THE STRONG GEOMETRIC LEMMA IN THE HEISENBERG GROUP

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ABSTRACT. We prove that in the first Heisenberg group, unlike Euclidean spaces and higher dimensional Heisenberg groups, the best possible exponent for the strong geometric lemma for intrinsic Lipschitz graphs is 4 instead of 2. Combined with earlier work from [CLY22b] and [CLY22a], our result completes the proof of the strong geometric lemma in Heisenberg groups. One key tool in our proof, and possibly of independent interest, is a suitable refinement of the foliated coronizations which first appeared in [NY22].

1. INTRODUCTION

According to Rademacher's theorem, Lipschitz surfaces in $\mathbb{R}^n$ infinitesimally resemble planes. However, for some more global questions, like the boundedness of singular integrals, the information we get from Rademacher's theorem is too qualitative. To answer these questions, we need to know that Lipschitz graphs can be effectively approximated by affine planes "at most places and scales." The traditional way to quantify such a statement is via the notion of β–numbers, introduced by Jones in [Jon89, Jon90]. Since Jones's work, β–numbers and several variants have played an important role in geometric harmonic analysis and geometric measure theory in $\mathbb{R}^n$ as well as in other spaces [Oki92, Sch07, LS16a, LS16b, FFP07, CLZ19, Li19, DS17, CLY22b, CLY22a].

Given $E \subset \mathbb{R}^n$ and a ball $B(x,r)$, the m-dimensional β–number is defined as

$$\beta_E(x,r) = \inf_L r^{-m} \int_{B(x,r)\cap E} \frac{d(y,L)}{r} \, \mathrm{d}\mathscr{H}^m(y)$$

where L is taken over all m–dimensional planes and $\mathscr{H}^m$ denotes the m–dimensional Hausdorff measure. Thus, $\beta_E(x,r)$ is an measures how flat the is the set $E \cap B(x,r)$ on average. This function is scale-invariant in the sense that $\beta_{tE}(tx,tr) = \beta_E(x,r)$ for all $t > 0$. By a fundamental result of Dorronsoro [Dor85], if E is an m–dimensional Lipschitz graph, then the squares of the β–numbers satisfy the following Carleson packing condition: there is a $c > 0$ depending only on the Lipschitz constant such that

$$\int_0^R \int_{E\cap B(x,R)} \beta_E(y,r)^2 \, \mathrm{d}\mathscr{H}^m(y) \frac{dr}{r} \le cR^m, \quad \text{for } R>0, x \in E. \tag{1}$$

This is known as the *strong geometric lemma* with exponent 2.

The strong geometric lemma for Lipschitz graphs has been one of the cornerstones in the theory of uniform rectifiability, which was originally developed by David and Semmes in the early 90s. In particular, an Ahlfors m–regular subset of $\mathbb{R}^n$ is called uniformly m–rectifiable if and only if it satisfies (1) for some $c > 0$. David and Semmes in

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[DS91, DS93] introduced an extensive suite of geometric and analytic notions in order to provide various characterizations of uniformly rectifiable sets in Euclidean spaces. They thus provided a rich and influential geometric foundation for the study of singular integral operators on lower dimensional subsets of $\mathbb{R}^n$.

In the last 20 years there have been systematic efforts toward the development of geometric measure theory on sub-Riemannian spaces, see e.g. the lecture notes [SC16, Mat23] for some recent surveys. In particular, this research program has grown substantially on Carnot groups, a class of nilpotent Lie groups that admit dilations, i.e., maps that scale the metric by a constant. Indeed, by a result of LeDonne [LD15], Carnot groups are the *only* locally compact geodesic spaces which admit dilations and are isometrically homogeneous. This makes these groups ideal environments to study geometric and analytic questions involving many different scales. The simplest examples of nonabelian Carnot groups are the $(2n+1)$–dimensional Heisenberg groups $\mathbb{H}_n$.

To introduce meaningful notions of rectifiability in $\mathbb{H}_n$ (or any other Carnot group), one should first come up with the analogue of a Lipschitz graph in that setting. This is rather subtle because, for example, Heisenberg groups cannot be viewed as Cartesian products of subgroups. One might consider Lipschitz images $\mathbb{R}^k \to \mathbb{H}_n$, but this approach only works when $k \le n$. Ambrosio and Kirchheim [AK00] proved that any Lipschitz image $f(\mathbb{R}^k) \subset \mathbb{H}_n$ has vanishing $\mathscr{H}^k$–measure for $k > n$. Therefore one should introduce a notion of *intrinsic* graphs fitting the sub-Riemannian group structure. This was achieved by Franchi, Serapioni, and Serra Cassano in [FSSC06]. Similarly to Euclidean Lipschitz graphs, intrinsic Lipschitz graphs in $\mathbb{H}_n$ satisfy a cone condition, which will be defined in the next section.

Intrinsic Lipschitz graphs feature prominently in the newly emerging theories of low-codimensional rectifiability [MSSC10, FSSC11, Vit22, SC16]. One reason for this is that, like Lipschitz graphs in $\mathbb{R}^n$, intrinsic Lipschitz graphs in $\mathbb{H}_n$ satisfy a version of Rademacher's theorem; they infinitesimally resemble planes almost everywhere [FSSC11]. As in $\mathbb{R}^n$, answering questions about singular integrals and uniform rectifiability in $\mathbb{H}_n$ requires more quantitative bounds, which has led to the study of various notions of quantitative rectifiability in intrinsic Lipschitz graphs [CFO19b, NY18, NY22, FOR18, Rig19, CLY22b, CLY22a].

Broadly speaking, the aforementioned works seek to obtain quantitative bounds on how intrinsic Lipschitz graphs can be approximated by vertical planes and vertical sets. As in the Euclidean case, these works are partly motivated by a singular integral operator which can be viewed as the natural analogue of the the (Euclidean) 1–codimensional Riesz transform. Notably, and as in $\mathbb{R}^n$, this new singular integral is related to removability for Lipschitz harmonic functions in $\mathbb{H}_n$, see [CFO19a, FO21, CLY22a].

One of the key objects in the study of uniform rectifiability in Heisenberg groups is the following codimension–1 version of the β–numbers. For $E \subset \mathbb{H}_n$, $x \in \mathbb{H}_n$, and $r > 0$, we define

$$\beta_E(x,r) = \inf_{L \in \mathsf{VP}} r^{-2n-1} \int_{B(x,r)\cap E} \frac{d(y,L)}{r} \, \mathrm{d}\mathscr{H}^{2n+1}(y) \tag{2}$$

where VP denotes the set of codimension–1 planes which are parallel to the z-axis. Note that the quantities $\beta_E(x,r)$ are scale-invariant since $\beta_{\delta_t(E)}(\delta_t(x), tr) = \beta_E(x,r)$ for any $t > 0$. We record that an L_∞ version of the codimension–1 β–numbers in the Heisenberg group was introduced in [CFO19b], where a weaker qualitative (Heisenberg) analogue of (1), known as the weak geometric lemma, was obtained for intrinsic Lipschitz graphs in $\mathbb{H}_n$.

In [CLY22b] we established the direct analogue of the strong geometric lemma with exponent 2 for intrinsic Lipschitz graphs in $\mathbb{H}_n$ for $n \geq 2$.

Theorem 1.1. [CLY22b] *Let $n \geq 2$ and let Γ be an intrinsic L–Lipschitz graph in $\mathbb{H}_n$. Then, for any $y \in \Gamma$ and any $R > 0$,*

$$\int_0^R \int_{B(y,R)\cap\Gamma} \beta_\Gamma(x,r)^2 \,\mathrm{d}\mathcal{H}^{2n+1}(x)\,\frac{\mathrm{d}r}{r} \lesssim_L R^{2n+1}. \tag{3}$$

However, in [CLY22a] we showed that unlike the Euclidean case, the strong geometric lemma fails in $\mathbb{H}_1$ for all exponents $s \in [2,4)$.

Theorem 1.2. [CLY22a] *There exist a constant $L > 0$, a radius $R > 0$, and a sequence of L–intrinsic Lipschitz graphs $(\Gamma_n)_{n\in\mathbb{N}}$ in $\mathbb{H}_1$ such that $\mathbf{0} \in \Gamma_n$ for all n and*

$$\lim_{n\to\infty} \int_0^R \int_{B(0,R)\cap\Gamma_n} \beta_{\Gamma_n}(x,r)^s \,\mathrm{d}\mathcal{H}^3(x)\,\frac{\mathrm{d}r}{r} = +\infty \tag{4}$$

for all $s \in [2,4)$.

In this paper we complete this line of research by showing that the strong geometric lemma holds in $\mathbb{H}_1$ with exponent 4. Our main theorem is the following:

Theorem 1.3. *Let Γ be an intrinsic L–Lipschitz graph in $\mathbb{H}_1$. Then for any $y \in \Gamma$ and any $R > 0$,*

$$\int_{B(y,R)\cap\Gamma} \int_0^R \beta_\Gamma(x,r)^4 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\mathcal{H}^3(x) \lesssim_L R^3. \tag{5}$$

We prove Theorem 1.3 by using the foliated corona decompositions constructed in [NY22]. These decompose an intrinsic Lipschitz graph into rectangular regions called pseudoquads, whose aspect ratio depends on the shape of the corresponding intrinsic Lipschitz graph. At points and scales where the graph is flat, the pseudoquads are short and wide (large aspect ratio), and at points where the graph is bumpy, the pseudoquads are tall and skinny (small aspect ratio).

The decomposition satisfies a weighted Carleson condition, which bounds the total size of the pseudoquads in the decomposition, weighted by the inverse fourth power of the aspect ratio (see Definition 2.9). This parallels the construction of bumpy surfaces in [NY22] and [CLY22a], where Theorem 1.2 was proved by constructing surfaces with bumps with large aspect ratio. Each bump increases the area of the surface in proportion to the inverse fourth power of its aspect ratio, which leads to the exponent 4 in Theorem 1.2. The proof of Theorem 1.3 relates the geometry of a foliated corona decomposition to the β–numbers of the associated graph and uses the weighted Carleson condition to prove the inequality (5).

One can generalize some of these questions by defining analogues of β–numbers for other subsets and other Carnot groups. For instance, quantitative rectifiability of 1–dimensional subsets of Carnot groups has been studied in [Li19, CLZ19], where relative beta numbers where introduced in order to study the travelling salesman problem and its connections to singular integrals. Little is known, however, about quantitative rectifiability for higher-dimensional subsets, even surfaces in $\mathbb{H}_n$ with topological dimension between 2 and $2n-1$.

Similarly, though one can define intrinsic Lipschitz graphs in arbitrary Carnot groups (see [SC16, Section 4.5]), and these objects play an important role in rectifiability, the quantitative rectifiability of these graphs has not been systematically studied. For instance, given $p \in \mathbb{N}$, is there a Carnot group G such that intrinsic Lipschitz graphs in G satisfy an analogue of (5) only for exponents greater than or equal to p?

1.1. **Outline of paper.** In Section 2, we give some basic definitions and theorems that we will use in the rest of the paper. We also recap the results of [NY22] on foliated corona decompositions. In Section 3, we prepare to prove Theorem 1.3 by reducing (5) to an inequality involving the intrinsic Lipschitz function ψ that parametrizes Γ.

In order to use foliated corona decompositions to prove Theorem 1.3, we must improve some of the bounds from [NY22]. To achieve this, we first (in Section 4) introduce foliated coronizations and paramonotone-stopped foliated coronizations (PSFCs), which are decompositions with some convenient additional bounds. Then, in Section 5, we prove a bound on the L_4 distance between a pseudoquad in an intrinsic Lipschitz graph and a plane. This improves the corresponding L_1 bound in [NY22]. Finally, in Section 6, we prove Theorem 1.3.

2. Preliminaries

2.1. **The Heisenberg group.** The Heisenberg group $\mathbb{H} := \mathbb{H}_1$ is the nonabelian Lie group whose elements are points in $\mathbb{R}^3$ and whose group operation is given by

$$(x, y, z)(x', y', z') = \left(x + x', y + y', z + z' + \frac{xy' - x'y}{2}\right). \tag{6}$$

The identity element is $\mathbf{0} := (0,0,0)$ and the inverse of $v = (x, y, z)$ is the element $v^{-1} = (-x,-y,-z)$. Let $X = (1,0,0), Y = (0,1,0), Z = (0,0,1) \in \mathbb{H}$ and let $x, y, z \colon \mathbb{H} \to \mathbb{R}$ be the coordinate functions. Given any $p \in \mathbb{H}$, $p \neq 0$, we will denote by $\langle p\rangle$ the one-parameter subgroup containing p; in these coordinates, $\langle p\rangle$ is the subspace spanned by p. This lets us write $w^t = tw$ for $w \in \mathbb{H}$ and $t \in \mathbb{R}$; when $t \in \mathbb{Z}$, this agrees with the usual notion of exponentiation.

The center of the group is $\langle Z\rangle = \{(0,0,z) \mid z \in \mathbb{R}\}$. An element $p \in \mathbb{H}$ such that $z(p) = 0$ is called a *horizontal vector*, and we denote by A the set of horizontal vectors. Let $\pi \colon \mathbb{H} \to \mathbb{R}^2$ be the projection $\pi(x, y, z) = (x, y)$.

The *Korányi metric* on $\mathbb{H}$ is the left-invariant metric defined by

$$d_{\mathrm{Kor}}(p, p') := \|p^{-1}p'\|_{\mathrm{Kor}},$$

where

$$\|(x, y, z)\|_{\mathrm{Kor}} := \sqrt[4]{(x^2 + y^2)^2 + 16z^2}.$$

We note that the family of automorphisms: $s_t \colon \mathbb{H} \to \mathbb{H}$, $t \in \mathbb{R}$,

$$s_t(x, y, z) = (tx, ty, t^2 z),$$

dilate the Korányi metric metric; for $t \geq 0$ and $p, q \in \mathbb{H}$,

$$d_{\mathrm{Kor}}(s_t(p), s_t(q)) = t d_{\mathrm{Kor}}(p, q).$$

Let $I \subset \mathbb{R}$ be an open interval. The map $\gamma \colon I \to \mathbb{H}$ is a *horizontal curve* if $x \circ \gamma, y \circ \gamma, z \circ \gamma \colon I \to \mathbb{R}$ are Lipschitz (and thus γ' is defined almost everywhere on I) and

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\gamma(t)^{-1}\gamma(s)\right]\Big|_{s=t} \in A,$$

for almost every $t \in I$. Cosets of the form $L = v\langle aX + bY\rangle$ where $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and $v \in \mathbb{H}$, are called *horizontal lines*. The *slope* of a horizontal line L is the slope of its projection $\pi(L)$, i.e., if $L = v\langle aX + bY\rangle$, then $\operatorname{slope} L = \frac{b}{a}$ when $a \neq 0$ and $\operatorname{slope} L = \infty$ when $a = 0$.

A plane parallel to the z–axis is called a *vertical plane*. Note that if V is a vertical plane, the projection $\pi(V)$ is a line and we define the slope of V as $\operatorname{slope}(V) := \operatorname{slope}(\pi(V))$. We will frequently use the vertical plane $V_0 := \{y = 0\}$.

2.2. **Intrinsic Lipschitz graphs and characteristic curves.** Intrinsic graphs and intrinsic Lipschitz graphs are classes of surfaces in $\mathbb{H}$ that play an important role in the study of rectifiability. Similarly to [CLY22a] we will define intrinsic graphs in terms of functions from $\mathbb{H}$ to $\mathbb{R}$ that are constant along cosets of $\langle Y\rangle$. In particular, if $\phi\colon \mathbb{H} \to \mathbb{R}$ is constant on cosets of $\langle Y\rangle$, the *intrinsic graph* of ϕ is the set

$$\Gamma_\phi = \{vY^{\phi(v)} \mid v \in V_0\} = \{p \in \mathbb{H} \mid \phi(p) = y(p)\}.$$

(Many authors call these *entire intrinsic graphs* and use "intrinsic graph" to refer to closed subsets of Γ_ϕ.)

Note that left-translations and dilations of intrinsic graphs are also intrinsic graphs. We parametrize Γ_ϕ by the map $\Psi_\phi\colon V_0 \to \Gamma_\phi$,

$$\Psi_\phi(p) = pY^{\phi(p)-y(p)}.$$

This map projects V_0 to Γ_ϕ along cosets of $\langle Y\rangle$, and if ϕ is continuous, it is a homeomorphism from V_0 to Γ_ϕ.

Conversely, we let $\Pi\colon \mathbb{H} \to V_0$ be the (nonlinear) projection along cosets of $\langle Y\rangle$ given by $\Pi(v) = vY^{-y(v)}$, $v \in \mathbb{H}$. Equivalently,

$$\Pi(x, y, z) = \left(x, 0, z - \frac{1}{2}xy\right).$$

Although Π is not a homomorphism, it commutes with the dilations s_t:

$$\Pi(s_t(v)) = s_t(\Pi(v))$$

for all $v \in \mathbb{H}$ and $t \in \mathbb{R}$.

For $0 < L < 1$, the open double cone is defined as

$$\operatorname{Cone}_L = \{p \in \mathbb{H} \mid d_{\mathrm{Kor}}(\mathbf{0}, p) < L^{-1}|y(p)|\}.$$

This is a dilation-invariant set, and when L is close to 1, it is a small neighborhood of $\langle Y\rangle \setminus \{\mathbf{0}\}$. An *$L$–intrinsic Lipschitz graph* is an intrinsic graph Γ_ϕ such that $p\operatorname{Cone}_L \cap \Gamma_\phi = \emptyset$ for all $p \in \Gamma_\phi$. Equivalently, Γ_ϕ is L–intrinsic Lipschitz if and only if $\operatorname{Lip}(y|_{\Gamma_\phi}) \leq L$. We say that a function $\phi\colon \mathbb{H} \to \mathbb{R}$ is is an *L–intrinsic Lipschitz function* if it is constant on cosets of $\langle Y\rangle$ and Γ_ϕ is an L–intrinsic Lipschitz graph. (Again, some authors refer to Γ_ϕ as an *entire intrinsic Lipschitz graph*. By Theorem 27 of [NY18], any subset of $\mathbb{H}$ that satisfies the cone condition above can be extended to an entire intrinsic Lipschitz graph with the same intrinsic Lipschitz constant.)

If $A \subset V_0$ is a Borel set and $f\colon A \to \mathbb{R}$ is a Borel function, $|A|$ will denote the Lebesgue measure of A and $\int_A f(x)\,\mathrm{d}x$ will denote the integral with respect to the Lebesgue measure. We will denote by $\mathscr{H}^3$ the 3–dimensional Hausdorff measure on $\mathbb{H}$ taken with respect to the Korányi metric. The following lemma is well known, see e.g. [FS16] and [CLY22a, Lemma 2.8].

Lemma 2.1. *Let $0 < L < 1$ and Γ be a L–intrinsic Lipschitz graph and let $A \subset \Gamma$ be a measurable subset. Then $|\Pi(A)| \approx_L \mathscr{H}^3(A)$.*

The following lemma will be used frequently in our proofs.

Lemma 2.2 ([NY22, Lemma 2.3])**.** *Let $0 < L < 1$ and let $\Gamma = \Gamma_\psi$ be an L–intrinsic Lipschitz graph of a function $\psi\colon U \to \mathbb{R}$. Then for all $v, w \in U$,*

$$|\psi(v) - \psi(w)| \le \frac{2}{1-L} d_{\mathrm{Kor}}(\Psi_\psi(v), w\langle Y\rangle). \tag{7}$$

In particular, for any $v \in U$ and any $s \in \mathbb{R}$ such that $vZ^s \in U$,

$$|\psi(v) - \psi(vZ^s)| \le \frac{4}{1-L}\sqrt{|s|}. \tag{8}$$

Given a function $\psi\colon \mathbb{H} \to \mathbb{R}$ which is constant on cosets of $\langle Y\rangle$ and a smooth function $f\colon V_0 \to \mathbb{R}$, we define

$$\partial_\psi f = \frac{\partial f}{\partial x} - \psi\frac{\partial f}{\partial z}. \tag{9}$$

The differential operator ∂_ψ defines a continuous vector field on V_0 whose x–coordinate is 1, so the Peano existence theorem implies that there is at least one flow line of ∂_ψ through every point of V_0. Such a flow line can be parametrized by $t \mapsto (t, 0, g(t))$, where $g\colon I \to \mathbb{R}$ (where I is an interval) satisfies

$$g'(t) + \psi(t, 0, g(t)) = 0, \quad \text{for all } t \in I. \tag{10}$$

These flow lines are called *characteristic curves of* Γ_ψ, and we say that g has a *characteristic graph.* When ψ is intrinsic Lipschitz, the characteristic curves of Γ_ψ are exactly the Π–projections of horizontal curves $\gamma\colon I \to \Gamma_\psi$ which satisfy $x(\gamma(t)) = t$ for all $t \in I$, see [NY22, Lemma 2.6]. Moreover, if ψ is smooth then the characteristic curves of Γ_ψ foliate V_0. However, this is not the case if ψ is not smooth as characteristic curves could branch and rejoin, see e.g. [BCSC15].

The following lemma provides bounds on characteristic curves for intrinsic Lipschitz graphs.

Lemma 2.3 ([NY22, Lemma 2.7])**.** *Let $0 < L < 1$ and Γ be an L–intrinsic Lipschitz graph. If $\gamma\colon I \to V_0$ is a characteristic curve of Γ parameterized such that $x(\gamma(t)) = t$, then letting $g(t) = z(\gamma(t))$ we have:*

$$|g(t) - g(s) - g'(s)\cdot(t-s)| \le \frac{L}{\sqrt{1-L^2}}\frac{(t-s)^2}{2}, \qquad \forall s, t \in I. \tag{11}$$

2.3. **Foliated corona decompositions.** Foliated corona decompositions were recently introduced in [NY22] to analyze the structure of intrinsic Lipschitz graphs in $\mathbb{H}$. These decompositions partition an intrinsic Lipschitz graph Γ_f into rectangular regions, called pseudoquads, on which f is close to an affine function. In this section, we summarize the results from [NY22] that we will need in this paper.

Let Γ be an intrinsic Lipschitz graph. A *pseudoquad* Q is a region of V_0 bounded by two vertical lines and two characteristic curves of Γ, i.e., a region of the form

$$\{(x,z) \in V_0 \mid x \in I, g_1(x) \le z \le g_2(x)\},$$

where $I = [a,b] \subset \mathbb{R}$ is a closed interval and the g_i are functions with characteristic graphs. We say that I is the *base* of Q and we call g_1 and g_2 the *lower* and *upper bounds* of Q.

Let V be a vertical plane. The horizontal curves of V are parallel lines, and their projections to V_0 are parallel parabolas. We call the pseudoquads of V *parabolic rectangles* for V; they are bounded by two parallel parabolas and two vertical lines. If

$$R = \{(x,z) \in V_0 \mid x \in I, h_1(x) \le z \le h_2(x)\}$$

is a parabolic rectangle, we define the *width* of R to be $\delta_x(R) = \ell(I)$ and the *height* to be $\delta_z(R) = h_2 - h_1$; since the graphs of h_1 and h_2 are parallel parabolas, these functions differ by a constant. The *slope* of R, denoted $\operatorname{slope}(R)$, is $\operatorname{slope}(V)$; note that, by (10) we have

$$h_i''(x) = -\operatorname{slope}(R) \tag{12}$$

for all i and x.

For $r > 0$ and an interval $I = [a, b]$, let rI be the scaling of I around its center by a factor of r, i.e.,

$$rI = \left[\frac{a+b}{2} - \frac{r\ell(I)}{2}, \frac{a+b}{2} + \frac{r\ell(I)}{2}\right].$$

For any $\rho > 0$, let

$$\begin{aligned}\rho R &= \left\{(x,z) \in V_0 : x \in \rho I, z \in \rho^2[h_1(x), h_2(x)]\right\}\\ &= \left\{(x,z) \in V_0 : x \in \rho I, \left|z - \frac{h_1(x)+h_2(x)}{2}\right| \le \frac{\rho^2\delta_z(R)}{2}\right\};\end{aligned}$$

this is the concentric parabolic rectangle which has width $\delta_x(\rho R) = \rho\delta_x(R)$ and height $\delta_z(\rho R) = \rho^2\delta_z(R)$.

For $0 < \mu \le \frac{1}{32}$, a *μ–rectilinear pseudoquad* is a tuple (Q, R), where Q is a pseudoquad and R is a parabolic rectangle with the same base I such that, if g_1 and g_2 (resp. h_1 and h_2) are the lower and upper bounds of Q (resp. R), then

$$\|g_i - h_i\|_{L_\infty(4I)} \le \mu\delta_z(R) \tag{13}$$

for $i = 1, 2$. We often omit R from the notation, referring to (Q, R) as simply Q. We define $\operatorname{slope} Q = \operatorname{slope} R$, $\delta_x(Q) = \delta_x(R)$, $\delta_z(Q) = \delta_z(R)$, and $\rho Q = \rho R$. Note that $1Q = R$, so $1Q \ne Q$ in general. When $\mu = \frac{1}{32}$, we simply say that Q is a rectilinear pseudoquad.

For any rectilinear pseudoquad Q, let $\alpha(Q) = \frac{\delta_x(Q)}{\sqrt{\delta_z(Q)}}$ be its *aspect ratio*. We use a square root here because the distance in the Heisenberg metric between the top and bottom of Q is proportional to $\sqrt{\delta_z(Q)}$; this aspect ratio is scale-invariant.

By the following lemma from [NY22], any pseudoquad that is sufficiently tall and skinny is rectilinear.

Lemma 2.4 ([NY22, Lemma 5.3])**.** *Let $\mu > 0$, let $0 < L < 1$, and let $\Gamma = \Gamma_f$ be an intrinsic L–Lipschitz graph. There is an $A > 0$ with the following property. Let Q be a pseudoquad for Γ and let $r = \delta_x(Q)$. Suppose that there is a point v in the lower boundary of Q and a point vZ^s in the upper boundary such that $\frac{r}{\sqrt{s}} \le A$. Then there is a parabolic rectangle R such that (Q, R) is μ–rectilinear.*

A foliated corona decomposition of a graph Γ is based on a collection of μ–rectilinear foliated patchworks. These are similar to dyadic partitions or cubical patchworks in that they consist of a hierarchy of partitions of Γ into pseudoquads, but an important difference is that different pieces can have different aspect ratios.

Definition 2.5. Let Q be a μ–rectilinear pseudoquad. A *μ–rectilinear foliated patchwork* for Q is a complete rooted binary tree (Δ, v_0) such that every v in its vertex set $\mathcal{V}(\Delta)$ is associated to a μ–rectilinear pseudoquad $(Q_v, R_v) \subset V_0$, $Q_{v_0} = Q$, and each vertex $v \in \mathcal{V}(\Delta)$ is either *vertically cut* or *horizontally cut* in the following sense.

Let w and w' be the children of v, let $I = [a, b]$ be the base of Q_v, and let g_1 and g_2 (resp. h_1 and h_2) be the lower and upper bounds of Q_v (resp. R_v).

(1) If v is vertically cut, then Q_w and $Q_{w'}$ are the left and right halves of Q_v, separated by the vertical line $x = \frac{a+b}{2}$. That is,

$$Q_w = \left\{(x,z) \in V_0 \mid a \le x \le \frac{a+b}{2}, g_1(x) \le z \le g_2(x)\right\}$$

$$Q_{w'} = \left\{(x,z) \in V_0 \mid \frac{a+b}{2} \le x \le b, g_1(x) \le z \le g_2(x)\right\}.$$

Similarly, $R_w = ([a, \frac{a+b}{2}] \times \mathbb{R}) \cap R_v$, $R_{w'} = ([\frac{a+b}{2}, b] \times \mathbb{R}) \cap R_v$. Then $\delta_x(Q_w) = \delta_x(Q_{w'}) = \frac{\delta_x(Q_v)}{2}$ and $\delta_z(Q_w) = \delta_z(Q_{w'}) = \delta_z(Q_v)$.

(2) If v is horizontally cut, then Q_w and $Q_{w'}$ are the top and bottom halves of Q_v, separated by a characteristic curve. That is, there is a function $c\colon \mathbb{R} \to \mathbb{R}$ with characteristic graph, a quadratic function $k\colon \mathbb{R} \to \mathbb{R}$, and a $d > 0$ such that

$$Q_w = \{(x,z) \in V_0 \mid a \le x \le b, g_1(x) \le z \le c(x)\}$$

$$Q_{w'} = \{(x,z) \in V_0 \mid a \le x \le b, c(x) \le z \le g_2(x)\}$$

$$R_w = \{(x,z) \in V_0 \mid a \le x \le b, k(x) - d \le z \le k(x)\}$$

$$R_{w'} = \{(x,z) \in V_0 \mid a \le x \le b, k(x) \le z \le k(x) + d\}.$$

Then $\delta_x(Q_w) = \delta_x(Q_{w'}) = \delta_x(Q_v)$ and $\delta_z(Q_w) = \delta_z(Q_{w'}) = d$. Furthermore, by the μ–rectilinearity of Q_w and $Q_{w'}$,

$$\max\left\{\|(k-d) - g_1\|_{L_\infty(4I)}, \|k - c\|_{L_\infty(4I)}, \|(k+d) - g_2\|_{L_\infty(4I)}\right\} \le \mu d. \tag{14}$$

In either case, $Q_v = Q_w \cup Q_{w'}$ and the two halves Q_w and $Q_{w'}$ have disjoint interiors. Let $\mathcal{V}_{\mathsf{v}}(\Delta) \subset \mathcal{V}(\Delta)$ be the set of vertically cut vertices and let $\mathcal{V}_{\mathsf{h}}(\Delta) \subset \mathcal{V}(\Delta)$ be the set of horizontally cut vertices.

As with pseudoquads, when μ is omitted, we take $\mu = \frac{1}{32}$, so any rectilinear foliated patchwork is at least $\frac{1}{32}$–rectilinear.

Lemma 2.6. *Let Q be a rectilinear pseudoquad. Then $\frac{2}{3}Q \subseteq Q \subseteq 2Q$.*

Proof. The upper bound is Lemma 4.1 of [NY22].

For the proof of the lower bound, let R be the parabolic rectangle corresponding to Q, let $g_1, g_2\colon \mathbb{R} \to \mathbb{R}$, $h_1, h_2\colon \mathbb{R} \to \mathbb{R}$, and $k_1, k_2\colon \mathbb{R} \to \mathbb{R}$ be functions whose graphs are the lower and upper bounds of Q, R, and $\frac{2}{3}Q$, respectively, and let I be the base of Q. Then h_i and k_i are quadratics that all differ by additive constants and $|g_i(x) - h_i(x)| \le \frac{1}{32}\delta_z(Q)$ for each $x \in I$ and $i \in \{1,2\}$.

We have that $k_1 - h_1 = \frac{5}{18}\delta_z(Q) = h_2 - k_2$. Thus, for each $x \in I$, we get

$$k_1(x) - g_1(x) \ge k_1(x) - h_1(x) - |g_1(x) - h_1(x)| > 0.$$

Likewise, $k_2(x) < g_2(x)$ for all $x \in I$ and so $\frac{2}{3}Q \subseteq Q$. □

For any rooted tree (T, v_0) and any $v \in \mathcal{V}(T)$, let $\mathscr{C}(v) = \mathscr{C}^1(v)$ denote the set of *children* of v and let

$$\mathscr{C}^n(v) = \bigcup_{w \in \mathscr{C}^{n-1}(v)} \mathscr{C}(w)$$

be the set of nth generation descendants, letting $\mathscr{C}^0(v) = \{v\}$. Let

$$\mathscr{C}^{\le n}(v) = \bigcup_{i=0}^{n} \mathscr{C}^i(v)$$

and let $\mathscr{D}(v)$ be the set of all descendants of v, including v itself. We equip $\mathcal{V}(T)$ with the usual partial order, so that $v \le w$ if v is a descendant of w.

We record the following bounds; see [NY22, Lemma 4.5]

Lemma 2.7. *Let $\epsilon > 0$. There is a $\mu > 0$ such that if Q is a μ–rectilinear pseudoquad, then*

$$1-\epsilon \le \frac{\delta_x(Q)\cdot\delta_z(Q)}{|Q|} \le 1+\epsilon. \tag{15}$$

Further, if Q is horizontally or vertically cut as in Definition 2.5 and Q' is a child of Q, then

$$\frac{1}{2}-\epsilon \le \frac{|Q'|}{|Q|} < \frac{1}{2}+\epsilon.$$

If Q is horizontally cut, then $\delta_z(Q') = \frac{\delta_z(Q)}{2}$ and $\alpha(Q') = \sqrt{2}\alpha(Q)$. If Q is vertically cut, then $\delta_x(Q') = \frac{\delta_x(Q)}{2}$ and

$$\frac{1}{2}-\epsilon \le \frac{\alpha(Q')}{\alpha(Q)} \le \frac{1}{2}+\epsilon.$$

In particular, when Q is a rectilinear pseudoquad (i.e., $\mu = \frac{1}{32}$), then these inequalities hold for $\epsilon = \frac{1}{4}$.

We produce rectilinear foliated patchworks by repeatedly cutting a rectilinear pseudoquad into smaller pseudoquads. Any rectilinear pseudoquad Q can be cut vertically into two μ–rectilinear pseudoquads of the same height and half the width, but not every pseudoquad can be cut horizontally. Cutting along a characteristic curve through the center of Q might not produce two rectilinear pseudoquads, since the upper and lower bounds of the new pseudoquads need not satisfy (13).

The main technical result of [NY22] shows that if a rectilinear pseudoquad is *paramonotone*, then it can be cut horizontally into two rectilinear pseudoquads. Paramonotonicity is a condition based on the monotonicity introduced in [CK10]. For any intrinsic Lipschitz graph Γ and any $R > 0$, there is a measure $\Omega^P_{\Gamma^+,R}$ on V_0 called the R–extended parametric normalized nonmonotonicity. The full definition of $\Omega^P_{\Gamma^+,R}$ can be found in Section 8 of [NY22], but for $U \subset V_0$, $\Omega^P_{\Gamma^+,R}(U)$ measures the horizontal lines L such that L intersects Γ multiple times in an R–neighborhood of $\Pi^{-1}(U)$.

This satisfies the kinematic formula

$$\sum_{i=-\infty}^{\infty} \Omega^P_{\Gamma^+,2^{-i}}(U) \lesssim_L |U| \tag{16}$$

for any measurable $U \subset V_0$ [NY22, Lem. 9.2]. Furthermore, by inspection of the definition [NY22, (155)],

$$\Omega^P_{\Gamma^+,R} \le \frac{R'}{R}\Omega^P_{\Gamma^+,R'} \tag{17}$$

for all $R < R'$.

Let Q be a pseudoquad of Γ. We say that Γ is (η,R)*–paramonotone* on rQ if the density of $\Omega^P_{\Gamma^+,R}$ on rQ satisfies

$$\frac{\Omega^P_{\Gamma^+,R\delta_x(Q)}(rQ)}{|Q|} \le \eta\alpha(Q)^{-4}. \tag{18}$$

This condition is invariant under scalings, stretch maps, and shear maps. The results of [NY22] show that paramonotone pseudoquads satisfy strong bounds on their characteristic curves.

Proposition 2.8 ([NY22, Prop. 7.2])**.** *There is a universal constant $r > 10$ such that for any $\lambda > 0$ and $0 < \zeta \le \frac{1}{32}$, there are $\eta, R > 0$ such that if $\Gamma = \Gamma_f$ is the intrinsic Lipschitz graph of $f\colon V_0 \to \mathbb{R}$, and if Q is a rectilinear pseudoquad for Γ such that Γ is (η, R)–paramonotone on rQ, then:*

(1) *There is a vertical plane $P \subset \mathbb{H}$ (a λ–approximating plane) and an affine function $F\colon V_0 \to \mathbb{R}$ such that P is the intrinsic graph of F and*

$$\frac{\|F - f\|_{L_1(10Q)}}{|Q|} \le \lambda \frac{\delta_z(Q)}{\delta_x(Q)}. \tag{19}$$

(2) *Let $u \in 4Q$ and let $g_\Gamma, g_P\colon \mathbb{R} \to \mathbb{R}$ be such that $\{z = g_\Gamma(x)\}$ (respectively $\{z = g_P(x)\}$) is a characteristic curve for Γ (respectively P) that passes through u. Then*

$$\|g_P - g_\Gamma\|_{L_\infty(4I)} \le \zeta \delta_z(Q).$$

We can thus produce a rectilinear foliated patchwork for a pseudoquad Q by inductively cutting Q into smaller and smaller pseudoquads. If a pseudoquad is paramonotone, we cut it horizontally. If not, we cut it vertically. We then repeat the process on the two new pseudoquads.

The resulting decomposition then satisfies certain bounds, which we describe below.

Definition 2.9. For $S \subset \mathcal{V}(\Delta)$, let $W(S) = \sum_{w \in S} \alpha(Q_w)^{-4}|Q_w|$; we call this the *weight* of S. We say that a rectilinear foliated patchwork Δ satisfies a *weighted Carleson packing condition* or that Δ is *C–weighted-Carleson* if every $v \in \mathcal{V}(\Delta)$ satisfies

$$W(\{w \in \mathcal{V}_{\mathsf{v}}(\Delta) \mid w \le v\}) \le C|Q_v|. \tag{20}$$

For any rectilinear pseudoquad R, $|R| \approx \delta_x(R)\delta_z(R)$, so

$$W(\{R\}) = \alpha(R)^{-4}|R| \approx \left(\frac{\delta_x(R)}{\sqrt{\delta_z(R)}}\right)^{-4} |R| \approx \frac{\delta_z(R)^3}{\delta_x(R)^3}. \tag{21}$$

Definition 2.10. A set of *λ–approximating planes* for Δ is a collection of vertical planes P_v with corresponding vertical affine functions $l_v\colon V_0 \to \mathbb{R}$, one for each $v \in \mathcal{V}_{\mathsf{h}}(\Delta)$, such that

$$|Q_v|^{-1}\|l_v - f\|_{L_1(10Q_v)} \le \lambda \frac{\delta_z(Q_v)}{\delta_x(Q_v)}. \tag{22}$$

Definition 2.11. Let $\mu_0 > 0$ and let $D\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$. We say that an intrinsic Lipschitz graph Γ admits a (D, μ_0)*–foliated corona decomposition* if for every $0 < \mu \le \mu_0$, every $\lambda > 0$ and every μ–rectilinear pseudoquad $Q_0 \subset V_0$, there is a μ–rectilinear foliated patchwork Δ for Q_0 such that Δ is $D(\mu, \lambda)$–weighted-Carleson and has a set of λ–approximating planes. When D and μ_0 are not important, we simply say that Γ admits a foliated corona decomposition.

Theorem 2.12 ([NY22, Thm. 7.5])**.** *Any intrinsic Lipschitz graph admits a foliated corona decomposition.*

That is, let $r > 10$ be as in Proposition 2.8. For every $0 < L < 1$ and for every $0 < \mu \le \frac{1}{32}$ and $\lambda > 0$, there are $D = D(L, \mu, \lambda)$, $\eta = \eta(\mu, \lambda)$, and $R = R(\mu, \lambda)$, all positive, with the following properties. Suppose that $\Gamma \subset \mathbb{H}$ is an intrinsic L–Lipschitz graph and $Q \subset V_0$ is a μ–rectilinear pseudoquad for Γ. Then there is a μ–rectilinear foliated patchwork Δ for Q such that Δ is D–weighted-Carleson and has a set of λ–approximating planes. Moreover, for all vertices $v \in \mathcal{V}(\Delta)$, the associated pseudoquad Q_v is horizontally cut if and only if Γ is (η, R)–paramonotone on rQ.

Furthermore, η and R satisfy Proposition 2.8 for $\zeta = \frac{1}{32r^2}$.

(The condition that η and R satisfy Proposition 2.8 is not part of the statement of Theorem 7.5 of [NY22], but in the proof, η and R are chosen to satisfy Proposition 2.8 for a suitable ζ and λ.)

3. Initial reductions

In this section, we reduce Theorem 1.3 to a bound on a parametric L_4 version of β_Γ on a pseudoquad for Γ. We first define L_p β–numbers. For $E \subset \mathbb{H}, x \in E, r > 0$, and $p \in [1,\infty)$, let

$$\beta_{p,E}(x,r) := \inf_{L\in\mathsf{VP}} \left[r^{-3} \int_{B(x,r)\cap E} \left(\frac{d(y,L)}{r}\right)^p \, \mathrm{d}\mathcal{H}^3(y)\right]^{1/p}. \tag{23}$$

Recalling (2), we note that $\beta_E = \beta_{1,E}$. Using Hölder's inequality one easily sees that if $1 \le p < q$ then

$$\beta_{p,E}(x,r) \le \left(\frac{\mathcal{H}^3(E\cap B(x,r))}{r^3}\right)^{1/p-1/q} \beta_{q,E}(x,r). \tag{24}$$

In particular, if E is Ahlfors 3–regular, then $\beta_{p,E}(x,r) \lesssim \beta_{q,E}(x,r)$. Consequently, (4) holds when $\beta_{1,\Gamma}$ is replaced by $\beta_{p,\Gamma}$ for $p > 1$.

For any measurable function $\psi\colon \mathbb{H} \to \mathbb{R}$ that is constant on cosets of $\langle Y\rangle$ and any $p \ge 1$ we define a parametric version of β by

$$\gamma_{p,\psi}(v,r) = r^{\frac{-3-p}{p}} \inf_{h\in\mathsf{Aff}} \|\psi - h\|_{L_p(V(\Psi_\psi(v),r))} \tag{25}$$

where

$$V(v,r) := \Pi(B(v,r)),$$

and Aff denotes all functions $h : \mathbb{H} \to \mathbb{R}$ of the form $h(v) = ax(v)+b$, for $a, b \in \mathbb{R}$. Observe that every vertical plane with finite slope is a graph of a function in Aff. Note also that $\gamma_{p,\psi}(\cdot,r)$ is constant on cosets of Y.

The sets $V(v,r)$ are shaped like parallelograms in V_0, with slope depending on $y(v)$.

Lemma 3.1. *For any $r > 0$ and $p \in \mathbb{H}$, let $(x_0,0,z_0) = \Pi(p)$. Then*

$$V(p,r) \subset \{(x,0,z) \mid |x - x_0| \le r, |z - z_0 + y(p)(x - x_0)| \le r^2\}. \tag{26}$$

Proof. We first consider the case $p = \mathbf{0}$. If $q \in B(\mathbf{0},r)$, then $\Pi(q) = qY^{-y(q)} \in B(\mathbf{0},2r)\cap V_0$. In particular, $|z(\Pi(q))| \le r^2$. Since $x(\Pi(q)) = x(q) \in [-r,r]$, this implies $V(\mathbf{0},r) \subset R_r$, where $R_r = [-r,r] \times 0 \times [-r^2,r^2]$.

Now consider an arbitrary $p \in \mathbb{H}$. Then $\Pi(pq) = pqY^{-y(p)-y(q)} = \Pi(p\Pi(q))$ for any $q \in \mathbb{H}$, so

$$V(p,r) = \Pi(B(p,r)) = \Pi(pB(\mathbf{0},r)) = \Pi(pV(\mathbf{0},r)) \subset \Pi(pR_r).$$

We write $p = (x_0,0,z_0)Y^{y(p)}$; then

$$V(p,r) \subset \Pi(pR_r) = (x_0,0,z_0)Y^{y(p)}R_rY^{-y(p)}.$$

By (6), $Y^{y(p)}(x,0,z)Y^{-y(p)} = (x,0,z - xy(p))$, so this implies (26). □

When ψ is intrinsic Lipschitz, $V(x,r)$ and $\Pi(B(x,r)\cap\Gamma_\psi)$ are comparable, and β_{Γ_ψ} and γ_ψ are comparable.

Lemma 3.2 ([CLY22a, Lemma 2.7])**.** *Let ψ be a L-intrinsic Lipschitz function, let $p \in \Gamma_\psi$, and let $r > 0$. There is a $c > 1$ depending on L such that*

$$V(p, c^{-1}r) \subseteq \Pi(B(p,r) \cap \Gamma_\psi) \subset V(p,r). \tag{27}$$

In particular, if $x, y \in V_0$ and $y \in V(\Psi_\psi(x), r)$, then $x \in V(\Psi_\psi(y), cr)$.

Lemma 3.3. *Let $0 < L < 1$ and $p \geq 1$. There is a $C := C(L) > 1$ such that for any L–intrinsic Lipschitz function $\psi \colon \mathbb{H} \to \mathbb{R}$, any $x \in \mathbb{H}$, and any $r > 0$,*

$$\gamma_{p,\psi}(x, r/C) \lesssim_{L,p} \beta_{p,\Gamma_\psi}(\Psi_\psi(x), r) \lesssim_{L,p} \gamma_{p,\psi}(x, Cr). \tag{28}$$

For $p = 1$, the previous lemma was essentially proved in [CLY22a, Lemma 4.2]. For $p > 1$ the proof is very similar and we omit it.

Then Theorem 1.3 is a special case of the following theorem.

Theorem 3.4. *Let Γ be an intrinsic L–Lipschitz graph in $\mathbb{H}$ and let $p \in [1,4]$. Then for any $y \in \Gamma$ and any $R > 0$,*

$$\int_{B(y,R)\cap\Gamma} \int_0^R \beta_{p,\Gamma}(x,r)^4 \, \frac{\mathrm{d}r}{r} \, \mathrm{d}\mathcal{H}^3(x) \lesssim_L R^3. \tag{29}$$

We have not considered if the range $[1,4]$ is sharp, and most likely it is not. By Dorronsoro's work [Dor85] the strong geometric lemma for m–dimensional Lipschitz graphs (or Lipschitz functions on $\mathbb{R}^m$) holds for β_p with $p < \infty$ in $\mathbb{R}^2$ and for $p < \frac{2m}{m-2}$ in $\mathbb{R}^n$, $n > 2$. Unpublished examples of Fang and Jones show that these ranges are sharp. Recently, Fässler and Orponen [FO20] extended Dorronsoro's techniques to Lipschitz functions from $\mathbb{H}_n$ to $\mathbb{R}$. Their approach suggests that (29) might hold for $1 \leq p < 6$, but we will not pursue this here.[1]

In the rest of this section, we will show that Theorem 3.4 is a consequence of the following bound.

Proposition 3.5. *Let $0 < L < 1$. There are $\tau > 0$ and $c > 0$ such that if $\Gamma = \Gamma_f$ is an L–intrinsic Lipschitz graph and $Q \subseteq V_0$ be a rectilinear pseudoquad with $\alpha(Q) \leq c$, then*

$$\int_{\frac{1}{3}Q} \int_0^{\tau\delta_x(Q)} \gamma_{4,f}(x,r)^4 \, \frac{\mathrm{d}r}{r} \, \mathrm{d}x \lesssim_L |Q|. \tag{30}$$

To show that Proposition 3.5 implies Theorem 3.4, we will need the following lemma.

Lemma 3.6. *Let $0 < \kappa < 1$ and $0 < L < 1$. There is a $c > 1$ depending on κ and L such that for any L–intrinsic Lipschitz graph $\Gamma = \Gamma_\psi$, any $x \in \Gamma$, and any $r > 0$, there is a rectilinear pseudoquad Q such that $V(x,r) \subset \kappa Q$, $\delta_x(Q) = 2\kappa^{-1}r$, and $\delta_z(Q) \leq cr^2$.*

Proof. After a left-translation, we may suppose that $x = \mathbf{0}$. Let $a = a(L) \in (0,1)$ be a small number to be chosen later and let $s = 2\kappa^{-2}a^{-2}r^2$. Let $g_1, g_2 \colon \mathbb{R} \to \mathbb{R}$ be functions with characteristic graphs such that $g_1(0) = -s$ and $g_2(0) = s$, let

$$Q = \{(x,0,z) : |x| \leq \kappa^{-1}r \text{ and } z \in [g_1(x), g_2(x)]\},$$

and let $R = [-\kappa^{-1}r, \kappa^{-1}r] \times [-s,s]$ so that $\alpha(R) = \frac{2\kappa^{-1}r}{\sqrt{2s}} = a$. We claim that (Q,R) is rectilinear and that $V(x,r) \subset Q$.

[1] In [CLY22b], we proved Theorem 1.1 for $\beta_{2,\Gamma}$, so (3) holds when $\beta_{1,\Gamma}$ is replaced by $\beta_{p,\Gamma}$ with $p \in [1,2]$. Likewise, this range is not sharp and for the same reasons it is likely that (3) holds for $\beta_{p,\Gamma}$ with $1 \leq p < \frac{2(2n+1)}{(2n+1)-2} = \frac{4n+2}{2n-1}$.

By (10) and Lemma 2.2,

$$|g_i'(0)| = |\psi(0,0,g_i(0))| \le \frac{4}{1-L}\sqrt{s} \le \frac{8r\kappa^{-1}a^{-1}}{1-L}.$$

By Lemma 2.3, for $t \in [-4\kappa^{-1}r, 4\kappa^{-1}r]$,

$$\frac{|g_i(t)-g_i(0)|}{s} \le \frac{1}{2\kappa^{-2}a^{-2}r^2}\left(|g_i'(0)|4\kappa^{-1}r + \frac{16L\kappa^{-2}}{\sqrt{1-L^2}}r^2\right) \le \frac{16a}{1-L} + \frac{8La^2}{\sqrt{1-L^2}}.$$

We choose $a(L) \in (0,1)$ small enough that

$$|g_i(t)-g_i(0)| \le \frac{s}{16} = \frac{1}{32}\delta_z(R),$$

so (Q,R) is rectilinear. By Lemma 3.1, $V(x,r) \subset [-r,r]\times[-r^2,r^2]$, so

$$V(x,r) \subset [-r,r]\times[-2a^{-2}r^2, -2a^{-2}r^2] = \kappa Q,$$

and $\delta_z(Q) = 4\kappa^{-2}a^{-2}r^2 \lesssim_{\kappa,L} r^2$. □

This lets us prove Theorem 3.4, assuming Proposition 3.5.

Proof of Theorem 3.4. Let $\Gamma = \Gamma_f$ be a L-intrinsic Lipschitz graph, $y \in \Gamma$, and $R > 0$. We first prove that

$$\int_{V(y,R)}\int_0^R \gamma_{4,f}(x,r)^4 \frac{\mathrm{d}r}{r}\,\mathrm{d}x \lesssim_L R^3. \tag{31}$$

Let $c, \tau > 0$ be the constants from Proposition 3.5. By Lemma 3.6, there is a rectilinear pseudoquad Q with $\delta_x(Q) \lesssim_L R$ and $\delta_z(Q) \lesssim_L R^2$ such that $V(y,R) \subset \frac{1}{3}Q$ and $\tau\delta_x(Q) \ge R$. Then

$$\begin{aligned}\int_{V(y,R)}\int_0^R \gamma_{4,f}(x,r)^4 \frac{\mathrm{d}r}{r}\,\mathrm{d}x &\le \int_{\frac{1}{3}Q}\int_0^{\tau\delta_x(Q)} \gamma_{4,f}(x,r)^4\frac{\mathrm{d}r}{r}\,\mathrm{d}x \\ &\overset{(30)}{\lesssim_L} |Q| \approx_L R^3.\end{aligned}$$

Let $C = C(L) > 1$ be the constant from Lemma 3.3. Lemma 2.1 tells us that $(\Psi_f)_*(|\cdot|) \approx_L \mathcal{H}^3$ on Γ, so

$$\begin{aligned}&\int_{B(y,R)\cap\Gamma}\int_0^R \beta_{4,\Gamma}(x,r)^4\frac{\mathrm{d}r}{r}\,\mathrm{d}\mathcal{H}^3(x)\\ &\quad\lesssim_L \int_{\Pi(B(y,R)\cap\Gamma)}\int_0^R \beta_{4,\Gamma}(\Psi_f(x),r)^4\frac{\mathrm{d}r}{r}\,\mathrm{d}x\\ &\quad\overset{(28)}{\lesssim_L} \int_{V(y,R)}\int_0^R \gamma_{4,f}(x,Cr)^4\frac{\mathrm{d}r}{r}\,\mathrm{d}x\\ &\quad\le \int_{V(y,CR)}\int_0^{CR}\gamma_{4,f}(x,r)^4\frac{\mathrm{d}r}{r}\,\mathrm{d}x\\ &\quad\overset{(31)}{\lesssim_L} R^3.\end{aligned}$$

By (24), this implies the theorem for $p \in [1,4]$.

□

4. Foliated coronizations

In this section, we will use foliated corona decompositions to define *foliated coronizations*, which are rectilinear foliated patchworks with some improved properties that make them easier to use for the arguments in this paper.

Definition 4.1. Let $\lambda > 0$, let $r > 10$ be the universal constant in Proposition 2.8, and let $0 < \mu \le \frac{1}{32r^2}$. If Δ is a μ–rectilinear foliated patchwork with horizontally cut root that is D_0–weighted-Carleson and has a set of λ–approximating planes, we call Δ a *foliated coronization*. When the constants are important, we will write that Δ is a (D_0, μ, λ)–foliated coronization. Furthermore, if there are $\eta, R > 0$ that satisfy Proposition 2.8 for $\zeta = \frac{1}{32r^2}$ and such that for all $v \in \mathcal{V}(\Delta)$, Q_v is horizontally cut if and only if Γ is (η, R)–paramonotone on rQ_v, then we say that Δ is (η, R)*–paramonotone stopped*. We will abbreviate paramonotone stopped foliated coronization as PSFC.

We can construct such coronizations using the following lemma.

Lemma 4.2. *Let $0 < L < 1$ and let $\eta, r, R, \mu > 0$. There is an $\alpha_{\min} \in (0,1)$ such that for any L–intrinsic Lipschitz graph Γ, if Q is a rectilinear pseudoquad for Γ and $\alpha(Q) \le \alpha_{\min}$, then Q is μ–rectilinear and Γ is (η, R)–paramonotone on rQ.*

Proof. Let A be as in Lemma 2.4, so that if $\alpha(Q) \le \frac{A}{2}$, then Q is μ–rectilinear.

Let $i > 0$ be such that $2^{i-1} \le R\delta_x(Q) < 2^i$. By (16) and (17), there is a $b > 0$ such that

$$\Omega^P_{\Gamma^+, R\delta_x(Q)}(rQ) \le 2\Omega^P_{\Gamma^+, 2^i}(rQ) \le b|rQ|.$$

Let

$$\alpha_{\min} = \min\{\frac{A}{2}, (4br^3\eta^{-1})^{-\frac{1}{4}}\}.$$

By Lemma 2.7, $|rQ| \le 2r^3|Q|$, so if $\alpha(Q) > \alpha_{\min}$, then

$$\Omega^P_{\Gamma^+, R\delta_x(Q)}(rQ) \le 2br^3|Q| \le \eta\alpha(Q)^{-4}|Q|,$$

and Γ is (η, R)–paramonotone on rQ. □

Combining Lemma 4.2 and Theorem 2.12 yields the following.

Lemma 4.3. *Let $0 < L < 1$. There is an $\alpha_{\min} \in (0,1)$ such that for any intrinsic L–Lipschitz graph Γ, and any rectilinear pseudoquad $Q \subset V_0$ with $\alpha(Q) \le \alpha_{\min}$, there is a PSFC with root Q.*

Specifically, let r be as in Proposition 2.8. Let $\lambda > 0$, and let $0 < \mu \le \frac{1}{32r^2}$. Let $D = D(L, \mu, \lambda)$, $\eta = \eta(\mu, \lambda)$, and $R = R(\mu, \lambda)$ be as in Theorem 2.12. Let $\alpha_{\min} = \alpha_{\min}(\eta, \mu, r, R, L)$ be as in Lemma 4.2. Then η and R satisfy Proposition 2.8 for $\zeta = \frac{1}{32r^2}$. Furthermore, for any intrinsic L–Lipschitz graph Γ, and any rectilinear pseudoquad $Q \subset V_0$ with $\alpha(Q) \le \alpha_{\min}$, there is a (D, μ, λ)–foliated coronization Δ of Q which is (η, R, r)–paramonotone stopped.

Proof. Let $Q \subset V_0$ be as above. By Lemma 4.2, Γ is (η, R)–paramonotone on rQ, and by Theorem 2.12, there is a μ–rectilinear foliated patchwork Δ for Q that is D–weighted-Carleson, has a set of λ–approximating planes, and is (η, R, r)–paramonotone stopped. In particular, Q is horizontally cut, so Δ is a PSFC as desired. □

These decompositions satisfy several nice properties. First, the pseudoquads of a PSFC have aspect ratios that are bounded below.

Lemma 4.4. *Let $0 < L < 1$, let Γ be an intrinsic L–Lipschitz graph Γ, and let $Q \subset V_0$ be a pseudoquad for Γ. Let Δ be a PSFC for Q and let $\alpha_{\min}$ be as in Lemma 4.2. Then:*

- *for all* $v \in \mathcal{V}(\Delta)$, $\alpha(Q_v) \geq \min\{\alpha(Q), \frac{\alpha_{\min}}{4}\}$,
- *there are only finitely many* $v \in \mathcal{V}(\Delta)$ *such that* $\alpha(Q_v) < \frac{\alpha_{\min}}{4}$, *and*
- *if* $\delta_z(Q_v) \leq \frac{16\delta_x(Q)^2}{\alpha_{\min}^2}$, *then* $\alpha(Q_v) \geq \frac{\alpha_{\min}}{4}$.

Proof. Let v_0 be the root of Δ.

Suppose that $v \neq v_0$ and $\alpha(Q_v) < \frac{\alpha_{\min}}{4}$. Let p be the parent of v. By Lemma 2.7, $\alpha(Q_v) < \alpha_{\min}$. Therefore, by Lemma 4.2, Γ is (η, R)–paramonotone on rQ_p and Q_p is horizontally cut. Therefore, $\alpha(Q_p) < \alpha(Q_v) < \frac{\alpha_{\min}}{4}$. The same argument thus holds for Q_p. By induction, every ancestor a of v is horizontally cut and $\alpha(Q_a) < \frac{\alpha_{\min}}{4}$. Consequently, $\alpha(Q_v) \geq \alpha(Q)$. This proves the first part of the lemma.

Furthermore, if $\alpha(Q_v) < \frac{\alpha_{\min}}{4}$, then $\delta_x(Q_v) = \delta_x(Q)$ and thus

$$\delta_z(Q_v) = \frac{\delta_x(Q_v)^2}{\alpha(Q_v)^2} > \frac{16\delta_x(Q)^2}{\alpha_{\min}^2}.$$

There are only finitely many such pseudoquads in Δ, so this proves the rest of the lemma. □

Second, expansions of the pseudoquads of Δ are nested.

Lemma 4.5 ([NY22, Lemma 4.7])**.** *Let* Δ *be a PSFC and suppose* $v, w \in \mathcal{V}(\Delta)$ *satisfy* $w \leq v$. *Let* $0 < r' \leq r$. *Then* $r'Q_w \subset r'Q_v$.

(This result is proved for sufficiently small μ in [NY22], but inspection of the proof shows that $\mu \leq \frac{1}{32r^2}$ is enough.)

Third, every pseudoquad has an approximating plane that satisfies Proposition 2.8.

Lemma 4.6. *Let* Δ *be a PSFC as above. For* $w \in \mathcal{V}(\Delta)$, *let* m *be the minimal horizontally-cut ancestor of* w, *where* $m = w$ *if* $w \in \mathcal{V}_{\mathsf{h}}(\Delta)$. *Since the root of* Δ *is horizontally-cut, such an ancestor exists. Let* $l_w := l_m$ *and* $P_m := P_m$, *where* l_m *and* P_m *are as in Definition 2.10, and let* I *be the base of* Q_w. *Then* Q_w *and* P_w *satisfy Proposition 2.8, i.e.,*

$$\|l_w - f\|_{L_1(10Q_w)} \lesssim \lambda \frac{\delta_z(Q_w)}{\delta_x(Q_w)} |Q_w|, \tag{32}$$

and for any $u \in 4Q$, *if* $g_\Gamma, g_{P_w} : \mathbb{R} \to \mathbb{R}$ *are functions such that* $\{z = g_\Gamma(x)\}$ *(respectively* $\{z = g_{P_w}(x)\}$*) are characteristic curves for* Γ *(respectively* P_w*) that pass through* u, *then*

$$\|g_{P_w} - g_\Gamma\|_{L_\infty(4I)} \leq \frac{1}{32r^2} \delta_z(Q). \tag{33}$$

Proof. For $w \in \mathcal{V}_{\mathsf{h}}(\Delta)$, the lemma follows from Proposition 2.8. We thus consider $w \in \mathcal{V}_{\mathsf{v}}$. Since Q_m is rectilinear, $\delta_z(Q_w) \approx \delta_z(Q_m)$. By (22),

$$\begin{aligned}\|l_w - f\|_{L_1(10Q_w)} \leq \|l_m - f\|_{L_1(10Q_m)} &\leq \lambda \frac{\delta_z(Q_m)}{\delta_x(Q_m)} |Q_m| \\ &\approx \lambda \delta_z(Q_m)^2 \approx \lambda \delta_z(Q_w)^2 \approx \lambda \frac{\delta_z(Q_w)}{\delta_x(Q_w)} |Q_w|.\end{aligned}$$

Suppose that $u \in 4Q_w$ and that $\{z = g_\Gamma(x)\}$ and $\{z = g_{P_w}(x)\}$ are characteristic curves through u. By Lemma 4.5, $u \in 4Q_m$, so (33) follows from Proposition 2.8 applied to Q_m. □

5. L_p APPROXIMATION BY PIECEWISE AFFINE FUNCTIONS

Now we prove L_p bounds on the pseudoquads of a foliated coronization for Γ_f. By Lemma 4.6, these pseudoquads have λ–approximating planes, but these planes only approximate f in L_1. In this section, we will a family of piecewise affine functions g_S that approximate f and we will bound $\|f - g_S\|_p$ for $1 \le p < 5$.

We will prove the following proposition.

Proposition 5.1. *Let $\Gamma = \Gamma_f$ be an L–intrinsic Lipschitz graph. Let Q be a pseudoquad of Γ and let Δ be a PSFC with root Q. Let $l_v, v \in \mathcal{V}(\Delta)$ be the approximating planes for Δ. There is a $c > 0$ depending on L and the parameters of Δ such that for every $v \in \mathcal{V}_{\mathsf{h}}(\Delta)$ and $1 \le p < 5$,*

$$\|l_v - f\|_{L_p(Q_v)} \le \frac{c}{5-p} \lambda \frac{\delta_z(Q_v)}{\delta_x(Q_v)} |Q_v|^{\frac{1}{p}}. \tag{34}$$

We prove Proposition 5.1 by approximating f by piecewise-affine functions.

We will need a few definitions. Let Δ be a rooted tree. A *coherent* set $S \subset \mathcal{V}(\Delta)$ is a subset with the following properties:

1. S has a unique maximal element $M = \max(S) \in S$.
2. If $v \in S$ and $w \in \mathcal{V}(\Delta)$ satisfies $v < w < M$, then $w \in S$.
3. If $v \in S$, then either all of the children of v are contained in S or none of them are.

A *partition* of a subset $U \subset V_0$ into pseudoquads is a finite collection $Q_1, \dots, Q_k$ of pseudoquads such that $\bigcup Q_i = U$ and such that the interiors of the Q_i's are pairwise disjoint. A coherent subset of a foliated patchwork corresponds to a partition of a set into pseudoquads. The following lemma is Lemma 6.3 of [NY22].

Lemma 5.2. *Let Δ be a rectilinear foliated patchwork and let $S \subset \mathcal{V}(\Delta)$ be coherent. Let $M = \max S$ be the maximal element of S and let $\min S$ be the set of minimal elements of S. Let*

$$F_1 = F_1(S) = \{p \in Q_M \mid \text{there are infinitely many } v \in S \text{ such that } p \in Q_v\}$$

and let $F_2 = F_2(S) = Q_M \setminus F_1$. Then

$$Q_M = F_1 \cup \bigcup_{w \in \min S} Q_w, \tag{35}$$

and the interiors of the Q_w's are pairwise disjoint and disjoint from F_1. If S is finite, then $\min S$ is a partition of Q_M.

We use these partitions to define approximations of f.

Definition 5.3 (piecewise-affine approximations). Let Δ be a paramonotone stopped foliated coronization (a PSFC) for Γ_f. By Lemma 4.6, there is an associated collection of affine functions $l_w, w \in \mathcal{V}(\Delta)$. For any coherent set $S \subset \mathcal{V}(\Delta)$, we define a function $g_S \in L_\infty(Q_{\max(S)})$ as follows. Let F_1, F_2 be as in Lemma 5.2 and define g_S so that $g_S|_{F_1} = f|_{F_1}$ and $g_S|_{Q_w} = l_w|_{Q_w}$ for all $w \in \min(S)$; this is well-defined away from the boundaries of the Q_w's.

We claim that as T gets larger and larger, g_T converges to f in L_p. We first prove convergence in L_1.

Lemma 5.4. *Let Δ be a PSFC for a pseudoquad Q. Let $v \in \mathcal{V}(\Delta)$ and let $S \subset \mathcal{V}(\Delta)$ be a coherent subset with* $\max S = v$. *Let $T_1, T_2, \dots \subset S$ be a sequence of coherent sets with* $\max T_i = v$ *such that $T_1 \subset T_2 \subset \dots$ and $\bigcup T_i = S$. Then*

$$\lim_{i\to\infty} \|g_{T_i} - g_S\|_{L_1(Q_v)} = 0.$$

Proof. Let T be a coherent set with $u = \max T$. We first show that

$$\|g_T - f\|_{L_1(Q_u)} \lesssim |Q_u|^{\frac{4}{3}}, \tag{36}$$

with implicit constant depending on the parameters of Δ. Let $C = \min\left\{\frac{\alpha_{\min}}{4}, \alpha(Q_{v_0})\right\}$ so that $\alpha(Q_w) \ge C$ for all $w \in \mathcal{V}(\Delta)$ by Lemma 4.4. Since Q_w is rectilinear, we have $|Q_w| \approx \delta_z(Q_w)\delta_x(Q_w)$, so

$$\delta_z(Q_w) \approx \alpha(Q_w)^{-\frac{2}{3}}|Q_w|^{\frac{2}{3}} \le C^{-\frac{2}{3}}|Q_w|^{\frac{2}{3}}$$

and

$$\delta_x(Q_w) \approx \alpha(Q_w)^{\frac{2}{3}}|Q_w|^{\frac{1}{3}} \ge C^{\frac{2}{3}}|Q_w|^{\frac{1}{3}}.$$

Let l_w be the λ–approximating planes for Δ. Then, by Lemma 4.6,

$$\|l_w - f\|_{L_1(Q_w)} \lesssim \lambda \frac{\delta_z(Q_w)}{\delta_x(Q_w)}|Q_w| \lesssim \lambda C^{-\frac{4}{3}}|Q_w|^{\frac{4}{3}}.$$

By Lemma 5.2,

$$\|g_T - f\|_{L_1(Q_u)} = \sum_{m\in\min T} \|g_T - f\|_{L_1(Q_m)} \lesssim \sum_{m\in\min T} \lambda C^{-\frac{4}{3}}|Q_u|^{\frac{1}{3}}|Q_m| \le \lambda C^{-\frac{4}{3}}|Q_u|^{\frac{1}{3}}|Q_u|,$$

which implies (36).

Now we consider g_S and g_{T_i}. Let $F_1(S)$ and $F_1(T_i)$ be as in Lemma 5.2 for S and T_i; note that $F_1(T) \subset F_1(S)$, so $g_{T_i} = g_S = f$ on $F_1(T)$. Let $M_i = \min T_i$. If $u \in M_i \cap \min S$, then $g_S = g_{T_i} = l_u$ on $\operatorname{int} Q_u$. Let $N_i = M_i \setminus \min S$. Then for each $u \in N_i$, the intersections $T_i \cap \mathcal{D}(u)$ and $S \cap \mathcal{D}(u)$ are coherent sets containing u, and

$$\|g_{T_i} - g_S\|_{L_1(Q_v)} = \sum_{u\in N_i} \|g_{T_i} - g_S\|_{L_1(Q_u)} = \sum_{u\in N_i} \|g_{T_i\cap\mathcal{D}(u)} - g_{S\cap\mathcal{D}(u)}\|_{L_1(Q_u)}.$$

Let $a_i = \max_{u\in N_i}|Q_u|$. By (36),

$$\|g_{T_i} - g_S\|_{L_1(Q_v)} \lesssim a_i^{\frac{1}{3}} \sum_{u\in N_i} |Q_u| \le a_i^{\frac{1}{3}}|Q_v|,$$

so it suffices to show that $\lim_i a_i = 0$.

Let $\epsilon > 0$. By Lemma 2.7, for any $\epsilon > 0$, there are only finitely many $w \in S$ such that $|Q_w| > \epsilon$. Suppose that i is large enough that T_i contains every such w. Then any $u \in N_i$ has a child $u' \in S$ with $|Q_{u'}| \le \epsilon$. By Lemma 2.7, $|Q_u| \le 4\epsilon$, so $a_i \le 4\epsilon$. Letting ϵ go to zero, we find that $\lim_i a_i = 0$ and thus $\lim_i \|g_{T_i} - g_S\|_{L_1(Q_v)} = 0$. □

We thus consider how g_S changes when we enlarge S. First, we consider adding the children of a vertex $w \in \min(S)$ to S. This corresponds to cutting a pseudoquad in the corresponding partition into two pieces.

Lemma 5.5. *Let S be a coherent set and let $w \in \min(S)$. Then $S' = S \cup \mathcal{C}(w)$ is coherent,* $\operatorname{supp}(g_S - g_{S'}) \subset Q_w$, *and*

$$\|g_S - g_{S'}\|_{L_\infty(Q_w)} \lesssim \lambda \frac{\delta_z(Q_u)}{\delta_x(Q_u)}.$$

Proof. It follows from the definitions that g_S and $g_{S'}$ agree outside of Q_w. Let u be a child of w. Then $\delta_x(Q_u) \approx \delta_x(Q_w)$, $\delta_z(Q_u) \approx \delta_z(Q_u)$, and $|Q_u| \approx |Q_w|$, so

$$\|l_w - l_u\|_{L_1(10Q_u)} \le \|l_w - f\|_{L_1(10Q_u)} + \|f - l_u\|_{L_1(10Q_u)} \lesssim \lambda \frac{\delta_z(Q_w)}{\delta_x(Q_w)} |Q_w|.$$

Since $Q_w \subset 10Q_u$ and since l_w and l_u are affine functions,

$$\|l_w - l_u\|_{L_\infty(Q_w)} \le \|l_w - l_u\|_{L_\infty(10Q_u)} \lesssim \lambda \frac{\delta_z(Q_w)}{\delta_x(Q_w)}. \tag{37}$$

Let u and u' be the children of w. Then

$$\|g_S - g_{S'}\|_{L_\infty(Q_w)} \le \max\{\|l_w - l_u\|_{L_\infty(Q_w)}, \|l_w - l_{u'}\|_{L_\infty(Q_w)}\} \lesssim \lambda \frac{\delta_z(Q_w)}{\delta_x(Q_w)},$$

as desired. □

Next, we consider adding coherent subsets to S. For a horizontally-cut vertex $w \in \mathcal{V}_{\mathsf{h}}(\Delta)$ and a descendant $v \in \mathcal{D}(w)$, we say that v is an *h–descendant* of w if every vertex on the path from w to v, except possibly v itself, is horizontally cut. We let the *h–subtree* $D^h(w) \subset \mathcal{D}(w)$ to be the set of h–descendants of w. Note that $D^h(w)$ is coherent, and it corresponds to a partition of Q_w into a stack of pseudoquads, all with the same width as Q_w.

Lemma 5.6. *Let $S \subset \mathcal{V}(\Delta)$ be a coherent set and let $w \in \min(S) \cap \mathcal{V}_{\mathsf{h}}(\Delta)$. Let $S' = S \cup D^h(w)$. Then* $\operatorname{supp}(g_S - g_{S'}) \subset Q_w$ *and*

$$\|g_{S'} - g_S\|_{L_\infty(Q_w)} \lesssim \lambda \frac{\delta_z(Q_w)}{\delta_x(Q_w)}.$$

Proof. It follows from the definitions that g_S and $g_{S'}$ agree outside of Q_w. For $i \ge 0$, let $D^h_i(w) = D^h(w) \cap \mathcal{C}^i(w)$, $D^h_{\le i}(w) = D^h(w) \cap \mathcal{C}^{\le i}(w)$, and $h_i = g_{S \cup D^h_i(w)}$. Then $h_0 = g_S$. By Lemma 5.4, h_i converges to g_S pointwise almost everywhere.

Every pseudoquad in $D^h(w)$ has width $\delta_x(Q_w)$, and by Lemma 2.7, for any $i \ge 0$ and $v \in D^h_i(w)$,

$$\delta_z(Q_v) \le \left(\frac{3}{4}\right)^i \delta_z(Q_w).$$

By Lemma 5.5,

$$\|h_i - h_{i+1}\|_{L_\infty(Q)} \lesssim \lambda \left(\frac{3}{4}\right)^i \frac{\delta_z(Q_w)}{\delta_x(Q_w)}.$$

Thus

$$\|g_S - g_{S'}\|_{L_\infty(Q)} \le \lim_i \|h_i - h_0\|_{L_\infty(Q)} \lesssim \sum_{i=0}^{\infty} \lambda \left(\frac{3}{4}\right)^i \frac{\delta_z(Q_w)}{\delta_x(Q_w)}.$$

□

We will use these bounds let us construct a sequence of approximations of f that converge in L_p. For $v \in \mathcal{V}(\Delta)$ and $j \ge 0$, let

$$R_j(v) = \left\{ w \in \mathcal{D}(v) \mid \delta_x(Q_w) \ge 2^{-j} \delta_x(Q_v) \right\}. \tag{38}$$

By Lemma 5.4, the $g_{R_j(v)}$ converge to f in $L_1(Q_v)$; we will show that they converge in $L_p(Q_v)$ too.

We will need to estimate the weight of the descendants of v. Let $P_j(v) = \min R_j(v)$. Then

$$P_j(v) = \left\{ w \in \mathcal{D}_{\mathsf{v}}(v) \mid \delta_x(Q_w) = 2^{-j} \delta_x(Q_v) \right\}.$$

The following Vitali-type covering lemma is proved in [NY22]. (In [NY22], it is stated for $\frac{1}{32r^2}$–rectilinear foliated patchworks, but all PSFCs are $\frac{1}{32r^2}$–rectilinear.)

Lemma 5.7 ([NY22, Lem. 9.4])**.** *Let Δ be a PSFC. For any $j \geq 0$, there is a subset $V_j(v) \subset P_j(v)$ such that the sets $rQ_w, w \in V_j(v)$ are pairwise disjoint, and $W(V_j(v)) \approx W(Q_j(v))$. It follows that*

$$W(\mathscr{D}_{\mathsf{v}}(v)) \approx_r \sum_j W(V_j(v)). \tag{39}$$

This lets us bound $W(P_i(v))$ for $v \in \mathcal{V}_{\mathsf{h}}(\Delta)$.

Corollary 5.8. *Let Δ be a PSFC and let $v \in \mathcal{V}_{\mathsf{h}}(\Delta)$. For all $i \geq 0$,*

$$W(P_i(v)) \lesssim 2^i \alpha(Q_v)^{-4}|Q_v| = 2^i W(\{v\}).$$

Proof. For every $w \in P_i(v)$, the pseudoquad Q_w is vertically cut, so Γ is not (η, R)–paramonotone on rQ_v. That is,

$$\Omega^P_{\Gamma^+,2^{-i}R\delta_x(Q_v)}(rQ_w) \geq \eta\alpha(Q_w)^{-4}|Q_w| = \eta W(\{Q_w\}).$$

Then, by Lemma 5.7,

$$W(P_i(v)) \approx W(V_i(v)) \leq \eta^{-1} \sum_{w\in V_i(v)} \Omega^P_{\Gamma^+,2^{-i}R\delta_x(Q_v)}(rQ_w).$$

The rQ_w's are disjoint and, by Lemma 4.5, they are contained in rQ_v, so

$$\begin{aligned} W(P_i(v)) &\lesssim \eta^{-1}\Omega^P_{\Gamma^+,2^{-i}R\delta_x(Q_v)}(rQ_v) \\ &\overset{(17)}{\leq} 2^i\eta^{-1}\Omega^P_{\Gamma^+,R\delta_x(Q_v)}(rQ_v). \end{aligned}$$

Since Q_v is horizontally cut, Γ is (η, R)–paramonotone on rQ_v, so

$$\Omega^P_{\Gamma^+,R\delta_x(Q_v)}(rQ_v) \leq \eta\alpha(Q_v)^{-4}|Q_v|$$

and thus

$$W(P_i(v)) \lesssim 2^i\alpha(Q_v)^{-4}|Q_v|,$$

as desired. □

This leads to the desired L_p bounds.

Proof of Proposition 5.1. Let Δ be a PSFC and let D_0, μ, λ, η, R, and r be the parameters in Definition 4.1. Let $2 \leq p < 5$.

Let $v \in \mathcal{V}_{\mathsf{h}}(\Delta)$. Let $Q = Q_v$. Let $w = \delta_x(Q)$, $h = \delta_z(Q)$. (In fact, the problem has enough symmetry that it suffices to prove the proposition when $\delta_x(Q) = \delta_z(Q) = \alpha(Q) = 1$.) For $i \geq 0$, let $R_i = \{u \in \mathscr{D}(v) \mid \delta_x(Q_u) \geq 2^{-i}w\}$ as in (38). We consider the approximations g_{R_i}.

By Lemma 5.4, the functions g_{R_i} converge pointwise to f, so by Fatou's Lemma,

$$\|f - l_v\|_{L_p(Q)} \leq \liminf_i \|g_{R_i} - l_v\|_{L_p(Q)} \leq \|g_{R_0} - l_v\|_{L_p(Q)} + \liminf_i \|g_{R_i} - g_{R_0}\|_{L_p(Q)}.$$

First, note that $D^h(v) = R_0$, so by Lemma 5.6,

$$\|g_{R_0} - l_v\|_{L_\infty(Q)} = \|g_{R_0} - g_{\{v\}}\|_{L_\infty(Q)} \lesssim \lambda \cdot \frac{h}{w},$$

so

$$\|g_{R_0} - l_v\|_{L_p(Q)} \lesssim \lambda^p \frac{h^p}{w^p}|Q|.$$

Likewise, for every $u \in \min(R_i)$, Lemma 5.6 implies that

$$\|g_{R_{i+1}} - g_{R_i}\|_{L_\infty(Q_u)} \lesssim \lambda \frac{\delta_z(Q_u)}{\delta_x(Q_u)}.$$

Since $\delta_x(Q_u) = 2^{-i} w$ for all $u \in \min(R_i)$,

(40)
$$\|g_{R_{i+1}} - g_{R_i}\|^p_{L_p(Q)} \lesssim \sum_{u \in \min(R_i)} \lambda^p \frac{\delta_z(Q_u)^p}{\delta_x(Q_u)^p} |Q_u| \approx \lambda^p w^{-p+1} 2^{i(p-1)} \sum_{u \in \min(R_i)} \delta_z(Q_u)^{p+1}.$$

We bound the $\delta_z(Q_u)$'s using Corollary 5.8. For all $i \ge 0$, (21) implies that

$$W(\min(R_i)) \approx 2^{3i} w^{-3} \sum_{u \in \min(R_i)} \delta_z(Q_u)^3.$$

Then, by Corollary 5.8,

$$\sum_{u \in \min(R_i)} \delta_z(Q_u)^3 \approx 2^{-3i} w^3 W(\min(R_i)) \lesssim 2^{-3i} w^3 \cdot 2^i \alpha(Q)^{-4} |Q| \overset{(21)}{\approx} 2^{-2i} h^3.$$

Since $p \ge 2$, convexity implies

$$\sum_{u \in \min(R_i)} \delta_z(Q_u)^{p+1} \le \left(\sum_{u \in \min(R_i)} \delta_z(Q_u)^3 \right)^{\frac{p+1}{3}} \lesssim 2^{-2i \cdot \frac{p+1}{3}} h^{p+1}.$$

Then

$$\begin{aligned} \|g_{R_{i+1}} - g_{R_i}\|^p_{L_p(Q)} &\lesssim \lambda^p w^{-p+1} 2^{i(p-1)} \cdot 2^{-2i \cdot \frac{p+1}{3}} h^{p+1} \\ &= \lambda^p w^{-p+1} h^{p+1} 2^{\frac{ip}{3} - \frac{5i}{3}}, \end{aligned}$$

so by (40),

$$\|g_{R_{i+1}} - g_{R_i}\|_{L_p(Q)} \lesssim \lambda 2^{i \cdot \frac{p-5}{3p}} \frac{\delta_z(Q)}{\delta_x(Q)} |Q|^{\frac{1}{p}}.$$

Since $p < 5$, this decays exponentially in i, so for any n,

$$\begin{aligned} \|g_{R_n} - l_v\|_{L_p(Q)} &\le \|g_{R_0} - l_v\|_{L_p(Q)} + \sum_{i=0}^{j} \|g_{R_{i+1}} - g_{R_i}\|_{L_p(Q)} \\ &\lesssim \frac{1}{1 - 2^{\frac{p-5}{3}}} \cdot \lambda \cdot \frac{\delta_z(Q)}{\delta_x(Q)} |Q|^{\frac{1}{p}} \\ &\approx \frac{\lambda}{5-p} \cdot \frac{\delta_z(Q)}{\delta_x(Q)} |Q|^{\frac{1}{p}}. \end{aligned}$$

Therefore, when $v \in \mathcal{V}_{\mathsf{h}}(\Delta)$,

$$\|f - l_v\|_{L_p(Q)} \lesssim \frac{1}{5-p} \lambda \frac{\delta_z(Q)}{\delta_x(Q)} |Q|^{\frac{1}{p}}.$$

When $v \in \mathcal{V}_{\mathsf{v}}(\Delta)$, we proceed similarly to Lemma 4.6. Let $m \in \mathcal{V}_{\mathsf{h}}(\Delta)$ be the minimal horizontally-cut ancestor of v. Then $l_m = l_v$, $\delta_z(Q_m) \approx \delta_z(Q_v)$, and $\delta_x(Q_m) \gtrsim \delta_x(Q_v)$. By the horizontally-cut case,

$$\|f - l_m\|_{L_p(Q_m)} \lesssim \lambda \frac{1}{5-p} \frac{\delta_z(Q_m)}{\delta_x(Q_m)} |Q_m|^{\frac{1}{p}}.$$

Since $p \ge 1$, we have

$$\frac{\delta_z(Q_m)}{\delta_x(Q_m)} |Q_m|^{\frac{1}{p}} \approx \frac{\delta_z(Q_m)^{\frac{p+1}{p}}}{\delta_x(Q_m)^{\frac{p-1}{p}}} \lesssim \frac{\delta_z(Q_v)}{\delta_x(Q_v)} |Q_v|^{\frac{1}{p}},$$

so

$$\|l_v - f\|_{L_p(Q_v)} \le \|l_m - f\|_{L_p(Q_m)} \lesssim \lambda \frac{1}{5-p} \frac{\delta_z(Q_v)}{\delta_x(Q_v)} |Q_v|^{\frac{1}{p}}.$$

This proves the proposition for $2 \le p < 5$. If $p = 1+\theta$ for $\theta \in (0,1)$, then by Lyapunov's inequality and (22),

$$\begin{aligned} \|l_v - f\|_{L_p(Q_v)} &\le \|l_v - f\|_{L_1(Q_v)}^{\frac{1-\theta}{p}} \|l_v - f\|_{L_2(Q_v)}^{\frac{2\theta}{p}} \\ &\lesssim \left(\lambda \frac{\delta_z(Q_v)}{\delta_x(Q_v)} |Q_v|\right)^{\frac{1-\theta}{p}} \left(\lambda \frac{\delta_z(Q_v)}{\delta_x(Q_v)} |Q_v|^{\frac{1}{2}}\right)^{\frac{2\theta}{p}} = \lambda \frac{\delta_z(Q_v)}{\delta_x(Q_v)} |Q_v|^{\frac{1}{p}}, \end{aligned}$$

so the proposition holds for $p \in [1,5)$. □

6. PROOF OF PROPOSITION 3.5

In this section, we will prove Proposition 3.5. Recall that the proposition states that there is $\tau > 0$ depending only on the intrinsic Lipschitz constant L of $\Gamma = \Gamma_f$ such that if α_0 is sufficiently small (again depending on L) and Q is a rectilinear pseudoquad with $\alpha(Q) = \alpha_0$, then

$$\int_{\frac{1}{3}Q} \int_0^{\tau\delta_x(Q)} \gamma_{4,f}(x,r)^4 \frac{\mathrm{d}r}{r} \,\mathrm{d}x \lesssim |Q| \tag{41}$$

where $\gamma_{4,f}(x,r) = r^{-\frac{7}{4}} \inf_{h\in \mathsf{Aff}} \|f-h\|_{L_4(V(\Psi_f(x),r))}$. Most of the implicit constants in this section depend on the intrinsic Lipschitz constant of Γ, so we will write $\lesssim$ for $\lesssim_L$ and so on for brevity.

We take $\alpha_{\min}$ as in Lemma 4.3, let $0 < \alpha_0 < \alpha_{\min}$, and let Q be a rectilinear pseudoquad with $\alpha(Q) = \alpha_0$. Since the conclusion of Proposition 3.5 is invariant under scaling, we rescale so that $\delta_z(Q) = 1$ and $\delta_x(Q) = \alpha(Q) < 1$. By Lemma 4.3, there is a PSFC with root Q, which we call Δ.

We prove Proposition 3.5 by using this PSFC to construct a sequence of approximations of f. For $i \ge 0$, let

$$S_i := \{w \in \mathcal{V}(\Delta) \mid \delta_z(Q_w) \ge 2^{-2i}\}. \tag{42}$$

If w and w' are siblings, then $\delta_z(Q_w) = \delta_z(Q_{w'})$, and if v is the parent of w, then $\delta_z(Q_v) \ge \delta_z(Q_w)$, so S_i is a coherent set. Furthermore, by Lemma 4.4, we have

$$\delta_x(Q_v) \ge \frac{\alpha_{\min}}{4} \sqrt{\delta_z(Q_v)} \ge \frac{\alpha_{\min}}{4} 2^{-i}$$

for all but finitely many $v \in S_i$, so S_i is finite. We let $F_i := \min(S_i)$; this is a partition of Q. By Lemma 2.7, if $\delta_z(Q_v) \ge 4 \cdot 2^{-2i}$, then Q_v's children have height greater than 2^{-2i}, so $v \notin F_i$. That is,

$$\delta_z(Q_w) \in [2^{-2i}, 4\cdot 2^{-2i}) \text{ for all } w \in F_i. \tag{43}$$

For each i, let g_{S_i} be the approximation defined in Definition 5.3, which agrees with l_w on Q_w for each $w \in F_i$.

For $x \in V_0$, let

$$\sigma_i(x,r) = r^{-\frac{7}{4}} \inf_{h\in\mathsf{Aff}} \|g_{S_i} - h\|_{L_4(V(\Psi_f(x),r))}.$$

This is similar to $\gamma_{4,g_{S_i}}$ except that the norm is taken in $L_4(V(\Psi_f(x),r))$. By the triangle inequality, for any x and r such that $V(\Psi_f(x),r) \subset Q$,

$$\gamma_{4,f}(x,r) \le r^{-\frac{7}{4}} \|g_{S_i} - f\|_{L_4(V(\Psi_f(x),r))} + \sigma_i(x,r). \tag{44}$$

We will first state and prove bounds on the terms above, then prove Proposition 3.5. We can bound $g_{S_i} - f$ using Proposition 5.1. Since the F_i are minimal elements of S_i, every element of F_i is horizontally cut. Therefore, by Proposition 5.1,

$$\|l_w - f\|_{L_4(Q_w)} \lesssim \frac{\delta_z(Q_w)}{\delta_x(Q_w)}|Q_w|^{\frac{1}{4}} \tag{45}$$

for all $w \in F_i$. Then, since $\delta_z(Q_w) \approx 2^{-2i}$,

$$\left(\frac{\|g_{S_i} - f\|_{L_4(Q)}}{2^{-i}}\right)^4 \lesssim \sum_{w\in F_i} \frac{2^{4i}\delta_z(Q_w)^4}{\delta_x(Q_w)^4}|Q_w| \approx W(F_i), \tag{46}$$

where $W(F_i) = \sum_{w\in F_i} \alpha(Q_w)^{-4}|Q_w|$ is as in Definition 2.9.

We can use the following lemma to bound $\sigma_i(x, r)$.

Lemma 6.1. *There exists a constant $\nu > 0$ depending only on L such that $V(\Psi_f(x), \nu) \subset Q$ for all $x \in \frac{1}{3}Q$ and such that for any $i \geq 0$ such that $2^{-i} \leq \frac{\delta_x(Q)}{\alpha_{\min}}$, we have*

$$\int_{\frac{1}{3}Q} \sigma_i\big(x, \nu 2^{-i}\big)^4 \,\mathrm{d}x \lesssim W(F_i). \tag{47}$$

These two bounds will imply the proposition.

Before we prove Lemma 6.1, we state some lemmas that we will prove in Section 6.1.

Lemma 6.2. *For any $0 < L < 1$ and $0 < a < b \leq 4$, there exists $0 < \eta < 1$ so that the following holds. Let $\Gamma = \Gamma_f$ be an L–intrinsic Lipschitz graph, and let Q be a rectilinear pseudoquad for Γ. Let $r = \min\{\sqrt{\delta_z(Q)}, \delta_x(Q)\}$. If $V(p, \eta r)$ intersects aQ for some $p \in \Gamma$ then $V(p, \eta r) \subseteq bQ$.*

Lemma 6.3. *Let $v, w \in F_i$. If $\delta_x(Q_w) \leq \delta_x(Q_v)$ and $Q_w \cap 3Q_v \neq \emptyset$, then $Q_w \subset 10Q_v$.*

This lemma lets us prove the following bound.

Lemma 6.4. *Let $v, w \in F_i$. If $\delta_x(Q_w) \leq \delta_x(Q_v)$ and $Q_w \cap 3Q_v \neq \emptyset$, then*

$$\|l_w - l_v\|_{L_\infty(Q_w)} \lesssim \frac{\delta_z(Q_w)}{\delta_x(Q_w)}. \tag{48}$$

Proof. By Lemma 6.3, $Q_w \subset 10Q_v$, so

$$\frac{1}{|Q_w|}\|f - l_{Q_v}\|_{L_1(Q_w)} \leq \frac{1}{|Q_w|}\|f - l_{Q_v}\|_{L_1(10Q_v)} \overset{(32)}{\leq} \frac{1}{|Q_w|}\lambda\frac{\delta_z(Q_v)}{\delta_x(Q_v)}|Q_v| \lesssim \frac{\delta_z(Q_w)}{\delta_x(Q_w)}.$$

In the last inequality, we used the fact that $\delta_z(Q_v) \approx \delta_z(Q_w)$ and so Lemma 2.7 gives that $|Q_v|/|Q_w| \approx \delta_x(Q_v)/\delta_x(Q_w)$.

Likewise,

$$\frac{1}{|Q_w|}\|f - l_{Q_w}\|_{L_1(Q_w)} \overset{(32)}{\leq} \lambda\frac{\delta_z(Q_w)}{\delta_x(Q_w)},$$

so by the triangle inequality,

$$\frac{1}{|Q_w|}\|l_{Q_w} - l_{Q_v}\|_{L_1(Q_w)} \lesssim \frac{\delta_z(Q_w)}{\delta_x(Q_w)}.$$

It remains to show that $|Q_w|\|l_v - l_w\|_{L_\infty(Q_w)} \lesssim \|l_v - l_w\|_{L_1(Q_w)}$. In fact, we claim that if L is any affine function and Q is a rectilinear pseudoquad, then

$$\|L\|_{L_\infty(Q)} \leq 24\frac{\|L\|_{L_1(Q)}}{|Q|}. \tag{49}$$

Let $I = x(Q)$ be the base of Q. After a scaling and translation, we may suppose that $I = [-1,1]$. Since Q is rectilinear, there are functions $g_1, g_2 \colon I \to \mathbb{R}$ such that $Q = \{(x,0,z) | x \in I, z \in [g_1(x), g_2(x)]\}$ and quadratic polynomials $h_1, h_2 \colon I \to \mathbb{R}$ such that $h_2 = h_1 + \delta_z(Q)$ and $\|g_i - h_i\|_\infty \le \frac{1}{32}\delta_z(Q)$. In particular, $|Q| \le 3\delta_z(Q)$.

Let $M = \|L\|_{L_\infty(Q)}$. Since $L(x,0,z) = ax + b$ for some a and b, we have $M = |L(1,0,0)|$ or $M = |L(-1,0,0)|$; suppose $M = |L(1,0,0)|$. (The other case is similar.)

Then $|L(x,0,0)| \ge \frac{M}{2}$ for $x \ge \frac{1}{2}$, so

$$\|L\|_{L_1(Q)} \ge \int_{\frac{1}{2}}^{1} \int_{g_1(x)}^{g_2(x)} \frac{M}{2}\, \mathrm{d}z\, \mathrm{d}x \ge \frac{M}{8}\delta_z(Q) \ge \frac{M}{24}|Q|.$$

This proves the lemma. □

Now we prove Lemma 6.1.

Proof of Lemma 6.1. Let $\zeta \colon Q \to \mathbb{R}$ be the function such that for any $w \in F_i$ and $x \in Q_w$, $\zeta(x) = \frac{\delta_z(Q_w)}{\delta_x(Q_w)}$. If $w \ne w'$, then Q_w and $Q_{w'}$ intersect in a set of measure zero, so we break ties arbitrarily. We will bound $\gamma_{4, g_{S_i}}$ in terms of $\|\zeta\|_4$.

For all $v \in F_i$, we have $\delta_z(Q_v) \in [2^{-2i}, 4 \cdot 2^{-2i}]$. By Lemma 4.4, this implies $\alpha(Q_v) \ge \frac{\alpha_{\min}}{4}$, so $\delta_x(Q_v) \ge \frac{\alpha_{\min}}{4} 2^{-i}$. Then

$$\frac{\alpha_{\min}}{4} 2^{-i} \le \min\{\delta_x(Q_v), \sqrt{\delta_z(Q_v)}\}.$$

By Lemma 2.6 and Lemma 6.2, there is an $0 < \eta_0 < 1$ such that if $v \in F_i$, $p \in \Gamma$, and $Q_v \cap V(p, \frac{\alpha_{\min}}{4}\eta_0 2^{-i}) \ne \emptyset$, then $V(p, \frac{\alpha_{\min}}{4}\eta_0 2^{-i}) \subset 3Q_v$. Likewise, there is a $0 < \eta_1 < 1$ such that if $x \in \frac{1}{3}Q_v$, then

$$V(\Psi_f(x), \frac{\alpha_{\min}}{4}\eta_1 2^{-i}) \subset \frac{2}{3}Q_v \subset Q_v.$$

Let $\nu = \frac{\alpha_{\min}}{4}\min\{\eta_0, \eta_1\}$; we can choose ν to depend only on L. Then $V(\Psi_f(x), \nu) \subset Q$ for all $x \in \frac{1}{3}Q$.

Let $x \in \frac{1}{3}Q$ and let $D(x) = V(\Psi_f(x), \nu 2^{-i})$. Let $v_1, \dots, v_n \in F_i$ be the elements such that Q_{v_i} intersects $D(x)$ and suppose that $\delta_x(Q_{v_1}) \ge \delta_x(Q_{v_i})$ for all i. For any $y \in D(x)$, we have $y \in 3Q_{v_1}$ and there is a j such that $y \in Q_{v_j}$. By Lemma 6.4,

$$|g_{S_i}(y) - l_{v_1}(y)| = |l_{v_j}(y) - l_{v_1}(y)| \lesssim \zeta(y).$$

Therefore,

$$\sigma_i(x, \nu 2^{-i}) \le (\nu 2^{-i})^{-\frac{7}{4}} \|g_{S_i} - l_{v_1}\|_{L_4(D(x))} \lesssim 2^{\frac{7i}{4}} \|\zeta\|_{L_4(D(x))} \tag{50}$$

and

$$\int_{\frac{1}{3}Q} \sigma_i(x, \nu 2^{-i})^4\, \mathrm{d}x \lesssim 2^{7i} \int_{\frac{1}{3}Q} \int_{D(x)} \zeta^4(y)\, \mathrm{d}y\, \mathrm{d}x =: I.$$

Let c be as in Lemma 3.2. Then for every $x \in \frac{1}{3}Q$ and $y \in D(x)$, we have that $x \in V(\Psi_f(y), c\nu 2^{-i})$. Our choice of ν implies that $y \in Q$, so by Fubini's Theorem,

$$I \le 2^{7i} \int_Q \int_{V(\Psi_f(y), c\nu 2^{-i})} \zeta^4(y)\, \mathrm{d}x\, \mathrm{d}y \lesssim 2^{4i} \|\zeta\|^4_{L_4(Q)}.$$

Furthermore, since $\delta_z(Q_w) \approx 2^{-2i}$ for all $w \in F_i$,

$$\|\zeta\|^4_{L_4(Q)} = \sum_{w \in F_i} |Q_w| \frac{\delta_z(Q_w)^4}{\delta_x(Q_w)^4} = \sum_{w \in F_i} \frac{|Q_w| \delta_z(Q_w)^2}{\alpha^4(Q_w)} \approx 2^{-4i} W(F_i),$$

so

$$\int_{\frac{1}{3}Q} \sigma_i(x, \nu 2^{-i})^4 \, \mathrm{d}x \lesssim 2^{4i} \|\zeta\|_{L_4(Q)}^4 \lesssim W(F_i)$$

as desired. □

Finally, we prove Proposition 3.5.

Proof of Proposition 3.5. Let Δ, S_i, and F_i be as above and let ν be as in Lemma 6.1. Let $\tau = \frac{\nu}{2\alpha_{\min}}$ and $i_0 = \left\lceil \log_2 \frac{\alpha_{\min}}{\delta_x(Q)} \right\rceil$ so that $2^{-i_0} \le \frac{\delta_x(Q)}{\alpha_{\min}}$ and $\nu 2^{-i_0} \ge \tau \delta_x(Q)$.

Suppose that $i \ge i_0$ and let $r_i = \nu 2^{-i}$. By (44) and Lemma 6.1,

$$\int_{\frac{1}{3}Q} \gamma_{4,f}(x, r_i)^4 \, \mathrm{d}x \lesssim \int_{\frac{1}{3}Q} r_i^{-7} \|g_{S_i} - f\|_{L_4(V(\Psi_f(x), r_i))}^4 \, \mathrm{d}x + W(F_i).$$

As in the proof of Lemma 6.1, Fubini's theorem shows

$$r_i^{-7} \int_{\frac{1}{3}Q} \|g_{S_i} - f\|_{L_4(V(\Psi_f(x), r_i))}^4 \, \mathrm{d}x \lesssim r_i^{-4} \|g_{S_i}(y) - f(y))\|_{L_4(Q)}^4,$$

and by (46),

$$\int_{\frac{1}{3}Q} \gamma_{4,f}(x, r_i)^4 \, \mathrm{d}x \lesssim W(F_i) + W(F_i) \lesssim W(F_i).$$

For $r \in [\frac{r_i}{2}, r]$, we have $\gamma_{4,f}(x, r) \lesssim \gamma_{4,f}(x, r_i)$, so

$$\int_{\frac{1}{3}Q} \int_{\frac{r_i}{2}}^{r_i} \gamma_{4,f}(x, r)^4 \frac{\mathrm{d}r}{r} \, \mathrm{d}x \lesssim \int_{\frac{1}{3}Q} \gamma_{4,f}(x, r)^4 \, \mathrm{d}x \lesssim W(F_i).$$

By (43), if $w \in F_i$, then $\delta_z(Q_w) \in [2^{-2i}, 4 \cdot 2^{-2i})$. It follows that the F_i's are disjoint, so we can sum over $i \ge i_0$ to obtain

$$\int_{\frac{1}{3}Q} \int_0^{\nu 2^{-i_0}} \gamma_{4,f}(x, r)^4 \frac{\mathrm{d}r}{r} \, \mathrm{d}x \lesssim \sum_{i=i_0}^{\infty} W(F_i) \le W(\mathcal{V}(\Delta)) \lesssim |Q|.$$

This proves the proposition. □

6.1. **Geometry of rectilinear pseudoquads.** In this section, we will prove some basic results and bounds for rectilinear pseudoquads of intrinsic Lipschitz graphs. Recall that a rectilinear pseudoquad for $\Gamma = \Gamma_f$ is a tuple (Q, R_Q) of a pseudoquad Q and a parabolic rectangle R_Q approximating Q. Then R_Q is a pseudoquad in a vertical plane of slope slope(Q), which we denote P_Q.

Our first set of results deals with arbitrary rectilinear pseudoquads, without any assumption on paramonotonicity; our goal is to prove Lemma 6.2. We first bound the slope of a rectilinear pseudoquad in terms of the intrinsic Lipschitz constant.

Lemma 6.5. *Let (Q, R) be a rectilinear pseudoquad of an L–intrinsic Lipschitz graph. Then*

$$|\operatorname{slope} Q| \le \frac{L}{\sqrt{1-L^2}} + \frac{\delta_z(Q)}{\delta_x(Q)^2} = \frac{L}{\sqrt{1-L^2}} + \alpha(Q)^{-2}. \tag{51}$$

Proof. Let I denote the base of Q. By translation and scaling, we may assume $I = [-\delta_x(Q)/2, \delta_x(Q)/2]$. Let $W = 2\delta_x(Q)$, so that $4I = [-W, W]$. Let g and h be the functions whose graphs are the characteristic curves forming the top boundaries of Q and

R, respectively. Since h is quadratic and slope $Q = -h''(x)$, there are $b, c \in \mathbb{R}$ such that $h(x) = -\operatorname{slope}(Q)\frac{x^2}{2} + bx + c$. Therefore,

$$|\operatorname{slope}(Q)| = \frac{|h(-W) - 2h(0) + h(W)|}{W^2}.$$

By Lemma 2.3,

$$\begin{aligned}|g(-W)-&2g(0)+g(W)| \\ &\le |g(-W) - g(0) + g'(0)W| + |g(W) - g(0) - g'(0)W| \\ &\overset{(11)}{\le} \frac{L}{\sqrt{1-L^2}} W^2.\end{aligned}$$

By the rectilinearity of Q,

$$\max_{t\in[-W,W]} |g(t) - h(t)| \le \frac{1}{32}\delta_z(Q),$$

so

$$|\operatorname{slope}(Q)| = \frac{|h(-W) - 2h(0) + h(W)|}{W^2} \le \frac{L}{\sqrt{1-L^2}} + \frac{1}{32}\frac{\delta_z(Q)}{\delta_x(Q)^2}.$$

□

We use this to bound the distance from P_Q to Γ_f.

Lemma 6.6. *For any $0 < L < 1$ there exists an $\eta > 0$ so that the following holds. Let Γ_f be an L–intrinsic Lipschitz graph and let Q be a rectilinear pseudoquad for Γ_f. Let λ be the affine function such that $P_Q = \Gamma_\lambda$. Then for all $p \in 4Q$,*

$$|f(p) - \lambda(p)| \le \eta \max\left\{\sqrt{\delta_z(Q)}, \frac{\delta_z(Q)}{\delta_x(Q)}\right\}. \tag{52}$$

Proof. Let I be the base of Q, and let $g_Q, g_R\colon \mathbb{R} \to \mathbb{R}$ be functions whose graphs are the characteristic curves going through the tops of Q and R, respectively. After a left translation, we may suppose that $I = [-\delta_x(Q)/2, \delta_x(Q)/2]$ and that P_Q goes through $\mathbf{0}$. Then $\lambda(x,z) = mx$ and $g_R(x) = g_R(0) - \frac{1}{2}mx^2$, where $m = \operatorname{slope}(Q)$.

We first show that if $\tau = \frac{3L}{\sqrt{1-L^2}} + 2$, then

$$\max_{t\in 4I} |f(t, 0, g_Q(t)) - mt| \le \tau \max\left\{\sqrt{\delta_z(Q)}, \frac{\delta_z(Q)}{\delta_x(Q)}\right\}. \tag{53}$$

Let $s \in 4I$, $q = (s, 0, g_1(s))$, and let $h = \min\{\sqrt{\delta_z(Q)}, \delta_x(Q)\}$. Without loss of generality, suppose that $s + h \in 4I$. By rectilinearity, $|g_Q(x) - g_R(x)| \le \frac{1}{32}\delta_z(Q)$ for all $x \in 4I$. By Lemma 2.3,

$$|g_Q(s+h) - g_Q(s) - hg_Q'(s)| \le \frac{L}{\sqrt{1-L^2}}\frac{h^2}{2},$$

and by Lemma 6.5,

$$|g_R(s+h) - g_R(s) - hg_R'(s)| = \frac{1}{2}|m|h^2 \le \frac{2L}{\sqrt{1-L^2}}h^2 + \frac{\delta_z(Q)}{\delta_x(Q)^2}h^2.$$

Since $|(g_R(s+h) - g_R(s)) - (g_Q(s+h) - g_Q(s))| \le \frac{1}{16}\delta_z(Q)$, these imply

$$|hg_Q'(s) - hg_R'(s)| \le \frac{3L}{\sqrt{1-L^2}}h^2 + \frac{\delta_z(Q)}{\delta_x(Q)^2}h^2 + \frac{1}{16}\delta_z(Q).$$

We divide both sides by h to find

$$\begin{aligned}|g_Q'(s)+ms| &\le \frac{3L}{\sqrt{1-L^2}}\sqrt{\delta_z(Q)}+\frac{\delta_z(Q)}{\delta_x(Q)}+\max\left\{\sqrt{\delta_z(Q)},\frac{\delta_z(Q)}{\delta_x(Q)}\right\}\\ &\le \tau\max\left\{\sqrt{\delta_z(Q)},\frac{\delta_z(Q)}{\delta_x(Q)}\right\}.\end{aligned}$$

By (10), $f(s,0,g_Q(s)) = -g_Q'(s)$, which proves (53).

Finally, let $(x,0,z)\in 4Q$. Then $x\in 4I$ and $|z-g_Q(x)|\le 16\delta_z(Q)$, so

$$\begin{aligned}|f(x,0,z)-mx| &\le |f(x,0,z)-f(x,0,g_Q(x))|+|f(x,0,g_Q(x))-mx|\\ &\overset{(8)\wedge(53)}{\le} \frac{16}{1-L}\sqrt{\delta_z(Q)}+\tau\max\left\{\sqrt{\delta_z(Q)},\frac{\delta_z(Q)}{\delta_x(Q)}\right\}.\end{aligned}$$

The lemma follows by taking $\eta=\tau+\frac{16}{1-L}$. □

Finally, we can prove Lemma 6.2.

Proof of Lemma 6.2. Let $x\in\Gamma$ and $\rho>0$ so that $V(x,\rho)$ intersects aQ. Let $p\in V(x,\rho)\cap aQ$. By Lemma 3.2, there is a $c>0$ such that

$$V(x,\rho)\subset\Pi(B(x,c\rho)\cap\Gamma)\subset\Pi(B(\Psi_f(p),2c\rho)\cap\Gamma)\subset V(p,2c\rho). \tag{54}$$

We claim that there is some $\eta'>0$ such that $V(p,\eta' r)\subset bQ$ for every $p\in\Psi_f(aQ)$. The full claim then follows by taking $\eta=(2c)^{-1}\eta'$.

Let I be the base of Q and let $m=\mathrm{slope}(Q)$. After a left translation, we may suppose that $I=[-\delta_x(Q)/2,\delta_x(Q)/2]$, that $P_Q=\Gamma_\lambda$, where $\lambda(x)=mx$, and that

$$R_Q=\left\{(x,0,z):x\in I,\left|z+\frac{m}{2}x^2\right|\le\frac{\delta_z(Q)}{2}\right\}.$$

Let $v=(x_0,0,z_0)\in aQ$. Then

$$\left|z_0+\frac{m}{2}x_0^2\right|\le\frac{a^2}{2}\delta_z(Q).$$

Let $c=z_0+\frac{m}{2}x_0^2$ and let $g(x)=c-\frac{m}{2}x^2$, so that the graph of g is the characteristic curve of P_Q through v. Let $d=\frac{b^2}{2}-\frac{a^2}{2}$ and let

$$T:=\{(x,0,z)|x\in bI,|z-g(x)|\le d\delta_z(Q)\}.$$

If $(x,0,z)\in T$, then

$$\left|z+\frac{m}{2}x^2\right|\le|c|+d\delta_z(Q)\le\frac{b^2}{2}\delta_z(Q),$$

so $T\subset bQ$. We claim that there is a ν depending only on L such that $V(p,\nu r)\subset T$.

By Lemma 3.1,

$$V(p,\nu r)\subset\left\{(x,0,z)||x-x_0|\le\nu r,|z-z_0+(x-x_0)f(v)|\le(\nu r)^2\right\}.$$

If $(x,0,z)\in V(p,\nu r)$, then

$$|z-g(x)|\le(\nu r)^2+|g(x)-z_0+(x-x_0)f(v)|\le\nu^2\delta_z(Q)+|\ell(x)|, \tag{55}$$

where $\ell(x)=g(x)-z_0+(x-x_0)f(v)$. Note that $\ell(x_0)=0$. By Lemma 6.6,

$$|\ell'(x_0)|=|g'(x_0)+f(v)|\overset{(10)}{=}|f(v)-\lambda(v)|\le\eta\max\left\{\sqrt{\delta_z(Q)},\frac{\delta_z(Q)}{\delta_x(Q)}\right\}=\eta\frac{\delta_z(Q)}{r}.$$

Finally, $\ell''(x_0)=-m$. Therefore, for $|x-x_0|\le\nu r$,

$$|\ell(x)|\le\nu r\eta\frac{\delta_z(Q)}{r}+(\nu r)^2\frac{|m|}{2}\overset{(51)}{\le}\nu\eta\delta_z(Q)+\nu^2\delta_z(Q)\left(\frac{L}{\sqrt{1-L^2}}+1\right). \tag{56}$$

If $0 < \nu < b - a$ is sufficiently small and $(x, 0, z) \in V(p, \nu r)$, then (55) and (56) imply $|z - g(x)| \le d\delta_z(Q)$ and thus $(x, 0, z) \in T$. Therefore, $V(p, \nu r) \subset bQ$, as desired.

□

Next, we consider pseudoquads that are part of a PSFC and prove Lemma 6.3. Let Δ be a PSFC, let S_i be as in (42), and let $F_i = \min S_i$. For $p \in V_0$, let $\kappa_{\Gamma,p}$ be a function whose graph is a characteristic curve of Γ going through p, i.e., a solution of (10) with the initial condition $\kappa_{\Gamma,p}(x(p)) = z(p)$. For $w \in F_i$, let $R_w = R_{Q_w}$ and $P_w = P_{Q_w}$. Let ρ_w be the affine function such that $R_w = \Gamma_{\rho_w}$. Since Q_w is part of a PSFC, it has a λ–approximating plane Λ_w that satisfies Proposition 2.8, and we let l_w be the corresponding affine function. The construction of P_w and Λ_w are different, so they need not be the same plane, but they must be close, as in the following lemma.

Lemma 6.7. *Let Δ be a PSFC and let $w \in \mathcal{V}(\Delta)$. Let $I = x(Q_w)$. Then, for any $q \in 4Q_w$,*

$$\|\kappa_{P_w,q} - \kappa_{\Lambda_w,q}\|_{L_\infty(4I)} \le \frac{3}{32}\delta_z(Q_w)$$

and

$$\|\kappa_{P_w,q} - \kappa_{\Gamma,q}\|_{L_\infty(4I)} \le \frac{1}{8}\delta_z(Q_w).$$

Indeed, this lemma holds for any pseudoquad Q that satisfies Proposition 2.8.

Proof. Let $g_1, g_2, h_1, h_2 \colon 4I \to \mathbb{R}$ be the functions parameterizing the boundary of Q_w, so that $Q_w = \{(x, 0, z) | x \in I, z \in [g_1(x), g_2(x)]\}$ and $R_w = \{(x, 0, z) | x \in I, z \in [h_1(x), h_2(x)]\}$.

Let $q \in 4Q_w$ and let $p = (x(q), 0, g_1(q))$ so that p lies in the graph of g_1; we can take $g_1 = \kappa_{\Gamma,p}$. Since Λ_w and P_w are vertical planes,

$$\kappa_{\Lambda_w,p} - \kappa_{P_w,p} = \kappa_{\Lambda_w,q} - \kappa_{P_w,q},$$

and we have

$$\|\kappa_{P_w,p} - \kappa_{\Lambda_w,p}\|_\infty \le \|\kappa_{P_w,p} - h_1\|_\infty + \|h_1 - g_1\|_\infty + \|g_1 - \kappa_{\Lambda_w,p}\|_\infty \tag{57}$$

where all norms are in $L_\infty(4I)$. By the rectilinearity of Q_w, the first two terms are each at most $\frac{1}{32}\delta_z(Q)$; the last term is at most $\frac{1}{32}\delta_z(Q)$ by Proposition 2.8. This proves the first inequality.

Thus, by (57) and Proposition 2.8,

$$\begin{aligned}\|\kappa_{P_w,q} - \kappa_{\Gamma,q}\|_\infty &\le \|\kappa_{P_w,q} - \kappa_{\Lambda_w,q}\|_\infty + \|\kappa_{\Lambda_w,q} - \kappa_{\Gamma,q}\|_\infty \\ &\le \frac{3}{32}\delta_z(Q_w) + \frac{1}{32}\delta_z(Q_w).\end{aligned}$$

This proves the lemma. □

Note that, by (10), this implies

$$\|\rho_w - l_w\|_{L_\infty(4Q_w)} \lesssim \frac{\delta_z(Q_w)}{\delta_x(Q_w)}.$$

We use Lemma 6.7 to prove Lemma 6.3, which states that if $v, w \in F_i$, $\delta_x(Q_w) \le \delta_x(Q_v)$, and $Q_w \cap 3Q_v \ne \emptyset$, then $Q_w \subset 10Q_v$.

Proof of Lemma 6.3. Let $I_v = x(Q_v)$ and $I_w = x(Q_w)$ be the bases of Q_v and Q_w. Since $Q_w \cap 3Q_v$ is nonempty, $I_w \subset 4I_v$. Let c_v be the center of R_v and let $\gamma_v = \kappa_{P_v,c_v}$, so that if $\ell = \frac{\delta_z(Q_v)}{2}$, then

$$rQ_v = \left\{(x, z) \mid x \in I_v, |z - \gamma_v(x)| \le r^2\ell\right\}.$$

Likewise, let c_w be the center of Q_w and $\gamma_w = \kappa_{P_w,c_w}$. We claim that $|\gamma_v(x)-\gamma_w(x)| \le 20\ell$ for $x \in I_w$.

Let $q = (x_0, z_0) \in Q_w \cap 3Q_v$. Since γ_v and $\kappa_{P_v,q}$ are parallel, we have $\gamma_v(x) = \kappa_{P_v,q}(x) + d$ for all x, where $d = \gamma_v(x_0) - z_0$. Since $q \in 3Q_v$, we have $|d| \le 9\ell$.

Similarly, $\gamma_w(x) = \kappa_{P_w,q}(x) + d'$, and since $q \in Q_w$ and $\delta_z(Q_w) \le 4\delta_z(Q_v)$,

$$|d'| = |\gamma_w(x_0) - z_0| \le \delta_z(Q_w) \le 8\ell.$$

Therefore, by Lemma 6.7, for all $x \in I_w$,

$$\begin{aligned}|\gamma_v(x)-\gamma_w(x)| &\le |d| + |d'| + |\kappa_{P_w,q}(x) - \kappa_{P_v,q}(x)| \\ &\le 17\ell + |\kappa_{P_w,q}(x) - \kappa_{\Gamma,q}(x)| + |\kappa_{\Gamma,q}(x) - \kappa_{P_v,q}(x)| \\ &\le 17\ell + \frac{\delta_z(Q_w)}{8} + \frac{\delta_z(Q_v)}{8} \le 20\ell.\end{aligned}$$

Let $p = (x, 0, z) \in Q_w$. Then $x \in I_w$ and $|z - \gamma_w(x)| \le \delta_z(Q_w) \le 8\ell$, so $|z-\gamma_v(x)| \le 28\ell$. Therefore $p \in 10Q_v$ and thus $Q_w \subset 10Q_v$. □

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