

REMOVABLE SETS FOR HOMOGENEOUS LINEAR PDE IN CARNOT GROUPS

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ABSTRACT. Let \mathcal{L} be a homogeneous left invariant differential operator on a Carnot group. Assume that both \mathcal{L} and \mathcal{L}^t are hypoelliptic. We study the removable sets for \mathcal{L} -solutions. We give precise conditions in terms of the Carnot–Carathéodory Hausdorff dimension for the removability for \mathcal{L} -solutions under several auxiliary integrability or regularity hypotheses. In some cases, our criteria are sharp on the level of the relevant Hausdorff measure. One of the main ingredients in our proof is the use of novel local self similar tilings in Carnot groups.

1. INTRODUCTION

A fundamental problem in partial differential equations is to understand the size of the sets which are removable for solutions to a given PDE. Such size is typically quantified in terms of both the order of the PDE and the regularity of the solution. Results of this type are well understood in the Euclidean setting, but have received less extensive treatment in more abstract contexts such as sub-Riemannian manifolds and metric spaces. In this paper we study the removability of sets for homogeneous left invariant differential operators on nilpotent stratified Lie groups (also known as Carnot groups).

Let us begin with some background and history of the subject in the Euclidean setting. Painlevé’s problem—to characterize the removability of sets for bounded analytic functions—motivated much work in geometric measure theory, complex analysis and harmonic analysis throughout the twentieth century. The eventual solution to the Painlevé problem was obtained by Tolsa [32] following extensive work by many people including Ahlfors, Denjoy, Vitushkin, Garnett, Calderón, Mattila, Jones, David, Mel’nikov and Verdera. Tolsa’s article [33], written for the proceedings of the 2006 ICM, is highly recommended for an informative history and survey on the Painlevé problem.

The complete solution to the Painlevé problem is quite subtle and relies on intricate geometric properties of the set to be removed related to the *Menger curvature*. Among sets $E \subset \mathbb{C}$ of finite Hausdorff 1-measure, the removable sets for bounded analytic functions are precisely the purely 1-unrectifiable sets, i.e, sets E which intersect all Lipschitz curves in sets of null \mathcal{H}^1 measure. This was *Vitushkin’s conjecture*, established by Guy David [14]. The same geometric criterion characterizes removable sets for bounded planar harmonic functions, see David and Mattila [15].

Removability for Hölder continuous harmonic functions (in any dimension) was characterized in an early paper of Carleson [8]. Unlike the more difficult case of bounded harmonic functions, there is a precise characterization of Hölder continuous harmonic removability at the level of the Hausdorff measure: *a compact set $E \subset \mathbb{R}^n$ is removable for δ -Hölder continuous harmonic functions if and only if E has zero Hausdorff $(n - 2 + \delta)$ -measure.*

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In connection with removability problems for functions in the class BMO, we mention Kaufman's paper [24] on removability for BMO analytic functions, and Ishchanov's results [23] on removability for BMO harmonic functions in higher dimensions.

A comprehensive treatment of removability for solutions of linear partial differential equations in Euclidean spaces of any dimension was given by Harvey and Polking [21]. They provide conditions on the level of Hausdorff measure for the removability of sets for solutions of homogeneous partial differential operators, under various ancillary integrability, continuity or regularity assumptions. Our results are strongly influenced by the paper of Harvey and Polking. The survey article [28] by Polking contains additional results and references to the literature.

The literature on the subject of removability in the Euclidean setting is vast, and the preceding is only a small sample. In particular, we have said nothing about removability for solutions of nonlinear PDE, including the p -harmonic or minimal surface equations, or for removability for, e.g., subharmonic functions or quasiregular mappings.

In recent years there has been significant emphasis on the extension of classical Euclidean analysis and geometry into more general non-Riemannian spaces, including sub-Riemannian manifolds and more abstract metric measure spaces. While there have been some extensions of the theory of removable sets as described above into such settings, the story appears to be less well understood. Removability for Hölder continuous p -harmonic functions in metric spaces was studied by Mäkäläinen [25]; for bounded p -harmonic functions see A. Björn [4].

In this paper we consider the category of nilpotent stratified Lie groups (also known as *Carnot groups*) equipped with the Carnot–Carathéodory metric. The theory of subelliptic PDE, dating back to the fundamental work of Hörmander, Folland, Stein, Nagel, Wainger and others, lies at the foundations of this subject. The book by Folland and Stein [19] remains an excellent introduction. Nowadays the study of sub-Riemannian spaces is a rich vein within analysis in metric spaces, with numerous applications across both pure and applied mathematics. Carnot groups appear naturally, both as model examples and as local tangent models for regular sub-Riemannian manifolds. In this paper we present an extensive removability theory for solutions to homogeneous left invariant partial differential equations, in the spirit of Harvey and Polking, on arbitrary Carnot groups.

The prototypical example of a Carnot group is the *Heisenberg group*. In [12], the first author together with Mattila considered the problem of removability for Lipschitz harmonic functions on the Heisenberg group. As in the Euclidean setting the Lipschitz case presents unique technical difficulties. In a follow-up paper [11] we will address the problem of Lipschitz removability in general Carnot groups.

We now turn to a description of the results of this paper. We quantify, in terms of the Carnot–Carathéodory Hausdorff dimension, the size of removable sets for solutions to homogeneous linear PDE in general Carnot groups, when the equations in question are defined by homogeneous left-invariant partial differential operators satisfying a hypoellipticity criterion. All of our results are sharp at the level of Hausdorff dimension, and some even indicate the sharp value of the corresponding Hausdorff measure for such removability. We study removability for BMO, Hölder, and L^p_{loc} solutions. Our main results are Theorem 4.4, Theorem 4.12 and Theorem 5.3. Related results appear in Remark 5.8.

The results in the paper [21] by Harvey and Polking rely essentially on the construction of a certain smooth partition of unity which has been used from then on by many authors. The resulting *Harvey–Polking Lemma* depends on the dyadic tilings of Euclidean space, hence in order to generalize the Harvey–Polking removability results to Carnot groups one must first construct analogous tilings. Although there are many analogues of dyadic cubes in metric spaces (see e.g. Christ [13]), none of them can be employed in order to build a useful sub-Riemannian smooth partition of unity.

One way to realize the usual local dyadic tilings in \mathbb{R}^n is in terms of a certain self similar set of full measure satisfying the open set condition (i.e. the n -dimensional cube of sidelength one is divided into 2^n cubes of sidelength $1/2$, and so on). We adopt the same approach in general Carnot groups and obtain self-similar tilings which can be employed efficiently for proving a sub-Riemannian Harvey-Polking Lemma.

This paper is organized as follows. In Section 2 we recall preliminary facts about Carnot groups and left invariant operators. In section 3 we are concerned with the existence of self-similar sets forming tilings of compact subsets. We use such tilings to develop a sub-Riemannian version of the Harvey–Polking machinery of smooth partitions of unity. In Section 4 we state and prove theorems on the removability of compact sets for BMO and Hölder continuous solutions of homogeneous left invariant PDE. These results are phrased in terms of the Hausdorff measure in the Carnot–Carathéodory metric, and are sharp. In Section 5 we consider the case of removability of compact sets for L^p_{loc} solutions. These results are no longer sharp on the level of the Hausdorff measure, although they remain sharp on the level of Hausdorff dimension. In Remark 5.8 we conclude with some related results for other function classes such as the horizontal Sobolev space $W^{k,p}_{H,\text{loc}}$ of functions with k th order horizontal derivatives in L^p_{loc} , and the Folland–Stein space C^k_H of functions with continuous k th order horizontal derivatives.

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2. DEFINITIONS AND NOTATION

2.1. Carnot groups. A *Carnot group* is a connected, simply connected, nilpotent Lie group \mathbb{G} of dimension at least two with graded Lie algebra

$$\mathfrak{g} = \mathfrak{v}_1 \oplus \cdots \oplus \mathfrak{v}_s$$

so that $[\mathfrak{v}_1, \mathfrak{v}_i] = \mathfrak{v}_{i+1}$ for $i = 1, 2, \dots, s-1$ and $[\mathfrak{v}_1, \mathfrak{v}_s] = 0$. The integer $s \geq 1$ is the *step* of G . We denote the group law in \mathbb{G} by \cdot and the identity element of \mathbb{G} by 0 . We identify elements of \mathfrak{g} with left invariant vector fields on \mathbb{G} in the usual manner.

We fix an inner product in \mathfrak{v}_1 and let X_1, \dots, X_{m_1} be an orthonormal basis for \mathfrak{v}_1 relative to this inner product. Using this basis, we construct the *horizontal subbundle* $H\mathbb{G}$ of the tangent bundle $T\mathbb{G}$ with fibers $H_p\mathbb{G} = \text{span}\{X_1(p), \dots, X_{m_1}(p)\}$, $p \in \mathbb{G}$. A left-invariant vector field X on \mathbb{G} is *horizontal* if it is a section of $H\mathbb{G}$. The inner product on \mathfrak{v}_1 defines a left invariant family of inner products on the fibers of the horizontal subbundle.

We denote by d the *Carnot–Carathéodory metric* on \mathbb{G} , defined by infimizing the lengths of horizontal paths joining two fixed points, where the horizontal length is computed using the aforementioned inner product.

For $t > 0$ we define $\delta_t : \mathfrak{g} \rightarrow \mathfrak{g}$ by setting $\delta_t(X) = t^i X$ if $X \in \mathfrak{v}_i$ and extending by linearity. Via conjugation with the exponential map, δ_t induces an automorphism of \mathbb{G} which we also denote by δ_t . Then $(\delta_t)_{t>0}$ is the one-parameter family of *dilations* of \mathbb{G} satisfying $d(\delta_t(p), \delta_t(q)) = td(p, q)$ for $p, q \in \mathbb{G}$. The Jacobian determinant of δ_t (relative to Haar measure) is everywhere equal to t^Q , where

$$Q = \sum_{i=1}^s i \dim \mathfrak{v}_i$$

is the *homogeneous dimension* of \mathbb{G} . **In this paper, we always assume $Q \geq 3$.**

As a simply connected nilpotent group, \mathbb{G} is diffeomorphic with $\mathfrak{g} = \mathbb{R}^D$, $D = \sum_{i=1}^s \dim \mathfrak{v}_i$, via the exponential map. The Haar measure on \mathbb{G} is induced by the exponential map from Lebesgue

measure on $\mathfrak{g} = \mathbb{R}^D$. It also agrees (up to a constant) with the Q -dimensional Hausdorff measure in the metric space (\mathbb{G}, d) . In this paper we will denote the Haar measure of a set $E \subset \mathbb{G}$ by $|E|$, and we will write integrals with respect to this measure as $\int_E f(x) dx$. We refer the reader to [27], [6] or [7] for further information on Carnot groups and their metric geometry.

2.2. Left invariant homogeneous operators. As usual for $E \subset \mathbb{G}$ we define $C_c^\infty(E)$ to be the set of all C^∞ functions with compact support contained in E . For U open we denote by $\mathcal{D}'(U)$ the space of distributions on U with the usual locally convex topology. Let also $\mathcal{D}' := \mathcal{D}'(\mathbb{G})$. We will denote by $\langle u, f \rangle$ the pairing of a distribution $u \in \mathcal{D}'(U)$ with a test function $f \in C_c^\infty(U)$.

A distribution $u \in \mathcal{D}'(U)$ is homogeneous of degree λ if $\langle u, f \circ \delta_r \rangle = r^{-Q-\lambda} \langle u, f \rangle$ for all $f \in C_c^\infty$. A distribution which is C^∞ away from 0 and homogeneous of degree $\lambda - Q$ will be called kernel of type λ . A differential operator \mathcal{L} will be called homogeneous of degree λ , or λ -homogeneous, if $\mathcal{L}(u \circ \delta_r) = r^\lambda (\mathcal{L}u) \circ \delta_r$ for all $u \in \mathcal{D}'$. By the Poincaré–Birkhoff–Witt theorem [19] and homogeneity, the space of λ -homogeneous, left invariant differential operators in \mathbb{G} is the linear span of the operators $X_{i_1} \dots X_{i_\lambda}$ with $i_k = 1, \dots, m$ and $k = 1, \dots, \lambda$.

A differential operator \mathcal{L} on \mathbb{G} is *hypoelliptic* if for any open set $U \subset \mathbb{G}$ and any two distributions u and v on U satisfying $\mathcal{L}u = v$, $v \in C^\infty(U)$ implies $u \in C^\infty(U)$. We call a function f satisfying $\mathcal{L}f = 0$ an \mathcal{L} -*solution*. (By hypoellipticity, such solutions are automatically smooth.)

Example 2.1. The operator $\mathcal{L} = \sum_{i=1}^m X_i^2$ is known as the *sub-Laplacian* on \mathbb{G} . The theory of sub-Laplacians on stratified Lie groups has been extensively developed; we refer to Folland [17] and to the books [19], [6]. Such operators are self-adjoint, 2-homogeneous and left invariant. By analogy with the Euclidean case, we call solutions of the equation $\mathcal{L}f = 0$ \mathcal{L} -*harmonic* functions.

Definition 2.2. Let \mathcal{L} be a linear partial differential operator defined on an open set U contained in a domain \mathbb{R}^D , and let E be a closed subset of U . Let $\mathcal{F}(U)$ be a class of distributions defined in U . The set E is said to be *removable for \mathcal{L} -solutions in the class \mathcal{F}* , or *removable for \mathcal{F} \mathcal{L} -solutions*, if whenever $f \in \mathcal{F}(U)$ satisfies $\mathcal{L}f|_{U \setminus E} = 0$, then $\mathcal{L}f = 0$.

3. LOCAL TILINGS AND SMOOTH PARTITIONS OF UNITY IN CARNOT GROUPS

3.1. Dyadic decompositions in Carnot groups. In this section, we construct essentially disjoint partitions of compact subsets of a Carnot group \mathbb{G} , which are analogous to the classical dyadic decomposition of \mathbb{R}^n .

Our starting point is the construction of a fundamental tile $T \subset \mathbb{G}$, analogous to the Euclidean unit cube. As in the Euclidean case, T is decomposed into a family of homothetic copies, each of which has size half the size of T . We make use of the following theorem on the existence of self-similar sets in Carnot groups. Theorem 3.1 is a special case of Proposition 4.14 in [3]. In case the group \mathbb{G} is of step two with rational structure constants (for instance, if \mathbb{G} is the Heisenberg group), explicit tilings of this type were constructed by Strichartz [30], [31]; the latter tilings were further studied in [1] and [34].

Theorem 3.1 (Existence of self-similar tiles). *Let \mathbb{G} be a Carnot group of homogeneous dimension Q . There exist $\frac{1}{2}$ -homotheties $f_1, \dots, f_M \in \mathbb{G}$, $M = 2^Q$, and a compact set $T \subset \mathbb{G}$ so that $T = \bigcup_{j=1}^M T_j$ where $T_j = f_j(T)$. Moreover,*

$$(3.2) \quad 0 < |T| < \infty$$

and $|T_j \cap T_{j'}| = 0$ whenever $j \neq j'$.

A t -homothety of \mathbb{G} is the composition of a fixed left translation with the dilation δ_t .

Let $W = \{1, \dots, M\}$. For $m \geq 0$ and $w = w_1 \cdots w_m \in W^m$ we introduce the notation

$$(3.3) \quad f_w = f_{w_1} \circ \cdots \circ f_{w_m}$$

and $T_w = f_w(T)$. We denote by \mathcal{D}_m the family of sets T_w as w ranges over W^m .

Proposition 3.4. *The set T is the closure of an open set.*

In view of Proposition 3.4 we may select a point $p \in T$ and radii $0 < R^i < R^o$ so that $B(p, R^i) \subset T \subset B(p, R^o)$. Fixing p , we assume that R^i has been chosen as large as possible and R^o has been chosen as small as possible subject to the preceding constraint. Then $R^o < \text{diam} T$. We call p the *center* of T and R^i and R^o the *inner* and *outer radii* of T , respectively. For the remainder of the paper, we fix this data.

Proof of Proposition 3.4. By [2, Theorem 3.1], the assertion in (3.2) implies that the iterated function system (IFS) $\{f_1, \dots, f_M\}$ satisfies the open set condition (OSC). Choose a bounded open set O so that $f_j(O) \subset O$ for all j and the sets $\{f_j(O)\}$ are pairwise disjoint. Note that $0 < |O| < \infty$. Since T is the invariant set for this IFS, $T \subset \bar{O}$. We claim that $T = \bar{O}$. If not, $V := O \setminus T$ is a nonempty open set. Since $|O|$ is positive and finite,

$$|O \setminus \cup_{j=1}^M f_j(O)| = 0.$$

For sufficiently large m , there exists $w \in W^m$ so that $f_w(O) \subset V$. Then the corresponding fixed point x_w for f_w is contained in V . However, since T is the invariant set for the IFS, $x_w \in T$. This is the desired contradiction. \square

For each $w \in W^m$, $m \geq 0$, we define the center of the *tile* T_w to be $p_w = f_w(p)$, and the inner and outer radii of T_w to be

$$R_w^i = 2^{-m} R^i \quad \text{and} \quad R_w^o = 2^{-m} R^o,$$

respectively. We have $B(p_w, R_w^i) \subset T_w \subset B(p_w, R_w^o)$.

The Hausdorff measures and dimensions of bounded subsets of \mathbb{G} can be computed using a ‘‘dyadic Hausdorff measure’’ constructed using the tiles $\{T_w\}$. By applying a dilation, we may restrict our attention to subsets of the initial tile T . Let $\mathcal{D}_* = \cup_{m \geq 0} \mathcal{D}_m$. For a set $A \subset T$ and $s, \epsilon > 0$, define

$$\mathcal{H}_{\mathcal{D}, \epsilon}^s(A) = \inf \sum_i (\text{diam } T_{w_i})^s$$

where the infimum is taken over all coverings of A by tiles $T_{w_i} \in \mathcal{D}_*$ with $\text{diam } T_{w_i} < \epsilon$. Define $\mathcal{H}_{\mathcal{D}}^s(A) = \lim_{\epsilon \rightarrow 0} \mathcal{H}_{\mathcal{D}, \epsilon}^s(A)$. For each $s > 0$, $\mathcal{H}_{\mathcal{D}}^s$ is a Borel measure on T [26]. If the intersection of two tiles has nonempty interior, then one of the tiles is contained inside the other. As a result we can without loss of generality restrict our attention in the definition of $\mathcal{H}_{\mathcal{D}}^s$ to essentially disjoint coverings.

Proposition 3.5 (Equivalence of Hausdorff and dyadic Hausdorff measure). *For each $s > 0$ there exists a constant $C = C(s) > 0$ so that $\mathcal{H}^s(A) \leq \mathcal{H}_{\mathcal{D}}^s(A) \leq C\mathcal{H}^s(A)$ for every $A \subset T$.*

Lemma 3.6. *There exists a constant $K > 0$ so that for any ball $B(q, r)$ with $r \leq 1$ and $m \geq 0$ chosen so that $2^{-m-1} \leq r < 2^{-m}$, it holds that the number of tiles $T_w \in \mathcal{D}_m$ which intersect $B(q, r)$ is at most K .*

Corollary 3.7. *For each $m \geq 0$, no point of \mathbb{G} lies in more than K of the tiles in \mathcal{D}_m .*

Proof of Lemma 3.6. Assume that T_{w_1}, \dots, T_{w_N} is a collection of tiles in \mathcal{D}_m , each of which intersects $B(q, r)$. Then $B(p_{w_1}, R_{w_1}^i), \dots, B(p_{w_N}, R_{w_N}^i)$ is a collection of disjoint balls, all contained in the ball $B(q, r + 2^{-m} \text{diam } T)$ and all with radius $2^{-m} R^i$. Since

$$r + 2^{-m} \text{diam } T < 2^{-m}(1 + 2R^o)$$

and \mathbb{G} is a doubling metric space, we conclude that N is bounded above by a constant depending only on $(1 + 2R^o)/R^i$. \square

Proof of Proposition 3.5. It suffices to prove the upper bound.

Since the spherical Hausdorff measure $\mathcal{H}_{\mathbb{B}}^s$ (defined using coverings by balls) and the usual Hausdorff measure \mathcal{H}^s are comparable, it suffices to prove that $\mathcal{H}_{\mathcal{D}, C\epsilon}^s(A) \leq C(\mathcal{H}_{\mathbb{B}, \epsilon}^s(A) + \epsilon)$ for a suitable constant C and for all $\epsilon > 0$.

Assume A is covered by balls $\{B_i\}_i$ with $\text{diam } B_i < \epsilon$ and $\sum_i (\text{diam } B_i)^s < \mathcal{H}_{\mathbb{B}, \epsilon}^s(A) + \epsilon$. By Lemma 3.6, each B_i is covered by tiles $\{T_{i,j} : 1 \leq j \leq N\}$, where N is bounded independent of i and $\text{diam } T_{i,j} \simeq \text{diam } B_i$ for all $1 \leq j \leq N$. Then A is covered by the tiles $\{T_{i,j}\}_{i,j}$ and $\text{diam } T_{i,j} \leq C\epsilon$ for all i and j , for some fixed C . We obtain

$$\mathcal{H}_{\mathcal{D}, C\epsilon}^s(A) \leq \sum_{i,j} (\text{diam } T_{i,j})^s \leq C \sum_i (\text{diam } B_i)^s \leq C(\mathcal{H}_{\mathbb{B}, \epsilon}^s(A) + \epsilon).$$

This completes the proof. \square

The dyadic Hausdorff measures $\mathcal{H}_{\mathcal{D}}^s$ are a special case of the general notion of ‘‘comparable net measures’’ introduced by Davies and Rogers, see for instance [29, §2.7] or [16, pp. 64–74].

3.2. Smooth partitions of unity in Carnot groups and a sub-Riemannian Harvey–Polking Lemma. The following lemma is inspired by [21, Lemma 3.1].

Lemma 3.8 (Harvey–Polking partition of unity). *Let $\{T_{w_i} : 1 \leq i \leq N\}$ be a finite collection of essentially disjoint tiles, with $T_{w_i} \in \mathcal{D}_{m(i)}$. For each i there is a function $\varphi_i \in C_0^\infty(\mathbb{G})$, supported in $B(p_{w_i}, 2R_{w_i}^o)$, so that*

$$\sum_{i=1}^N \varphi_i(q) = 1 \quad \text{for all } q \in \bigcup_{i=1}^N T_{w_i}.$$

Moreover, for each multi-index α there exists a constant $C_\alpha > 0$ so that

$$|X_\alpha \varphi_i(x)| \leq C_\alpha 2^{m(i)|\alpha|} \quad \text{for all } x \in \mathbb{G} \text{ and } 1 \leq i \leq N.$$

Here, for a multi-index $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \{1, \dots, m_1\}^\ell$ (recall that m_1 denotes the dimension of the horizontal layer \mathfrak{v}_1 of \mathbb{G}) we write

$$X_\alpha = X_{\alpha_1} X_{\alpha_2} \cdots X_{\alpha_\ell}$$

and $|\alpha| = \ell$.

Proof of Lemma 3.8. Let $s_i = \text{diam } T_{w_i}$, $i = 1, \dots, N$, and without loss of generality assume that $s_1 \geq s_2 \geq \dots \geq s_N$. Choose $\psi \in C_0^\infty(\mathbb{G})$ so that $\psi|_T \equiv 1$ and $\psi|_{\mathbb{G} \setminus B(p, 2R^o)} = 0$. For each i , define ψ_i by

$$\psi_i(q) = \psi(f_{w_i}^{-1}(q)),$$

where f_w is defined in (3.3). Then $\psi_i \in C_0^\infty(\mathbb{G})$, $\psi_i|_{T_{w_i}} \equiv 1$ and ψ_i is supported in the ball $B(p_{w_i}, 2R_{w_i}^o)$. For $1 \leq i \leq N$ define $\varphi_1 = \psi_1$ and $\varphi_{i+1} = \psi_{i+1} \prod_{k=1}^i (1 - \psi_k)$. Then $\varphi_i \in C_0^\infty(\mathbb{G})$ and φ_i is supported in the ball $B(p_{w_i}, 2R_{w_i}^o)$. Furthermore, $\sum_i \varphi_i = 1$ in the union of the tiles T_{w_i} . Let $\theta_i = \sum_{k=1}^i \varphi_k = 1 - \prod_{k=1}^i (1 - \psi_k)$. Since s_i is decreasing it is enough to show

$$(3.9) \quad \|X_\alpha \theta_i\|_\infty \leq C_\alpha s_i^{-|\alpha|}$$

for all $i = 1, \dots, N$. Indeed, $X_\alpha \theta_i = X_\alpha \theta_{i-1} + X_\alpha \varphi_i$, whence (3.9) implies that

$$\|X_\alpha \varphi_i\|_\infty \leq \|X_\alpha \theta_i\|_\infty + \|X_\alpha \theta_{i-1}\|_\infty \lesssim s_i^{-|\alpha|} + s_{i-1}^{-|\alpha|} \lesssim s_i^{-|\alpha|}.$$

Let

$$B_\alpha = \left\{ \bar{\beta} = (\beta_1, \dots, \beta_r) : \beta_j \text{ is a multi-index, } |\beta_j| \geq 1 \text{ and } \sum_{j=1}^r \beta_j = \alpha \right\}$$

For every r -tuple of multi-indices $\bar{\beta}$ one can assign $i(i-1)\dots(i-r+1)$ r -tuples (ν_1, \dots, ν_r) in $\{1, \dots, i\}^r$ with all ν_j distinct. Define the functions

$$g_{\nu_1, \dots, \nu_r} = \begin{cases} 0 & \text{if } \nu_j = \nu_k \text{ for some } j \neq k \\ \prod_{\substack{k \neq \nu_1, \dots, \nu_r \\ k \leq i}} (1 - \psi_k) & \text{if all } \nu_j \text{ are distinct and } r < k \\ -1 & \text{if all } \nu_j \text{ are distinct and } r = k. \end{cases}$$

Then there exist constants $C_{\beta_1, \dots, \beta_r}$ such that

$$X_\alpha \theta_i = \sum_{(\beta_1, \dots, \beta_r) \in B_\alpha} C_{\beta_1, \dots, \beta_r} \sum_{(\nu_1, \dots, \nu_r) \in \{1, \dots, i\}^r} g_{\nu_1, \dots, \nu_r} (X_{\beta_1} \psi_{\nu_1}) \dots (X_{\beta_r} \psi_{\nu_r}).$$

Therefore

$$|X_\alpha \theta_i(q)| \leq \sum_{(\beta_1, \dots, \beta_r) \in B_\alpha} C_{\beta_1, \dots, \beta_r} \left(\sum_{\nu_1=1}^i |X_{\beta_1} \psi_{\nu_1}(q)| \right) \dots \left(\sum_{\nu_r=1}^i |X_{\beta_r} \psi_{\nu_r}(q)| \right)$$

for all $q \in \mathbb{G}$.

Now consider any of the sums $\sum_{\nu=1}^i |X_\beta \psi_\nu(q)|$, where β is a multi-index and $\nu = 1, \dots, i$. Notice that $X_\beta \psi_\nu(q) = 0$ if $q \notin B(p_{w_\nu}, 2R_{w_\nu}^o)$. Furthermore for all $i = 1, \dots, m$ and $r > 0$

$$\begin{aligned} X_i(\psi \circ \delta_r)(q) &= \frac{d}{dt} \psi((\delta_r(q)) \exp(rtX_i))|_{t=0} \\ &= r \frac{d}{dt} \psi((\delta_r(q)) \exp(tX_i))|_{t=0} = r(X_i \psi \circ \delta_r)(q). \end{aligned}$$

Hence

$$X_i \psi_j(q) = X_i(\psi \circ f_j^{-1})(q) = 2((X_i \psi) \circ f_j^{-1})(q) \quad \text{for } j = 1, \dots, M \text{ and } i = 1, \dots, m$$

and $\|X_\beta \psi_\nu\|_\infty \leq C s_\nu^{-|\beta|}$ in $B(p_{w_\nu}, 2R_{w_\nu}^o)$. Therefore for all $q \in \mathbb{G}$ we have

$$\sum_{\nu=1}^i |X_\beta \psi_\nu(q)| \leq C_\beta \sum_{\substack{s_\nu \geq s_i \\ q \in B(p_{w_\nu}, 2R_{w_\nu}^o)}} |s_\nu|^{-|\beta|}.$$

Given q , as in the proof of Lemma 3.6 there exists a finite number $K := K(\mathbb{G})$ of tiles T_{w_1}, \dots, T_{w_K} in \mathcal{D}_m such that $q \in \bigcap_{l=1}^K B(p_{w_l}, 2R_{w_l}^o)$. Therefore for all $p \in \mathbb{N}$ there exist at most K tiles T_ν with $\text{diam } T_\nu = 2^p \text{ diam } T_i$ such that $q \in B(p_{w_\nu}, 2R_{w_\nu}^o)$. Hence

$$\sum_{\nu=1}^i |X_\beta \psi_\nu(q)| \leq K C_\beta \sum_{p=0}^{\infty} (2^p \text{diam } T_i)^{-|\beta|} \lesssim (\text{diam } T_i)^{-|\beta|}$$

and

$$|X_\alpha \theta_i(q)| \leq \sum_{(\beta_1, \dots, \beta_r) \in B_\alpha} C_{\beta_1, \dots, \beta_r} |s_i|^{-|\beta_1|} \dots |s_i|^{-|\beta_r|} \lesssim |s_i|^{-|\alpha|}.$$

This completes the proof. \square

4. REMOVABLE SETS FOR BMO AND HÖLDER CONTINUOUS \mathcal{L} -SOLUTIONS

Our first theorem characterizes the removable sets for BMO \mathcal{L} -solutions.

Definition 4.1. Let U be an open set in \mathbb{G} . A function $f \in L^1_{\text{loc}}(U)$ is in $\text{BMO}(U)$ if any of the two following equivalent conditions is satisfied:

$$(4.2) \quad \sup_{\substack{x \in \mathbb{G}, r > 0 \\ B(x,r) \subset U}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy < \infty$$

or

$$(4.3) \quad \sup_{\substack{x \in \mathbb{G}, r > 0 \\ B(x,r) \subset U}} \inf_{c \in \mathbb{R}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - c| dy < \infty.$$

As usual $f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy$ and $\text{BMO} := \text{BMO}(\mathbb{G})$.

Theorem 4.4. Fix $1 \leq \lambda < Q$ and let \mathcal{L} be a λ -homogeneous, left invariant operator on \mathbb{G} such that both \mathcal{L} and \mathcal{L}^t are hypoelliptic. Then a compact set E is removable for BMO \mathcal{L} -solutions if and only if $\mathcal{H}^{Q-\lambda}(E) = 0$.

Thus, for instance, when \mathcal{L} is the sub-Laplacian on \mathbb{G} , a compact set E is removable for BMO \mathcal{L} -harmonic functions if and only if $\mathcal{H}^{Q-2}(E) = 0$.

In the Euclidean setting \mathbb{R}^n , $n \geq 3$, Theorem 4.4 is due to Ishchanov [23]. Recall that we have assumed $Q \geq 3$ in this paper. The planar case is somewhat different; for results in that setting, see Kaufman [24].

Proof of Theorem 4.4. We first suppose that $\mathcal{H}^{Q-\lambda}(E) = 0$. Let a domain $\Omega \supset E$ and a function $f \in \text{BMO}(\Omega)$ be given. Let $d_0 = \text{dist}(E, \Omega^c)$. Let also $\psi \in C_0^\infty(\Omega)$ and $0 < \varepsilon < (d_0/4)^{Q-\lambda}$. By Proposition 3.5 and compactness there exists a finite collection $\{T_{w_j}\}_{j=1}^N$ of tiles such that $E \subset \bigcup_{j=1}^N T_{w_j}$ and

$$\sum_{j=1}^N (\text{diam } T_{w_j})^{Q-\lambda} < \varepsilon.$$

Note that $\bigcup_{j=1}^N B(p_{w_j}, 2R_{w_j}^o) \subset \Omega$, since $R_{w_j}^o \leq \text{diam } T_{w_j}$.

By Lemma 3.8 there exists a family of functions $\{\varphi_j\}_{j=1}^N$ such that $\text{supp } \varphi_j \subset B(p_{w_j}, R_{w_j}^o)$, $\sum_{j=1}^N \varphi_j \equiv 1$ on $\bigcup_{j=1}^N T_{w_j}$ and $\|X_\alpha \varphi_j\|_\infty \leq C(\text{diam } T_{w_j})^{-|\alpha|}$ for every multi-index α and every $j = 1, \dots, N$. Furthermore, as noted earlier, by the Poincaré–Birkhoff–Witt Theorem, \mathcal{L} is a linear combination of the operators X_{α_l} with $|\alpha_l| = \lambda$; therefore we can assume without loss of generality that $\mathcal{L} = X_\alpha$ with $|\alpha| = \lambda$. Since $\mathcal{L}f = 0$ in $\Omega \setminus E$, we conclude

$$(4.5) \quad \begin{aligned} \langle \mathcal{L}f, \psi \rangle &= \langle \mathcal{L}f, \psi \cdot 1 \rangle = \langle \mathcal{L}f, \sum_{j=1}^N \psi \varphi_j \rangle \\ &= \sum_{j=1}^N \langle \mathcal{L}f, \psi \varphi_j \rangle = \sum_{j=1}^n \langle \mathcal{L}(f - c_j), \psi \varphi_j \rangle \\ &= \sum_{j=1}^N (-1)^{|\alpha|} \langle f - c_j, X_\alpha(\psi \varphi_j) \rangle = \sum_{j=1}^N (-1)^{|\alpha|} \langle f - c_j, \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} X_\beta \psi X_{\alpha-\beta} \varphi_j \rangle, \end{aligned}$$

where $c_j = f_{B(p_{w_j}, 2R_{w_j}^o)}$, β is a multi-index with $|\beta| = |\alpha|$ and $\beta \leq \alpha$ denotes that $\beta_i \leq \alpha_i$ for all $i = 1, \dots, |\alpha|$. Therefore

$$\begin{aligned}
 |\langle \mathcal{L}f, \psi \rangle| &\leq \sum_{j=1}^N |\langle f - c_j, \psi X_\alpha \varphi_j \rangle| + \sum_{j=1}^n \left| \left\langle f - c_j, \sum_{\substack{\beta \leq \alpha \\ \beta \neq \alpha}} \binom{\alpha}{\beta} X_\beta \psi X_{\alpha-\beta} \varphi_j \right\rangle \right| \\
 &\lesssim \sum_{j=1}^N \int_{B(p_{w_j}, 2R_{w_j}^o)} |f(y) - c_j| \|X_\alpha(\varphi_j)\|_\infty dy \\
 &\quad + \sum_{j=1}^N \sum_{\substack{\beta \leq \alpha \\ \beta \neq \alpha}} \int_{B(p_{w_j}, 2R_{w_j}^o)} |f(y) - c_j| \|X_{\alpha-\beta} \varphi_j\|_\infty dy \\
 &\lesssim \sum_{j=1}^N (\text{diam } T_{w_j})^{-\lambda} |B(p_{w_j}, 2R_{w_j}^o)| + \sum_{j=1}^N \sum_{\substack{\beta \leq \alpha \\ \beta \neq \alpha}} (\text{diam } T_{w_j})^{|\beta|-\lambda} |B(p_{w_j}, 2R_{w_j}^o)| \\
 &\lesssim \sum_{j=1}^N \left((\text{diam } T_{w_j})^{Q-\lambda} + \sum_{\substack{\beta \leq \alpha \\ \beta \neq \alpha}} (\text{diam } T_{w_j})^{Q-\lambda+|\beta|} \right) \lesssim \sum_{j=1}^n (\text{diam } T_{w_j})^{Q-\lambda} < \varepsilon.
 \end{aligned}$$

Since ε was arbitrary we obtain that $\langle \mathcal{L}f, \psi \rangle = 0$, which means that f is a distributional solution to $\mathcal{L}f = 0$ in Ω . Therefore by hypoellipticity, $f \in C^\infty(\Omega)$ and $\mathcal{L}f = 0$ in Ω . Hence E is removable for BMO \mathcal{L} -solutions.

For the other direction it is enough to show that if $\mathcal{H}^{Q-\lambda}(E) > 0$, then E is not removable. By Frostman's Lemma there exists a non-trivial Borel measure μ with $\text{supp } \mu \subset E$, satisfying

$$(4.6) \quad \mu(B(x, r)) \leq Cr^{Q-\lambda} \quad \text{for all } x \in \mathbb{G} \text{ and } r > 0,$$

By [18, Theorem 2.1] there is a unique kernel k of type λ which is a fundamental solution for \mathcal{L} . Denote by $\|q\| = d(p, 0)$ the homogeneous norm associated to the Carnot–Carathéodory metric d on \mathbb{G} . Since k is continuous on $\Sigma := \{q \in \mathbb{G} : \|q\| = 1\}$, there exists $q_1 \in \Sigma$ such that $|k(q)| \leq |k(q_1)|$ for all $q \in \Sigma$. Consequently, if $p \in \mathbb{G} \setminus \{0\}$ then

$$(4.7) \quad |k(p)| = |k(\delta_{\|p\|} \circ \delta_{\|p\|^{-1}}(p))| \leq \|p\|^{\lambda-Q} |k(\delta_{\|p\|^{-1}}(p))| \leq c_1 \|p\|^{\lambda-Q},$$

where we set $c_1 := k(q_1)$.

Now let

$$f = k * \mu := \int k(x \cdot y^{-1}) d\mu(y).$$

Since k is a fundamental solution of the hypoelliptic operator \mathcal{L} we deduce that $\mathcal{L}f = 0$ in $\mathbb{G} \setminus E$.

Furthermore $f \in \text{BMO}$. To see this, first notice that $f \in L_{\text{loc}}^1$ with respect to the Haar measure. Let $x_0 \in \mathbb{G}$, $r > 0$ and let $\mu_1 := \mu \llcorner B(x_0, 2r)$ and $\mu_2 := \mu \llcorner B(x_0, 2r)^c$ denote the restrictions of μ

to $B(x_0, 2r)$ and $B(x_0, 2r)^c$ respectively. By (4.7) and Fubini's theorem,

$$\begin{aligned}
\left| \int_{B(x_0, r)} \int k(x \cdot y^{-1}) d\mu_1(y) dx \right| &\leq \int_{B(x_0, r)} \int_{B(x_0, 2r)} \|x \cdot y^{-1}\|^{\lambda-Q} d\mu(y) dx \\
&= \int_{B(x_0, 2r)} \int_{B(x_0, r)} \|x \cdot y^{-1}\|^{\lambda-Q} dx d\mu(y) \\
&\leq \int_{B(x_0, 2r)} \left(\sum_{j=0}^{\infty} \int_{B(x, 2^{-(j-1)}r) \setminus B(x, 2^{-j}r)} \|x \cdot y^{-1}\|^{\lambda-Q} dx \right) d\mu(y) \\
&\lesssim \int_{B(x_0, 2r)} r^\lambda d\mu(y) \lesssim r^Q.
\end{aligned}$$

Hence

$$(4.8) \quad \int_{B(x_0, r)} |k * \mu_1(x)| dx \lesssim r^Q.$$

For $x \in B(x_0, r)$,

$$|k * \mu_2(x) - k * \mu_2(x_0)| \leq \int_{B(x_0, 2r)^c} |k(x \cdot y^{-1}) - k(x_0 \cdot y^{-1})| d\mu(y).$$

Since k agrees with a C^∞ , $(\lambda - Q)$ -homogeneous function on $\mathbb{G} \setminus \{0\}$,

$$(4.9) \quad |k(Y \cdot X) - k(X)| \leq C \|Y\| \|X\|^{\lambda-Q-1} \quad \text{for all } \|Y\| \leq \|X\|/2.$$

This follows exactly as in [19, Proposition 1.7] using the smoothness of the map $y \rightarrow y \cdot x$. Therefore if $x \in B(x_0, r)$, $y \in B(x_0, 2r)^c$, letting $X = x_0 \cdot y^{-1}$, $Y = x \cdot x_0^{-1}$ we have that $\|Y\| \leq r \leq \|X\|/2$ and we can apply (4.9) to obtain

$$\begin{aligned}
|k(x \cdot y^{-1}) - k(x_0 \cdot y^{-1})| &= |k(Y \cdot X) - k(X)| \\
&\leq C \|Y\| \|X\|^{\lambda-Q-1} = r \|x_0 \cdot y^{-1}\|^{\lambda-Q-1}.
\end{aligned}$$

Therefore for $x \in B(x_0, r)$,

$$\begin{aligned}
|k * \mu_2(x) - k * \mu_2(x_0)| &\lesssim r \int_{B(x_0, 2r)^c} \|x_0 \cdot y^{-1}\|^{\lambda-Q-1} d\mu(y) \\
&= r \sum_{k=1}^{\infty} \int_{B(x_0, 2^{k+1}r) \setminus B(x_0, 2^k r)} \|x_0 \cdot y^{-1}\|^{\lambda-Q-1} d\mu(y) \\
&\leq r \sum_{k=1}^{\infty} \int_{B(x_0, 2^{k+1}r)} (2^k r)^{\lambda-Q-1} d\mu(y) \\
&= r \sum_{k=1}^{\infty} (2^k r)^{\lambda-Q-1} \mu(B(x_0, 2^{k+1}r)) \leq C,
\end{aligned}$$

and

$$(4.10) \quad \int_{B(x_0, r)} |k * \mu_2(x) - k * \mu_2(x_0)| dx \leq Cr^Q.$$

Combining (4.8) and (4.10) we get that for all $x_0 \in \mathbb{G}$ and $r > 0$ that

$$\begin{aligned} \int_{B(x_0, r)} |k * \mu(x) - k * \mu_2(x)| dx &= \int_{B(x_0, r)} |k * \mu_1(x) + k * \mu_2(x) - k * \mu_2(x)| dx \\ &\leq \int_{B(x_0, r)} |k * \mu_1(x)| dx + \int_{B(x_0, r)} |k * \mu_2(x) - k * \mu_2(x)| dx \leq Cr^Q. \end{aligned}$$

Recalling the equivalent definition (4.3) for BMO functions we deduce that $f = k * \mu \in \text{BMO}$.

Now suppose that $\mathcal{L}f = 0$. Then $f \in C^\infty(\mathbb{G})$ by hypoellipticity and moreover, by Geller's Liouville type theorem [20], f is a polynomial. By (4.7),

$$|f(p)| \leq \int_E \|p \cdot q^{-1}\|^{\lambda-Q} d\mu(q)$$

for all $p \in \mathbb{G}$. Since E is compact this implies that $\lim_{\|p\| \rightarrow \infty} |f(p)| = 0$, hence $f \equiv 0$. Since f cannot be constant, we have reached a contradiction. It follows that E is not removable for BMO \mathcal{L} -solutions. \square

Next, we consider the case of Hölder continuous functions. Again, we characterize the removable sets for \mathcal{L} -solutions in this class.

Definition 4.11. Let $U \subset \mathbb{G}$ open and $\delta \in (0, 1)$. A function $f : U \rightarrow \mathbb{R}$ belongs to $\text{Lip}_\delta(U)$ if there exists some constant C such that

$$|f(x) - f(y)| \leq Cd(x, y)^\delta$$

for all $x, y \in U$.

Theorem 4.12. Fix $1 \leq \lambda < Q$ and let \mathcal{L} be a λ -homogeneous, left invariant operator on \mathbb{G} such that both \mathcal{L} and \mathcal{L}^t are hypoelliptic. Then a compact set E is removable for $\text{Lip}_\delta \mathcal{L}$ -solutions, $\delta \in (0, 1)$, if and only if $\mathcal{H}^{Q-\lambda+\delta}(E) = 0$.

In particular, if \mathcal{L} is the sub-Laplacian on \mathbb{G} , then a compact set E is removable for $\text{Lip}_\delta \mathcal{L}$ -harmonic functions if and only if $\mathcal{H}^{Q-2+\delta}(E) = 0$.

Proof of Theorem 4.12. We first suppose that $\mathcal{H}^{Q-\lambda+\delta}(E) = 0$. The proof in this case is very similar to the proof of the same implication in Theorem 4.4. The only difference is that now in (4.5) we have to choose $c_j = f(p_{w_j})$.

For the other direction it is enough to show that if $\mathcal{H}^{Q-\lambda+\delta}(E) > 0$, then E is not removable. Again using Frostman's Lemma we find a non-trivial Borel measure μ with $\text{supp } \mu \subset E$, satisfying

$$\mu(B(x, r)) \leq Cr^{Q-\lambda+\delta} \quad \text{for all } x \in \mathbb{G} \text{ and } r > 0,$$

If $f = k * \mu$ where k is a fundamental solution of the hypoelliptic operator \mathcal{L} , it follows that $\mathcal{L}f = 0$ in $\mathbb{G} \setminus E$.

We are going to show that $f \in \text{Lip}_\delta(\mathbb{G})$. Let $x, z \in \mathbb{G}$. Then

$$\begin{aligned} |f(x) - f(z)| &= \left| \int k(x \cdot y^{-1}) d\mu(y) - \int k(z \cdot y^{-1}) d\mu(y) \right| \\ &\leq \int_{\{\|z \cdot y^{-1}\| > 2\|z \cdot x^{-1}\|\}} |k(x \cdot y^{-1}) - k(z \cdot y^{-1})| d\mu(y) \\ &\quad + \int_{\{\|z \cdot y^{-1}\| \leq 2\|z \cdot x^{-1}\|\}} |k(x \cdot y^{-1}) - k(z \cdot y^{-1})| d\mu(y) \\ &:= I_1 + I_2. \end{aligned}$$

To estimate I_1 we use (4.9) to obtain

$$I_1 \lesssim \|z \cdot x^{-1}\| \int_{\{\|z \cdot y^{-1}\| > 2\|z \cdot x^{-1}\|\}} \|z \cdot y^{-1}\|^{\lambda-Q-1} d\mu(y).$$

Moreover,

$$\begin{aligned} & \int_{\{\|z \cdot y^{-1}\| > 2\|z \cdot x^{-1}\|\}} \|z \cdot y^{-1}\|^{\lambda-Q-1} d\mu(y) \\ &= \sum_{j=1}^{\infty} \int_{B(z, 2^{j+1}\|z \cdot x^{-1}\|) \setminus B(z, 2^j\|z \cdot x^{-1}\|)} \|z \cdot y^{-1}\|^{\lambda-Q-1} d\mu(y) \\ &\lesssim \sum_{j=1}^{\infty} \frac{(2^{j+1}\|z \cdot x^{-1}\|)^{Q-\lambda+\delta}}{(2^j\|z \cdot x^{-1}\|)^{Q-\lambda+1}} \lesssim \|z \cdot x^{-1}\|^{\delta-1} \sum_{j=1}^{\infty} (2^{\delta-1})^j \lesssim \|z \cdot x^{-1}\|^{\delta-1}. \end{aligned}$$

Hence $I_1 \lesssim \|z \cdot x^{-1}\|^\delta$.

For I_2 , recalling the pseudo triangle inequality $\|p \cdot q\| \leq C(\|p\| + \|q\|)$ for $p, q \in \mathbb{G}$, we see that

$$\{y \in \mathbb{G} : \|z \cdot y^{-1}\| \leq 2\|z \cdot x^{-1}\|\} \subset \{y \in \mathbb{G} : \|x \cdot y^{-1}\| \leq 3C\|z \cdot x^{-1}\|\}.$$

Hence

$$I_2 \leq \int_{\{\|x \cdot y^{-1}\| \leq 3C\|z \cdot x^{-1}\|\}} |k(x \cdot y^{-1})| d\mu(y) + \int_{\{\|z \cdot y^{-1}\| \leq 2\|z \cdot x^{-1}\|\}} |k(z \cdot y^{-1})| d\mu(y).$$

Also

$$\begin{aligned} & \int_{\{\|x \cdot y^{-1}\| \leq 3C\|z \cdot x^{-1}\|\}} |k(x \cdot y^{-1})| d\mu(y) \\ &= \sum_{j=0}^{\infty} \int_{B(x, 2^{-j}3C\|z \cdot x^{-1}\|) \setminus B(x, 2^{-j-1}3C\|z \cdot x^{-1}\|)} \|x \cdot y^{-1}\|^{\lambda-Q-1} d\mu(y) \\ &\lesssim \sum_{j=0}^{\infty} \frac{(2^{-j}3C\|z \cdot x^{-1}\|)^{Q-\lambda+\delta}}{(2^{-j-1}3C\|z \cdot x^{-1}\|)^{Q-\lambda}} \\ &\lesssim \|z \cdot x^{-1}\|^\delta \sum_{j=0}^{\infty} (2^\delta)^{-j} \lesssim \|z \cdot x^{-1}\|^\delta \end{aligned}$$

and in the same way

$$\int_{\{\|z \cdot y^{-1}\| \leq 2\|z \cdot x^{-1}\|\}} |k(z \cdot y^{-1})| d\mu(y) \lesssim \|z \cdot x^{-1}\|^\delta.$$

Therefore $I_1 + I_2 \lesssim \|z \cdot x^{-1}\|^\delta$ and $f \in \text{Lip}_\delta(\mathbb{G})$.

Finally arguing as in Theorem 4.4 and using Geller's Liouville theorem we deduce that $\mathcal{L}f \not\equiv 0$ in \mathbb{G} which implies that E is not removable for $\text{Lip}_\delta \mathcal{L}$ -solutions. \square

5. REMOVABLE SETS FOR $L_{\text{loc}}^p \mathcal{L}$ -SOLUTIONS

The following lemma is a direct consequence of Lemma 3.8, nevertheless we provide the proof which follows as in [21].

Lemma 5.1. *Let $E \subset \mathbb{G}$ compact, $p \geq 1$ and $\varepsilon > 0$. Then there exists $\varphi_\varepsilon \in C_0^\infty(\mathbb{G})$ such that $\varphi_\varepsilon \equiv 1$ in a neighborhood of E , $\text{supp } \varphi_\varepsilon \subset E_\varepsilon := \{x : \text{dist}(x, E) < \varepsilon\}$ and for all multi-indices α such that $|\alpha| \leq \ell$,*

$$\|X_\alpha \varphi_\varepsilon\|_p \lesssim \varepsilon^{\ell-|\alpha|} (\mathcal{H}^{Q-\ell p}(E) + \varepsilon)^{1/p}.$$

Proof. We may assume without loss of generality that $\mathcal{H}^{Q-\ell p}(E) < \infty$. For any $\varepsilon > 0$ there exist disjoint tiles $\{T_k\}_{k=1}^N$ which cover E such that $\text{diam } T_k \leq \varepsilon$, $\bigcup_{k=1}^N B(p_{w_k}, R_{w_k}^o) \subset E_\varepsilon$ and

$$(5.2) \quad \sum_{k=1}^N (\text{diam } T_k)^{Q-\ell p} \leq C (\mathcal{H}^{Q-\ell p}(E) + \varepsilon).$$

We can assume that $\text{diam } T_1 \geq \text{diam } T_2 \geq \dots \geq \text{diam } T_N$. Let $\{\varphi_k\}_{k=1}^N$ be the partition of unity, in the sense of Lemma 3.8, associated to the tiles $\{T_k\}_{k=1}^N$. Then if $\varphi_\varepsilon = \sum_{k=1}^N \varphi_k$ we have that $\text{supp } \varphi_\varepsilon \subset \bigcup_{k=1}^N B(p_{w_k}, R_{w_k}^o)$ and $\varphi_\varepsilon \equiv 1$ on $\bigcup_{k=1}^N T_k$.

Let

$$D_k = B(p_{w_k}, R_{w_k}^o) \setminus \bigcup_{j=k+1}^N B(p_{w_j}, R_{w_j}^o), \quad \text{for } k = 1, \dots, N.$$

The sets D_k are disjoint and $\bigcup_{k=1}^N D_k = \bigcup_{k=1}^N B(p_{w_k}, R_{w_k}^o)$. Notice that if $x \in D_k$ then $x \notin \bigcup_{j=k+1}^N B(p_{w_j}, R_{w_j}^o)$ which implies that $\varphi_j(x) = 0$ for all $x \in D_k$ and $j > k$. Hence

$$\varphi_\varepsilon(x) = \sum_{j=1}^k \varphi_j(x) = \theta_k(x)$$

for all $x \in D_k$, where θ_k is as in Lemma 3.8, and consequently

$$|X_\alpha \varphi_\varepsilon(x)| \leq C_\alpha (\text{diam } T_k)^{-|\alpha|}.$$

Therefore by (5.2)

$$\begin{aligned} \|X_\alpha \varphi_\varepsilon\|_p^p &= \sum_{k=1}^N \int_{D_k} |X_\alpha \varphi_\varepsilon(x)|^p dx = \sum_{k=1}^N \int_{D_k} |X_\alpha \theta_k(x)|^p dx \\ &\lesssim \sum_{k=1}^N \int_{D_k} (\text{diam } T_k)^{-|\alpha|p} dx \lesssim \sum_{k=1}^N (\text{diam } T_k)^Q (\text{diam } T_k)^{-|\alpha|p} \\ &\lesssim \varepsilon^{(\ell-|\alpha|)p} \sum_{k=1}^N (\text{diam } T_k)^{Q-\ell p} \lesssim \varepsilon^{(\ell-|\alpha|)p} (\mathcal{H}^{Q-\ell p}(E) + \varepsilon). \end{aligned}$$

The proof is complete. \square

In our final theorem we consider removable sets for L_{loc}^p \mathcal{L} -solutions. Our characterization in the following theorem is sharp on the level of the Hausdorff dimension, but not on the level of the Hausdorff measure.

We denote by $\dim E$ the Hausdorff dimension of a set $E \subset \mathbb{G}$ with respect to the metric d .

Theorem 5.3. *Fix $1 \leq \lambda < Q$ and let \mathcal{L} be a λ -homogeneous, left invariant operator on \mathbb{G} such that both \mathcal{L} and \mathcal{L}^t are hypoelliptic. Then if $1 < p < \infty$:*

- (1) *any compact set E with $\mathcal{H}^{Q-\lambda p'}(E) < \infty$ is removable for L_{loc}^p \mathcal{L} -solutions, while*
- (2) *any compact set E with $\dim E > Q - \lambda p'$ is not removable for L_{loc}^p \mathcal{L} -solutions.*

Here as usual p' denotes the conjugate exponent of p . If $p = \infty$:

- (3) *any compact set E with $\mathcal{H}^{Q-\lambda}(E) = 0$ is removable for L_{loc}^∞ \mathcal{L} -solutions, while*
- (4) *any compact set E with $\dim E > Q - \lambda$ is not removable for L_{loc}^∞ \mathcal{L} -solutions.*

Finally,

- (5) *if E is compact with $\mathcal{H}^{Q-\lambda}(E) < \infty$, $\Omega \supset E$ is a domain and $f \in L_{\text{loc}}^\infty(\Omega)$ such that $\mathcal{L}f = 0$ in $\Omega \setminus E$, then $\mathcal{L}f$ is a measure supported in E .*

Proof of Theorem 5.3. We are first going to prove (1), (3) and (5) using Lemma 5.1 and following [21]. Suppose that $\mathcal{H}^{Q-\lambda p'}(E) < \infty$. Let a domain $\Omega \supset E$ and a function $f \in L^p_{\text{loc}}(\Omega)$ be given. Let also $\psi \in C_0^\infty(\Omega)$ and $\varepsilon > 0$. The Poincaré–Birkhoff–Witt Theorem implies that \mathcal{L} is a linear combination of the operators X_{α_l} with $|\alpha_l| = \lambda$; therefore we can assume without loss of generality that $\mathcal{L} = X_\alpha$ with $|\alpha| = \lambda$. Since $\mathcal{L}f = 0$ in $\Omega \setminus E$, if φ_ε is as in Lemma 5.1,

$$(5.4) \quad \langle \mathcal{L}f, \psi \rangle = \langle \mathcal{L}f, \psi \varphi_\varepsilon \rangle = (-1)^\lambda \langle f, X_\alpha(\psi \varphi_\varepsilon) \rangle.$$

Since $X_\alpha(\psi \varphi_\varepsilon) = \sum_{\beta \leq \alpha} c_{\alpha, \beta} X_\beta(\psi) X_{\alpha-\beta}(\varphi_\varepsilon)$ Lemma 5.1 implies that

$$\|X_{\alpha-\beta}(\varphi_\varepsilon)\|'_p \lesssim \varepsilon^{\lambda-|\alpha-\beta|} (\mathcal{H}^{Q-\lambda p'}(E) + \varepsilon)^{1/p'}$$

and by Hölder's inequality

$$(5.5) \quad |\langle f, X_\alpha(\psi \varphi_\varepsilon) \rangle| \leq \int_{E_\varepsilon} |f X_\alpha(\psi \varphi_\varepsilon)| \lesssim \|f \chi_{E_\varepsilon}\|_p (\mathcal{H}^{Q-\lambda p'}(E) + \varepsilon)^{1/p'}.$$

Since $f \in L^p_{\text{loc}}(\Omega)$ and $m(E) = 0$ (as $\dim(E) < Q$) the monotone convergence theorem implies that $\|f \chi_{E_\varepsilon}\|_p \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence $\langle \mathcal{L}f, \psi \rangle = 0$, which means that f is a distributional solution to $\mathcal{L}f = 0$ in Ω . Therefore by hypoellipticity, $f \in C^\infty(\Omega)$ and $\mathcal{L}f = 0$ in Ω . Hence E is removable for L^p_{loc} \mathcal{L} -solutions.

Let $f \in L^\infty_{\text{loc}}(\Omega)$, ψ and φ_ε as in the proof of (1). The proof of (3) follows in the same manner noticing that as in (5.5) Lemma 5.1 implies

$$(5.6) \quad |\langle f, X_\alpha(\psi \varphi_\varepsilon) \rangle| \leq \|f \chi_{E_\varepsilon}\|_\infty \|X_\alpha(\psi \varphi_\varepsilon)\|_1 \lesssim \mathcal{H}^{Q-\lambda}(E) + \varepsilon.$$

For the proof of (5), as in (5.4) we get

$$\langle \mathcal{L}f, \psi \rangle = |-1|^\lambda \langle f \mathcal{L}\varphi_\varepsilon, \psi \rangle + |-1|^\lambda \sum_{\beta < \alpha} c_{\alpha, \beta} \langle f, X_\beta(\psi) X_{\alpha-\beta}(\varphi_\varepsilon) \rangle.$$

By Lemma 5.1

$$|\langle f, X_\beta(\psi) X_{\alpha-\beta}(\varphi_\varepsilon) \rangle| \lesssim \|X_{\alpha-\beta}(\varphi_\varepsilon)\|_1 \lesssim \varepsilon^{\lambda-|\alpha-\beta|} (\mathcal{H}^{Q-\lambda}(E) + \varepsilon)$$

hence the distributions

$$(5.7) \quad D_\varepsilon := |-1|^\lambda f \mathcal{L}\varphi_\varepsilon$$

converge weakly to $\mathcal{L}f$ in $\mathcal{D}'(\Omega)$ as $\varepsilon \rightarrow 0$. Furthermore $\|\mathcal{L}\varphi_\varepsilon\|_1 \leq C$ for $\varepsilon > 0$ hence the distributions D_ε are uniformly bounded in $L_1(\Omega)$ and their weak limit is a measure.

The proof of (2) is as in Carleson's book [9, §7]. We provide the details for completeness. By the assumption in (2) there exists $\eta > \max\{Q - \lambda p', 0\}$ such that $\mathcal{H}^\eta(E) > 0$. Frostman's Lemma yields a non-trivial Borel measure μ with $\text{supp } \mu \subset E$, satisfying

$$\mu(B(x, r)) \leq Cr^\eta \quad \text{for all } x \in \mathbb{G} \text{ and } r > 0.$$

For $x \in \mathbb{G}$, let

$$u(x) = \int \|x \cdot y^{-1}\|^{\lambda-Q} d\mu(y).$$

Let $K \subset \mathbb{G}$ be compact and let g be a nonnegative function such that $\int_{\mathbb{G}} g(y)^{p'} dy = 1$. Now if $\theta := \eta p^{-1} + (Q - \lambda p') p'^{-1}$ then $Q - \lambda p' < \theta < \eta$. Define $\varphi : \mathbb{C} \rightarrow \mathbb{R}$ by

$$\varphi(\zeta) = \int_K \int \frac{g(x)^{p'(1-\zeta)}}{\|x \cdot y^{-1}\|^{\theta+(Q-\eta)\zeta}} d\mu(y) dx$$

and notice that φ is analytic. For $t \in \mathbb{R}$, and assuming without loss of generality that $\text{diam}(E) + \text{diam}(K) \leq 1$, we have

$$\begin{aligned} |\varphi(it)| &= \int_K g(x)^{p'} \left(\int \|x \cdot y^{-1}\|^\theta d\mu(y) \right), dx \\ &\lesssim \int_K g(x)^{p'} \left(\sum_{j=0}^{\infty} \frac{(2^{-j})^\eta}{(2^{-(j+1)})^\theta} \right) dx \leq M \end{aligned}$$

and by Fubini's theorem,

$$\begin{aligned} |\varphi(1+it)| &= \int_K \int \|x \cdot y^{-1}\|^{\theta+Q-\eta} d\mu(y) dx \\ &= \int_E \int_K \|x \cdot y^{-1}\|^{\theta+Q-\eta} dx d\mu(y) \lesssim \int_E \sum_{j=0}^{\infty} \frac{(2^{-j})^Q}{(2^{-(j+1)})^{\theta+Q-\eta}} dx \leq M. \end{aligned}$$

By the maximum modulus principle, $|\varphi(\zeta)| \leq M$ for all ζ with $0 < \text{Re}(\zeta) < 1$, and in particular,

$$\varphi(p^{-1}) = \int_K g(x)u(x) dx \leq M.$$

Consequently, if k is the fundamental solution of \mathcal{L} , then by (4.7) we have that

$$\int_K g(x) |k * \mu(x)| dx \lesssim M.$$

Finally by duality

$$\int_K |k * \mu(x)|^p dx \lesssim M^p$$

and $f = k * \mu \in L_{\text{loc}}^p$. As in Theorem 4.4 Geller's Liouville theorem implies that $\mathcal{L}f \neq 0$ in \mathbb{G} , which in turn implies that E is not removable for L_{loc}^p \mathcal{L} -solutions. This finishes the proof of (2). The proof of (4) is much simpler, again using Frostman's lemma and integrating on annuli. \square

In the following remarks we assume that \mathcal{L} is a λ -homogeneous left invariant partial differential operator on \mathbb{G} such that \mathcal{L} and \mathcal{L}^t are hypoelliptic.

Remarks 5.8. (1) Analogous results hold when L_{loc}^p is replaced with the Sobolev space $W_{H,\text{loc}}^{k,p}$ consisting of L_{loc}^p functions whose iterated horizontal partial distributional derivatives of order at most k exist as functions in L_{loc}^p . Such results extend to Carnot groups the analogous Euclidean results due to Harvey and Polking [21, Theorem 4.3]. For instance, *any compact set E with $\mathcal{H}^{Q-(\lambda-k)p'}(E) < \infty$ is removable for $W_{H,\text{loc}}^{k,p}$ \mathcal{L} -solutions*, provided $1 < p < \infty$. Moreover, *any compact E with $\mathcal{H}^{Q-\lambda+k}(E) = 0$ is removable for $W_{H,\text{loc}}^{k,\infty}$ \mathcal{L} -solutions*. Finally, *if E is compact with $\mathcal{H}^{Q-\lambda+k}(E) < \infty$, $\Omega \supset E$ is a domain and $f \in W_{H,\text{loc}}^{k,\infty}(\Omega)$ such that $\mathcal{L}f = 0$ in $\Omega \setminus E$, then $\mathcal{L}f$ is a measure supported in E* . The proofs are easy variants on the preceding argument; one may retain k derivatives on the function f and transfer only $\lambda - k$ derivatives onto the Harvey–Polking cutoff functions φ_ϵ .

(2) Removability for continuous \mathcal{L} -solutions can also be treated as in [21, Theorem 4.2(b)]. One obtains the conclusion that *compact sets E with $\mathcal{H}^{Q-\lambda}$ are removable for continuous \mathcal{L} -solutions*. The rough idea of the proof is as follows. Returning to the proof of part (3) in Theorem 5.3, note that $(\mathcal{L}\varphi_\epsilon)$ subconverges weakly to zero in $C(\Omega)'$. Multiplication by a continuous function f is a continuous operation on $C(\Omega)'$, so also (D_ϵ) (defined as in (5.7)) subconverges weakly to zero. Since also (D_ϵ) converges weakly to $\mathcal{L}f$ we conclude that $\mathcal{L}f = 0$ as desired.

(3) As in [21, Theorem 4.3(d) and Theorem 4.5] one also obtains results for the spaces C_H^k and $C_H^{k,\delta}$ consisting, respectively, of functions on \mathbb{G} whose iterated horizontal partial derivatives of order at most k are continuous, or of functions in C_H^k whose iterated horizontal partial derivatives of order equal to k are δ -Hölder continuous. In the former case, one has that *compact sets E with $\mathcal{H}^{Q-\lambda+k}(E) < \infty$ are removable for C_H^k \mathcal{L} -solutions*. In the latter case, for $0 < \delta < 1$, one has that *compact sets E with $\mathcal{H}^{Q-\lambda+k+\delta}(E) = 0$ are removable for $C_H^{k,\delta}$ \mathcal{L} -solutions*.

(4) As in the Euclidean setting, the analysis of removability for Lipschitz \mathcal{L} -solutions is more subtle. We will pursue this topic in more detail in a subsequent work [11].

We conclude with a few questions for further study.

Remark 5.9. Removability for PDE solutions, as considered in this paper, is traditionally quantified in terms of an appropriate notion of capacity. For Painlevé’s problem, the relevant capacity is the well known *analytic capacity*. In [22], Harvey and Polking develop a capacity framework which encodes their removability results from [21]. We have not developed the capacity theory associated to our removability theorems, although such a theory should be straightforward to implement.

Remark 5.10. Moving even beyond the sub-Riemannian framework, it is natural to inquire about similar removability questions in the general setting of metric measure spaces. Second-order PDE in divergence form can be developed using a weak formulation and the notion of Cheeger differentiability in doubling metric measure spaces admitting a Poincaré inequality. See [10] for Cheeger’s differentiation theory, and [5] for the basic machinery of second-order PDE in such spaces. We anticipate that some of our results carry over into this framework.

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