + d_j}.$$

For each j, g_j is a contraction on its domain of definition. However, this is not yet a CGDMS as it does not satisfy the open set condition. To rectify this, consider the $2n(2n-1)$ maps

$$g_{j,i} : \overline{B_i} \to B_j, \text{ where } g_{j,i} = g_j|_{\overline{B_i}}$$

all of which are defined when $i \neq -j$. The incidence matrix A is then just a matrix of 1's whenever $i \neq -j$, and zeros everywhere else. Moreover, extending g_j to the whole Riemann Sphere, it is apparent that $|Dg_j(z)| \geq 1$ only when $z \in \bar{B}_{-j}$, so uniform contractivity follows from the finiteness of the system. We now provide a specific example of a Schottky group for which our theory applies. Consider the initial, paired balls

$$B_1 = B(-0.7 - i, .4), \quad B_{-1} = B(0, 0.5);$$
$$B_2 = B(-0.8, 0.2), \quad B_{-2} = B(0.8, 0.2);$$
$$B_3 = B(0.9i, 0.3), \quad B_{-3} = B(0.8 - 0.9i, 0.6).$$

Visually, these circles are shown below:

The mappings between them were found computationally and are certainly not unique. Because of this, Schottky groups give a great example CGDMSs whose 1-cylinder sets agree but whose limit sets have different Hausdorff dimension. The maps used in this paper are as follows:

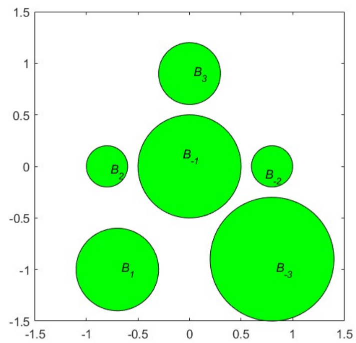

FIGURE 10. The initial circles for our Schottky group example.

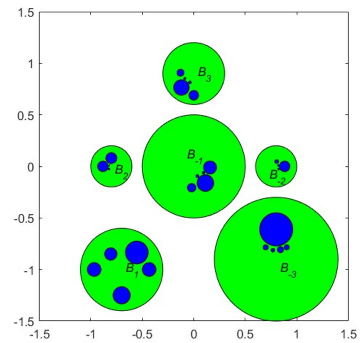

FIGURE 11. The first iteration of our Schottky system. The original disks are in green.

$$
\begin{aligned}
g_1(z) &= \frac{(-0.7-i)z+0.2}{z}\\
g_{-1}(z) &= \frac{0.2}{z+0.7+i}\\
g_2(z) &= \frac{0.8z+0.68}{z+0.8}\\
g_{-2}(z) &= \frac{-0.8z+0.68}{z-0.8}\\
g_3(z) &= \frac{(0.8-0.9i)z-0.63-0.72i}{z-0.9i}\\
g_{-3}(z) &= \frac{0.9iz-0.63-0.72i}{z-0.8i-0.9}.
\end{aligned}
$$

A graphic representing the first iteration is in Figure 11.

6.4. **The Apollonian gasket.** We now focus our attention on one of the most famous fractals, the Apollonian packing. To fully describe the packing as the limit of a conformal IFS, suppose that $k \in \{1, 2, \ldots, 6\}$ and consider the angles

$$\theta_k = (-1)^k \frac{2\pi}{3} \text{ and } \theta'_k = \frac{2\pi k}{3} \mod 2\pi.$$

The generators of the system then have a representation via the maps

$$f(z) = \frac{(\sqrt{3}-1)z+1}{-z+\sqrt{3}+1},\ R_{\theta_k}, \text{ and } R_{\theta'_k}$$

where R_θ is the standard complex rotation by angle θ. With this notation, the infinite set of maps generating the Apollonian packing is

$$\{\phi_{k,n} : k = 1, \ldots, 6 \text{ and } n \in \mathbb{N}\}$$

where

$$\phi_{k,n} = R_{\theta'_k} \circ f^n \circ R_{\theta_k} \circ f.$$

For the rest of this section, we let $\lambda = \sqrt{3}$.

Proposition 6.7 ($W = B(0, 1+\lambda)$)**.** The maximal domain furnishing a conformal extension for the Apollonian IFS is $B(0, 1+\lambda)$.

Proof. We will show that $X_\delta := B(0,\delta)$ with $\delta = 1+\lambda$ satisfies $\phi_{k,n}(X_\delta) \subset X_\delta$. Writing

$$f(z) = \frac{(\lambda-1)z+1}{-z+(\lambda+1)},$$

consider the matrix representation of $f(z)$, M given by

$$M = \begin{pmatrix} \lambda-1 & 1 \\ -1 & \lambda+1 \end{pmatrix}.$$

Notice that

$$M = VJV^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

as λ is the single eigenvalue of multiplicity 2 for M. By nilpotence

$$J^n = \left[\lambda I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]^n = \lambda^n + n\lambda^{n-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

and so the matrix representation of $f^n(z)$ is

$$M^n = \lambda^n \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & n/\lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.$$

Using this representation, and the matrix representation for the rotation

$$R_{\theta_k} = \begin{pmatrix} e^{i\theta_k} & 0 \\ 0 & 1 \end{pmatrix},$$

we see that the map

$$\phi_{k,n}(z) = R_{\theta'_k} \circ f^n \circ R_{\theta_k} \circ f(z)$$

has the matrix representation $\Phi_{k,n}$ given by

$$\Phi_{k,n} = \lambda^n \begin{pmatrix} e^{i\theta'_k} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & n/\lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{i\theta_k} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda-1 & 1 \\ -1 & \lambda+1 \end{pmatrix}. \tag{6.4}$$

Now we consider the action of each map on X_δ. Start with the image of $f(X_\delta)$. Since $1+\delta = 1+\sqrt{3}$ is a pole of $f(z)$, $f(z)$ maps the ball X_δ onto the right part of the plane of the vertical line

$$L(t) = -\frac{1}{2(\lambda+1)} + it, \quad t \in \mathbb{R}.$$

This is easy to see, for

$$f(-1-\sqrt{3}) = \frac{-(\sqrt{3}-1)(\sqrt{3}+1)+1}{(\sqrt{3}+1)+\sqrt{3}+1} = -\frac{1}{2(\sqrt{3}+1)}$$

and

$$f((1+\sqrt{3})i) = \frac{(\sqrt{3}-1)(1+\sqrt{3})i+1}{-(1+\sqrt{3})i+\sqrt{3}+1} = \frac{(2i+1)(1+i)}{2(\sqrt{3}+1)} = \frac{-1+3i}{2(\sqrt{3}+1)} = -\frac{1}{2(\sqrt{3}+1)} + \frac{3i}{2(\sqrt{3}+1)}.$$

The equality follows by noticing that the real parts of both these points are equal.

This is followed by a rotation by $2\pi/3$ — that is, finding the image after R_{θ_k}. By the symmetry of the gasket maps under the complex conjugation, we will only need to consider a rotation by $2\pi/3$. Under the rotation $e^{2\pi i/3}$, the line $L(t)$ becomes

$$\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2(\sqrt{3}+1)}+it\right)=-\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)t+\frac{1}{4(\sqrt{3}+1)}-i\frac{\sqrt{3}}{4(\sqrt{3}+1)}.$$

Solving for real and imaginary parts to be zero, we see that the new line $\tilde{L}(t)$ passes through the points $\frac{1}{1+\lambda}$ and $-\frac{i}{\lambda(1+\lambda)}$.

The image after V^{-1} is given by the inversion by $\frac{-1}{z-1}$. This is a Möbius transformation, so it maps the line $\tilde{L}(t)$ into a circle. To compute the center and the radius of this circle, notice that

$$\begin{aligned} f_V(\pm\infty) &= 0 \\ f_V\left(\frac{1}{1+\lambda}\right) &= \alpha, \quad \text{where} \quad \alpha=\frac{1+\lambda}{\lambda}, \\ f_V(-i\beta) &= \frac{1}{1+\beta^2}-i\frac{\beta}{1+\beta^2}, \quad \text{where} \quad \beta=\frac{1}{\lambda(1+\lambda)}. \end{aligned}$$

Thus, we need to compute the center and radius of circle passing through three points $(0,0)$, $(1+1/\lambda,0)$, and $(\frac{1}{1+\beta^2},-\frac{\beta}{1+\beta^2})$, which is equivalent of solving a 3×3 linear system with the matrix

$$A=\begin{pmatrix} 0 & 0 & 1 \\ 2\alpha & 0 & 1 \\ \frac{2}{1+\beta^2} & -\frac{2\beta}{1+\beta^2} & 1 \end{pmatrix}$$

and the right hand side

$$b=-\begin{pmatrix} 0 \\ \alpha^2 \\ \frac{1}{1+\beta^2} \end{pmatrix}$$

Solving, we obtain that the desired center of the circle is $(\frac{1+\lambda}{2\lambda},\frac{\lambda+1}{2})$ and the radius $\rho=\frac{\lambda+1}{\lambda}$.

The image after J is simply the translation by $\frac{n}{\lambda}$, corresponding to the matrix

$$\begin{pmatrix} 1 & n/\lambda \\ 0 & 1 \end{pmatrix}.$$

Alternatively, this is the map $z\to z+n/\lambda$, which is just a translation by n/λ and the new image is just a circle centred at $(\frac{1+\lambda}{2\lambda}+\frac{n}{\lambda},\frac{\lambda+1}{2})$ of radius $\rho=\frac{\lambda+1}{\lambda}$. We can represent it as

$$C(t)=\frac{1+\lambda}{2\lambda}+\frac{n}{\lambda}+\frac{\lambda+1}{\lambda}\cos(t)+i\left(\frac{\lambda+1}{2}+\frac{\lambda+1}{\lambda}\sin(t)\right), \quad t\in(0,2\pi).$$

We could now proceed with the next map $1-\frac{1}{z}$, but we will use a different approach. Due to the elementary fact that for functions $g:X\to Y$ and $h:Y\to X$, $g(X)\subset h^{-1}(X)$ implies $h(g(X))\subset X$, a splitting argument for $\phi_{k,n}$ may be used to show that $\phi_{k,n}(X_\delta)\subset X_\delta$. Here the map $g(z)$ is the map corresponding to the product of matrices

$$\begin{pmatrix} 1 & n/\lambda \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e^{i\theta_k} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda-1 & 1 \\ -1 & \lambda+1 \end{pmatrix}$$

and the map $h(z)$ to the product of matrices

$$\begin{pmatrix} e^{i\theta'_k} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

Naturally the rotation leaves X_δ invariant. Since

$$\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix}^{-1} = \begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix},$$

we need to find the image of $B(0,\delta)$ under the Möbius map $f_V(z) = -\frac{1}{z-1}$. We proceed similarly when we treated V^{-1}, consider the image of three points $(1+\lambda)$, $(-1-\lambda)$, and $i(1+\lambda)$.

$$\begin{aligned} f_V(1+\lambda) &= -\frac{1}{\lambda}\\ f_V(-1-\lambda) &= \frac{1}{2+\lambda}\\ f_V(i(1+\lambda)) &= \frac{1}{1+\beta^2} + i\frac{\beta}{1+\beta^2}, \quad \text{where} \quad \beta = 1+\lambda. \end{aligned}$$

Thus, we need to compute the center and radius of circle passing through three points $(-\frac{1}{\lambda},0)$, $(\frac{1}{2+\lambda},0)$, and $(\frac{1}{1+\beta^2},\frac{\beta}{1+\beta^2})$, which is equivalent to solving a 3×3 linear system with the matrix

$$A = \begin{pmatrix} -\frac{2}{\lambda} & 0 & 1\\ \frac{2}{2+\lambda} & 0 & 1\\ \frac{2}{1+\beta^2} & -\frac{2\beta}{1+\beta^2} & 1 \end{pmatrix}$$

and the right hand side

$$b = -\begin{pmatrix} \frac{1}{\lambda^2}\\ \frac{1}{(2+\lambda)^2}\\ \frac{1}{1+\beta^2} \end{pmatrix}$$

with $\beta = 1+\lambda$. Solving, we obtain that the center of the circle is $(-\frac{1}{\lambda(2+\lambda)},0)$ and the radius is $\rho = \frac{1+\lambda}{\lambda(2+\lambda)}$.

To conclude $\phi_{k,n}(X_\delta) \subset X_\delta$, we only need to establish that the distance between centers of the circles $c_1 = (-\frac{1}{\lambda(2+\lambda)},0)$ and $c_2 = (\frac{1+\lambda}{2\lambda} + \frac{n}{\lambda}, \frac{\lambda+1}{2})$ is greater than the sum of the radii $\rho_1 = \frac{1+\lambda}{\lambda(2+\lambda)}$ and $\rho_2 = \frac{\lambda+1}{\lambda}$. Magically,

$$\rho_1 + \rho_2 = \frac{1+\lambda}{\lambda(2+\lambda)} + \frac{\lambda+1}{\lambda} = \frac{\lambda+1}{\lambda}\cdot\frac{3+\lambda}{2+\lambda} = 2$$

and the direct computations show that even for $n = 1$,

$$\mathrm{dist}(c_1, c_2) = 2.0442\cdots > 2,$$

and of course the above distance is even greater for $n \geq 2$. □

Remark 6.8. The decomposition in (6.4) is also of practical interest. If one were to naively compute M^n for higher powers of n (specifically, for $n \approx 1250$) the entries of M^n would become so large they could not be stored in memory. Using the fact that Möbius maps act on $PGL(2,\mathbb{C})$, the scaling factor λ^n outside of the decomposition of $\Phi_{k,n}$ may be ignored, avoiding the aforementioned exponential scaling.

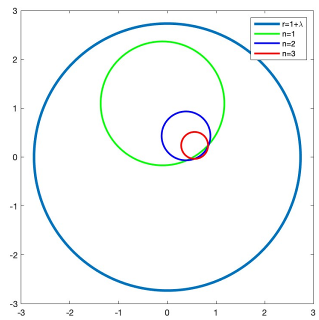

FIGURE 12. illustration of inclusion, for $n = 1,2,3$.

6.4.1. *Tail Bounds.* In this section we find tail bounds for the Apollonian gasket. As mentioned with continued fractions, such bounds are necessary for rigorous Hausdorff dimension estimates of infinite systems, though the structure of such bounds will change depending on the system. Generally, an ordering needs to be given on the maps of the system, which in this case is given in it's definition.

Recall that any Möbius transformation

$$g(z) = \frac{az+b}{cz+d}$$

has a matrix representation

$$M_g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and the norm of its derivative at $z \in \mathbb{C}$ is given by the formula

$$|Dg(z)| = \frac{|\det(M_g)|}{|cz+d|^2}. \tag{6.5}$$

As in the previous section, the matrix form for $\Phi_{k,n}$ is

$$\Phi_{k,n} = \underbrace{\lambda^n \begin{pmatrix} e^{i\theta'_k} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & n/\lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}}_{= R_{\theta'_k} \circ f^n} \overbrace{\begin{pmatrix} e^{i\theta_k} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda-1 & 1 \\ -1 & \lambda+1 \end{pmatrix}}^{= R_{\theta_k} \circ f}.$$

Finding tail bounds for the system will amount to applying (6.5) and the chain rule. Focusing on the rightmost matrices, note that R_{θ_k} is just a rotation by θ_k, and thus leaves the derivative unchanged. Taking the determinant,

$$\det \begin{pmatrix} \lambda-1 & 1 \\ -1 & \lambda+1 \end{pmatrix} = \lambda^2 - 1 + 1 = \lambda^2.$$

The c and d terms for the map are -1 and $\lambda+1$, respectively, so the derivative will be maximized when

$$|-z+\lambda+1|^2$$

is minimized. This is at $z = 1$, giving the derivative $\lambda^2/\lambda^2 = 1$. Hence we have that

$$\left\| D\Phi_{k,n}(z) \right\| \le \left\| DR_{\theta'_k} \circ f^n(R_{\theta_k}(f(z))) \right\| \left\| D(R_{\theta_k} \circ f) \right\|_\infty = \left\| D(R_{\theta'_k} \circ f^n(R_{\theta_k}(f(z))) \right\|.$$

We need to find $R_{\theta_k} \circ f(\mathbb{D})$. Since f is symmetric about the real axis, the points

$$f(-1) = \frac{2-\lambda}{2+\lambda} \quad \text{and} \quad f(1) = 1$$

are antipodal points on $f(\mathbb{D})$. Thus $f(\mathbb{D}) = B(\frac{2}{2+\lambda}, \frac{\lambda}{2+\lambda})$. Without loss of generality, suppose that $\theta_k = \frac{2\pi}{3}$. Then rotating $f(\mathbb{D})$ by $e^{2\pi i/3}$ gives

$$R_{\theta_k} \circ f(\mathbb{D}) = B\left(-\frac{1}{2+\lambda} + \frac{\lambda}{2+\lambda} i, \frac{\lambda}{2+\lambda}\right).$$

Moving onto the next three maps, note that the final map is just a rotation by θ'_k, and therefore doesn't change the norm of the derivative. Hence we can omit it from our calculations. Furthermore,

$$\lambda^n \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & n/\lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \lambda^n \begin{pmatrix} -\frac{n}{\lambda}+1 & \frac{n}{\lambda} \\ -\frac{n}{\lambda} & \frac{n}{\lambda}+1 \end{pmatrix},$$

implying that

$$\det\left(\lambda^n \begin{pmatrix} -\frac{n}{\lambda}+1 & \frac{n}{\lambda} \\ -\frac{n}{\lambda} & \frac{n}{\lambda}+1 \end{pmatrix}\right) = \det\left(\begin{pmatrix} \lambda^n & 0 \\ 0 & \lambda^n \end{pmatrix} \begin{pmatrix} -\frac{n}{\lambda}+1 & \frac{n}{\lambda} \\ -\frac{n}{\lambda} & \frac{n}{\lambda}+1 \end{pmatrix}\right) = \lambda^{2n}.$$

Referring back to (6.5), this implies that

$$\|D\Phi_{k,n}\| = \frac{\lambda^{2n}}{\lambda^{2n}} \max_{z \in R_{\theta_k} \circ f(\mathbb{D})} \frac{1}{|-\frac{n}{\lambda} z + 1 + \frac{n}{\lambda}|^2} = \left(\frac{\lambda}{n}\right)^2 \max_{z \in R_{\theta_k} \circ f(\mathbb{D})} \frac{1}{\left|z - 1 - \frac{\lambda}{n}\right|^2}.$$

Notice that the above maximum occurs at $z \in B\left(-\frac{1}{2+\lambda} + \frac{\lambda}{2+\lambda} i, \frac{\lambda}{2+\lambda}\right)$ that minimizes $\left|z - 1 - \frac{\lambda}{n}\right|$.

It is well known from basic complex analysis that the minimum of $|z-a|$ on the circle $|z - z_0| = r$ is attained for

$$z = a + \left(1 - \frac{r}{|z_0 - a|}\right)(z_0 - a),$$

with $a = 1 + \frac{\lambda}{n}$, $r = \frac{\lambda}{2+\lambda}$, and $z_0 = -\frac{1}{2+\lambda} + \frac{\lambda}{2+\lambda} i$, we have

$$\max_{z \in R_{\theta_k} \circ f(\mathbb{D})} \frac{1}{\left|z - 1 - \frac{\lambda}{n}\right|^2} = \frac{1}{(|z_0 - a| - r)^2} = \frac{(2+\lambda)^2}{(|-1 + \lambda i - (1+\lambda/n)(2+\lambda))| - \lambda)^2}.$$

Using that $\lambda = \sqrt{3}$, we compute,

$$\|D\Phi_{k,n}\| = \frac{\lambda^2}{n^2} \frac{(2+\lambda)^2}{(|-1+\lambda i - (1+\lambda/n)(2+\lambda))| - \lambda)^2} \le \frac{\lambda^2}{n^2} \frac{(2+\lambda)^2}{(|-1+\lambda i - 2 - \lambda))| - \lambda)^2} < \frac{3 \times 1.28}{n^2}.$$

After a simple application of the integral comparison test, one finds that

$$\sum_{k \in \{1,\dots,6\},\ n=N+1}^{\infty} \left\|D\Phi_{k,n}\right\|_\infty^t \le \sum_{k \in \{1,\dots,6\},\ n=N+1}^{\infty} \left\|D\Phi_{k,n}\right\|^t$$
$$\le 6(3 \times 1.28)^t \int_{N+1}^{\infty} x^{-2t} dx < 6 \times 4^t \times \frac{1}{2t-1} N^{-2t+1}.$$

7. Hausdorff dimension estimates

In this section, for the concrete example from the previous section, we provide the estimates for all the constants and parameters needed for computations and give reliable computational range the Hausdorff dimensions.

7.1. **2-dimensional continued fractions.** In two dimensions $\beta = (\beta_1, \beta_2)$, hence

$$\sum_{|\beta|=2} (\beta!)^{-2} = 1 + \frac{1}{4} + \frac{1}{4} = \frac{3}{2},$$

and as a result $C_{BH} = \sqrt{6}$ and by Bramble-Hilbert Lemma 5.1

$$\|\rho_t - \mathscr{I}_h \rho_t\|_{L^\infty(\tau)} \le 2\sqrt{6} h_\tau^2 |\rho_t|_{W^{2,\infty}(\tau)}.$$

By Theorem 4.1, for any $|\alpha| = 2$, and taking $\eta = 1$,

$$|D^\alpha \rho_t(x)| \le \frac{4}{s^2(1 - s(2+s))^t} \rho_t(x), \quad \forall x \in \tau. \tag{7.1}$$

Thus, we need to obtain an estimate for $\frac{1}{s^2(1-s(2+s))^t}$ which depends on the Hausdorff dimension t of the limit set. Although we do not know this exactly, good upper bounds on the quantity can be applied.

7.1.1. *Alphabet with four smallest generators.* For a simple illustration we consider the alphabet consisting with four generators,

$$E_4 = \{(1,0), (1,1), (1,-1), (2,0)\}.$$

Denoting the limit set of the system by J_{E_4}, the upper bound for the $\dim_{\mathscr{H}}(J_{E_4})$ is 1.15. As a result

$$\min_{s \in (0, \sqrt{2}-1)} \frac{1}{s^2(1 - s(2+s))^t} \le 41,$$

combining the estimates we obtain

$$\|\rho_t - \mathscr{I}_h \rho_t\|_{L^\infty(\tau)} \le 8 \cdot 41 \sqrt{6} h_\tau^2 \|\rho_t\|_{L^\infty(\tau)}.$$

Naturally, no tail bounds are required in this case. Using this estimate, we compute that

$$\dim_{\mathscr{H}}(J_{E_4}) \in [1.149571..., 1.149582....].$$

7.1.2. *Infinite lattice alphabet.* Now we consider the infinite alphabet $E = \mathbb{N} \times \mathbb{Z}$. For this example, we know that $t < 1.86$ and as a result

$$\min_{s \in (0, \sqrt{2}-1)} \frac{1}{s^2(1 - s(2+s))^t} \le 72,$$

combining, we obtain

$$\|\rho_t - \mathscr{I}_h \rho_t\|_{L^\infty(\tau)} \le 8 \cdot 72 \sqrt{6} h_\tau^2 \|\rho_t\|_{L^\infty(\tau)}.$$

For tail bound we use Lemma 6.5. Thus, since for $t < 1.86$,

$$\min_{s \in (0, \sqrt{2}-1)} \frac{1}{s(1 - s(2+s))^t} \le 14,$$

we have

$$\sum_{e \in E \setminus \tilde{E}} \|D\phi_e(x_j)\|^t \mathscr{I}_h \rho_t(\phi_e(x_j)) \le \frac{7\sqrt{2}\pi}{2(t-1)} R^{2-2t} \rho_t(0),$$

and to account for the tail, we need modify j-th column and the row of the matrix $\tilde{B}_t$ that corresponds to the zero node.

Denoting the limit set of this system by J_E, our computation found that

$$\dim_{\mathscr{H}}(J_E) \in [1.8488\ldots, 1.8572\ldots].$$

7.1.3. *Gaussian prime alphabet.* As an intermediate example, we consider the case when the alphabet consist of Gaussian prime with positive real parts. For this example, we know that $t < 1.515$ and as a result

$$\min_{s\in(0,\sqrt{2}-1)} \frac{1}{s^2(1-s(2+s))^t} \le 56,$$

combining, we obtain

$$\|\rho_t - \mathscr{I}_h\rho_t\|_{L^\infty(\tau)} \le 8\cdot 56\sqrt{6}h_\tau^2\|\rho_t\|_{L^\infty(\tau)}.$$

For tail bound we use Lemma 6.5. Thus, since for $t < 1.515$,

$$\min_{s\in(0,\sqrt{2}-1)} \frac{1}{s(1-s(2+s))^t} \le 12,$$

we have

$$\sum_{e\in E\setminus\tilde{E}} \|D\phi_e(x_j)\|^t \mathscr{I}_h\rho_t(\phi_e(x_j)) \le \frac{6\sqrt{2}\pi}{2(t-1)}R^{2-2t}\rho_t(0),$$

and to account for the tail, we need modify j-th column and the row of the matrix $\tilde{B}_t$ that corresponds to the zero node.

Denote the limit set of this system by J_{prime}. Then,

$$\dim_{\mathscr{H}} J_{prime} \in [1.5060..., 1.5140...].$$

7.2. 3-**dimensional continued fractions.** In three dimensions $\beta = (\beta_1, \beta_2, \beta_3)$, hence

$$\sum_{|\beta|=2} (\beta!)^{-2} = 1+1+1+\frac{1}{4}+\frac{1}{4}+\frac{1}{4} = \frac{15}{4},$$

and as a result $C_{BH} = \sqrt{15}$ and by Bramble-Hilbert Lemma 5.1

$$\|\rho_t - \mathscr{I}_h\rho_t\|_{L^\infty(\tau)} \le 2\sqrt{15}h_\tau^2|\rho_t|_{W^{2,\infty}(\tau)}.$$

By Theorem 4.1, for any $|\alpha| = 2$, and taking $\eta = 1$,

$$|D^\alpha\rho_t(x)| \le \frac{6}{s^2(1-s(2+s))^t}\rho_t(x), \quad \forall x\in\tau. \tag{7.2}$$

7.2.1. *Alphabet with five smallest generators.* First, we consider the alphabet consisting with five generators,

$$E_5 = \{(1,0,0),(1,1,0),(1,-1,0),(1,0,1),(1,0,-1)\}.$$

Denoting the limit set of the system by J_{E_5}, the upper bound for the $\dim_{\mathscr{H}}(J_{E_5})$ is 1.46. As a result

$$\min_{s\in(0,\sqrt{2}-1)} \frac{1}{s^2(1-s(2+s))^t} \le 54,$$

combining the estimates we obtain

$$\|\rho_t - \mathscr{I}_h\rho_t\|_{L^\infty(\tau)} \le 12\cdot 54\sqrt{15}h_\tau^2\|\rho_t\|_{L^\infty(\tau)}.$$

Naturally, no tail bounds are required in this case. Using this estimate, we compute that

$$\dim_{\mathscr{H}}(J_{E_5}) \in [1.4423,..., 1.4617...].$$

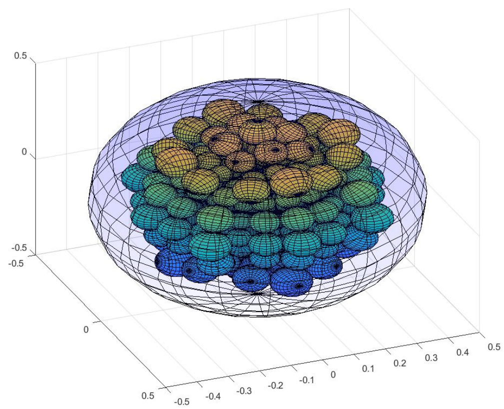

FIGURE 13. The first iteration of the 3D continued fraction IFS.

7.2.2. *Infinite lattice alphabet.* Now we consider the infinite alphabet $E = \mathbb{N}\times\mathbb{Z}^2$. For this example, we know that $t < 2.6$ and as a result

$$\min_{s\in(0,\sqrt{2}-1)} \frac{1}{s^2(1-s(2+s))^t} \le 112,$$

combining all estimates we obtain

$$\|\rho_t - \mathscr{I}_h\rho_t\|_{L^\infty(\tau)} \le 12\cdot 112\sqrt{15}h_\tau^2\|\rho_t\|_{L^\infty(\tau)}.$$

To account for the tail bound, similarly to 2D case, we use Lemma 6.5. Thus, since for $t < 2.6$,

$$\min_{s\in(0,\sqrt{2}-1)} \frac{1}{s(1-s(2+s))^t} \le 18,$$

we have

$$\sum_{e\in E\setminus\tilde{E}} \|D\phi_e(x_j)\|^t \mathscr{I}_h\rho_t(\phi_e(x_j)) \le \frac{36\sqrt{3}\pi}{2t-3}R^{3-2t}\rho_t(0),$$

and again to account for the tail, we need modify j-th column and the row of the matrix $\tilde{B}_t$ that corresponds to the zero node.

Suppose that $J_{E_{3D}}$ is the limit set for the above 3-dimensional continued fraction system. Using a mesh size of $1.7e-02$ we found that

$$\dim_{\mathscr{H}}(J_{E_{3D}]}) \in [2.56...,2.58...].$$

7.3. **Quadratic perturbations of linear maps.** Similarly to Section 7.1, $C_{BH} = \sqrt{6}$ and by Bramble-Hilbert Lemma 5.1

$$\|\rho_t - \mathscr{I}_h\rho_t\|_{L^\infty(\tau)} \le 2\sqrt{6}h_\tau^2|\rho_t|_{W^{2,\infty}(\tau)}.$$

However, since this system does not consist of Möbius transformations, to estimate $|\rho_t|_{W^{2,\infty}}$ we will use (3) from Theorem 4.1, namely

$$|D^\alpha\rho_t(x)| \le \alpha!\left(\frac{ML}{sd_2}\right)^{|\alpha|}\exp\left(tC_R\left(\frac{L}{L-2}\right)^2\right)\rho_t(x), \quad \text{for all } x\in X, \tag{7.3}$$

where $d_2 = \text{dist}(X, \partial W)$, R, s, M, L can be any numbers such that $R \in (0, r)$, $s \in (0, R)$, $M > 1$, $L > 2$ and $C_R = \log\left(\frac{1+Rd_2}{(1-Rd_2)^3}\right) + 2\log\left(\frac{1+Rd_2}{1-Rd_2}\right)$. Since $d_2 = 1$,

$$C_R := \log\left(\frac{1+Rd_2}{(1-Rd_2)^3}\right) + 2\log\left(\frac{1+Rd_2}{1-Rd_2}\right) = \log\left(\frac{1+R}{(1-R)^3}\right) + 2\log\left(\frac{1+R}{1-R}\right).$$

Setting $s = r = 0.2$ and $|\alpha| = 2$, we need to optimize the expression

$$\min_{L>2,\ M>1} \left(\frac{ML}{0.1}\right)^2 \exp\left(tC_{0.2}\left(\frac{L}{L-2}\right)^2\right).$$

As before, this varies depending on the parameter t we are using. Setting $t = 0.633$, an upper bound for our system, we find that

$$|D^2\rho_t(x)| \le 1833\rho_t(x).$$

for all $x \in X$. Combining this with the Bramble-Hilbert Lemma, we see that

$$\left\|\rho_t - \mathscr{I}_h\right\|_{L^\infty(\tau)} \le 2\sqrt{6} \cdot 1833 \left\|\rho_t\right\|_{L^\infty(\tau)}.$$

For our computations, we used a mesh of size $8.1e-07$. Denoting the limit set by J_{abc}, a resulting computation gave

$$\dim_{\mathscr{H}}(J_{abc}) \in [0.6327142857142860\ldots, 0.6327142857142869\ldots].$$

This is up to standard MATLAB long precision and may be given as an equality, due to the computed upper and lower bounds being equal. Expanding the precision of the computation would therefore yield more digits.

7.4. **Schottky groups.** Error estimates for 2-dimensional, classical Schottky groups are slightly different than continued fractions. Since we are in two dimensions, $\beta = (\beta_1, \beta_2)$ implying $C_{BH} = \sqrt{6}$. However, the optimization problem involving $c(s)$ will necessarily change, as $\eta < 1$ in many cases. The corresponding minimization problem then is

$$\min_{s \in (0, \sqrt{2}-1)} 4\left(\frac{1}{s\eta}\right)^2 \frac{1}{(1-s(2+s))^t}.$$

Note that η can become arbitrarily small, implying different bounds are needed for these cases. In the provided example, it is straightforward to verify that $\eta > 0.32$. The subsequent minimization is

$$\min_{s \in (0, \sqrt{2}-1)} 4\left(\frac{1}{0.32s}\right)^2 \frac{1}{(1-s(2+s))^t}.$$

This, like the other minimization, is changed for each t we consider. That said, since an upper bound on the dimension of our Schottky group is 0.78, one finds that

$$|D\rho_t(x)| \le \min_{s \in (0, \sqrt{2}-1)} 4\left(\frac{1}{0.32s}\right)^2 \frac{1}{(1-s(2+s))^t} < 1082\rho_t(x).$$

Completing our bounds, just recall (4.1), and so

$$\left\|\rho_t - \mathscr{I}_h\rho_t\right\|_{L^\infty(\tau)} \le 2\sqrt{6} \cdot 1082 h_\tau^2 \left\|\rho_t\right\|_{L^\infty(\tau)}.$$

Denote the limit set of the Schottky group by J_{schott}. Then

$$\dim_{\mathscr{H}}(J_{\text{schott}}) \in [0.7753714283\ldots, 0.7753714286\ldots].$$

The maximum mesh size used for this computation was $1.67e-05$.

7.5. **Apollonian gasket.** The bounds for the Apollonian packing are similar to those on complex continued fractions. Since the generating IFS consists of Möbius maps, the bounds from the Bramble-Hilbert Lemma remain the same. Specifically, we have that $C_{BH} = \sqrt{6}$ so

$$\left\|\rho_t - \mathscr{I}_h\rho_t\right\|_{L^\infty(\tau)} \le 2\sqrt{6}|\rho_t|_{W^{2,\infty}}.$$

Applying (4.1), we need to optimize the expression

$$\min_{s\in(0,\sqrt{2}-1)} \frac{6}{s^2(1-s(2+s))^t}$$

when $|\alpha| = 2$ and t is an upper bound for the Hausdorff dimension of $J_{\mathscr{A}}$. Since $t < 1.306$, one finds that

$$|D^\alpha \rho_t(x)| \le 283\rho_t(x).$$

Excluding the tail, we find that

$$\left\|\rho_t - \mathscr{I}_h\rho_t\right\|_{L^\infty(\tau)} \le 2\sqrt{6}\cdot 283 h_\tau^2 \left\|\rho_t\right\|_{L^\infty(\tau)}.$$

Adding in the tail bounds,

$$\sum_{n=N+1,\ k=1,..,6}^{\infty} \left\|D\Phi_{k,n}\right\|_\infty \mathscr{I}_h\rho_t(\phi_e(x)) \le 6\times 4^t \times \frac{1}{2t} N^{-2t+1}\rho_t(0).$$

As shown below, similar bounds will hold for each subsystem we consider. For the limit set $J_{\mathscr{A}}$ of the Apollonian gasket, we have that

$$\dim_{\mathscr{H}}(J_{\mathscr{A}}) \in [1.305675..., 1.3057...].$$

This bound was obtained using a mesh of size $1.1e-03$.

7.5.1. *A Finite Apollonian Subsystem.* The first subsystem of the Apollonian gasket we consider is a finite subsystem consisting of the first 12 maps in it's standard enumeration. In particular, this is given by

$$\mathscr{A}|_{12} = \left\{\phi_{k,n} : k = 1,\ldots,6 \text{ and } n = 1,2\right\},$$

with corresponding limit set $J_{\mathscr{A}|_{12}}$. A visual of a system and its corresponding eigenfunction is found at the end of section 3 2. This system exhibits our methods capabilities to estimate systems without the quadratic decaying tails seen in other examples. As such, the bounds for it are similar to the original gasket. In this case, $t < 1.03$, so optimizing the expression yields

$$\min_{s\in(0,\sqrt{2}-1)} \frac{6}{s^2(1-s(2+s))^{1.03}} < 37$$

and so the Bramble-Hilbert lemma implies

$$\left\|\rho_t - \mathscr{I}_h\rho_t\right\|_{L^\infty(\tau)} \le 2\sqrt{6}\cdot 37 h_\tau^2 \left\|\rho_t\right\|_{L^\infty(\tau)}.$$

Using a mesh size of $1.17e-04$, our numerics found that

$$\dim_{\mathscr{H}}(\mathscr{A}|_{12}) \in [1.0285714285712..., 1.0285714285714...].$$

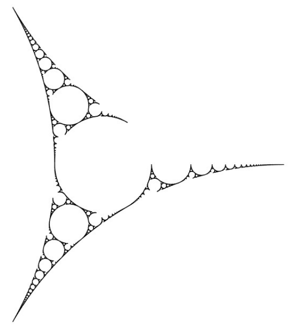

FIGURE 14. The Apollonian gasket with one generator removed.

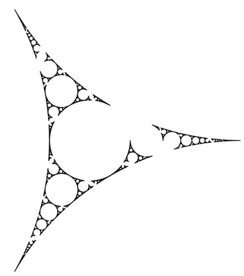

FIGURE 15. The Apollonian gasket IFS without a spiral.

7.5.2. *The Packing without a Generator.* Due to the flexibility of our method, we can find rigorous Hausdorff dimension estimates for infinite subsystems of the Apollonian gasket. Starting with one of the simplest subsystems, we consider the fractal generated from the Apollonian gasket without a generator. Specifically, let

$$\mathscr{A}_5 = \left\{\phi_{k,n} : k = 2,\dots,6 \text{ and } n \in \mathbb{N}\right\},$$

with corresponding limit set $J_{\mathscr{A}_5}$. Being a subsystem, all of the previous bounds carry over. In this case, taking $t < 1.24$ one finds

$$\min_{s\in(0,\sqrt{2}-1)} \frac{6}{s^2(1-s(2+s))^{1.24}} < 267$$

and hence, excluding the tail

$$\left\|\rho_t - \mathscr{I}_h\rho_t\right\|_{L^\infty(\tau)} \le 2\sqrt{6}\cdot 267 h_\tau^2 \left\|\rho_t\right\|_{L^\infty(\tau)}.$$

The appropriate tail bounds in this situation are

$$\sum_{n=N+1,\ k=2,..,6}^{\infty} \left\|D\Phi_{k,n}\right\|_\infty \mathscr{I}_h\rho_t(\phi_e(x)) \le 5\times 4^t \times \frac{1}{2t} N^{-2t+1}\rho_t(0).$$

Moving onto our numerics, using a mesh size of $1.1e-03$ we found that

$$\dim_{\mathscr{H}}(J_{\mathscr{A}_5}) \in [1.3056..., 1.3057...].$$

7.5.3. *The Packing without a Spiral.* Another interesting subsystem of $\mathscr{A}$ occurs when a map from a different generator is removed for each level n. Taking from concurrent maps, this yields a spiral of disks taken away from the fractal. Specifically, consider the CIFS

$$\mathscr{A}_{spiral} = \{\phi_{k,n} : k \in \{1,\dots,6\} \setminus 1 + (n \mod 6), n \in \mathbb{N}\}.$$

It is clear that the tail bounds for this system are the same as in the previous example, being

$$\sum_{n=N+1,\ (k,n)\in E_{spiral}}^{\infty} \left\|D\Phi_{k,n}\right\|_\infty \mathscr{I}_h\rho_t(\phi_e(x)) \le 5\times 4^t \times \frac{1}{2t} N^{-2t+1}\rho_t(0),$$

where $(k,n) \in E_{spiral}$ if and only if $\Phi_{k,n} \in \mathscr{A}_{spiral}$. The specific eigenfunction bounds differ at a certain level due to the higher dimension of the limit set for $\mathscr{A}_{spiral}$ compared to the system missing a generator. In this case $t < 1.25$, which is close enough to 1.24 so that the same full bound applies

$$\left\| \rho_t - \mathscr{I}_h \rho_t \right\|_{L^\infty(\tau)} \leq 2\sqrt{6} \cdot 267 h_\tau^2 \left\| \rho_t \right\|_{L^\infty(\tau)}.$$

Denote the limit set of this system by $J_{\mathscr{A}_{spiral}}$. Then our numerics find that

$$\dim_{\mathscr{H}}(J_{\mathscr{A}_{spiral}}) \in [1.2346..., 1.2357...]$$

with mesh size $1.1e-03$.

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