

THE STRONG GEOMETRIC LEMMA FOR INTRINSIC LIPSCHITZ GRAPHS IN HEISENBERG GROUPS

VASILEIOS CHOUSIONIS, SEAN LI, AND ROBERT YOUNG

ABSTRACT. We show that the β -numbers of intrinsic Lipschitz graphs of Heisenberg groups \mathbb{H}_n are locally Carleson integrable when $n \geq 2$. Our main bound uses a novel slicing argument to decompose intrinsic Lipschitz graphs into graphs of Lipschitz functions. A key ingredient in our proof is a Euclidean inequality that bounds the β -numbers of the original graph in terms of the β -numbers of many families of slices. This allows us to use recent work of Fässler and Orponen [FO19] which asserts that Lipschitz functions satisfy a Dorrnsoro inequality.

1. INTRODUCTION

In [Jon90], Jones characterized subsets $E \subset \mathbb{R}^2$ that lie on finite length rectifiable curves in the now-famous traveling salesman theorem. Given a ball $B(x, r) \subset \mathbb{R}^2$, he introduced the quantity

$$\beta_E(x, r) = \inf_L \sup_{z \in B(x, r) \cap E} \frac{d(z, L)}{r},$$

where the infimum is over all affine lines L in \mathbb{R}^2 . This quantity, known as the β -number, is a scale-invariant measure of how close $E \cap B(x, r)$ is to a line. Jones proved the following theorem.

Theorem 1.1. *A subset $E \subset \mathbb{R}^2$ lies on a finite length rectifiable curve if and only if*

$$(1) \quad \text{diam}(E) + \int_0^\infty \int_{\mathbb{R}^2} \beta_E(x, r)^2 dx \frac{dr}{r^2} < \infty.$$

Note that the measure $\frac{dr}{r^2}$ has a singularity at $r = 0$, so (1) can be viewed as saying E is close to a line ($\beta_E(x, r)$ is small) for “most” balls according to the measure $dx \frac{dr}{r^2}$. Thus, this can be viewed as a quantitative version of the classical Rademacher’s theorem, which implies that rectifiable curves are close to affine on infinitesimal scales. Integrals against measures of the form $dx \frac{dr}{r^n}$ are known as Carleson integrals.

Jones, in [Jon89], used the quantitative geometric information provided by (1) to give another proof of the L_2 -boundedness of the Cauchy transform on 1-dimensional Lipschitz graphs. Since Jones’s work, β -numbers and their variants have appeared frequently in harmonic analysis, geometric measure theory, and related fields. In particular, Theorem 1.1 has been generalized to various other spaces and settings [Oki92, Sch07, LS16a, LS16b, FFP07, CLZ19, Li19, DS17]. Moreover, the introduction of Carleson integrals of β -numbers was a starting point for the theory of quantitative rectifiability of David and Semmes [DS91, DS93]. Briefly, this theory provides several characterizations of Ahlfors regular sets of \mathbb{R}^n satisfying a Carleson bound similar to (1) via a suite of

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geometric and analytic conditions. Notably, it serves as a broad geometric framework for singular integrals acting on lower dimensional subsets of \mathbb{R}^n .

One of the settings where generalizations of the traveling salesman theorem have been studied is the setting of Carnot groups [LS16a, LS16b, FFP07, CLZ19, Li19]. The Carnot groups are a class of nilpotent Lie groups whose abelian members are precisely Euclidean spaces. Thus, they can be viewed as generalizations of \mathbb{R}^n . The simplest examples of nonabelian Carnot groups are the Heisenberg groups \mathbb{H}_n . Extending Theorem 1.1 and other aspects of quantitative rectifiability in Carnot groups contributes to the development of geometric measure theory on these sub-Riemannian spaces. For a recent overview of this research program, which started about 20 years ago, we refer the reader to the lecture notes [SC16].

An interesting challenge for researchers in sub-Riemannian geometric measure theory is to develop a robust theory of sub-Riemannian rectifiability. For example, the attempt to define rectifiability in the $(2n + 1)$ -dimensional Heisenberg group \mathbb{H}_n modeled after Federer's classical definition using Lipschitz images is problematic; Ambrosio and Kirchheim [AK00] proved that a Lipschitz image $f(\mathbb{R}^k) \subset \mathbb{H}_n$ has zero k -dimensional Hausdorff measure for $n + 1 \leq k \leq 2n$. This indicates that the notion of rectifiability changes drastically between the low-dimensional and the low-codimensional case.

Franchi, Serapioni and Serra-Cassano [FSSC06] introduced intrinsic Lipschitz graphs in order to define rectifiable sets of codimension 1 in \mathbb{H}_n . These are sets satisfying a cone condition, similar to the one satisfied by Euclidean Lipschitz graphs. The cone condition will be defined in the next section, but a notable class of examples is given by the fact that level sets of C^1 functions are locally intrinsic Lipschitz graphs away from critical points. Furthermore, just as Rademacher's theorem implies that Lipschitz graphs in \mathbb{R}^n can be approximated by planes almost everywhere, intrinsic Lipschitz graphs can be locally approximated by vertical planes almost everywhere [FSSC11]. The approach initiated in [FSSC06] has been quite successful, as several important subsequent contributions [MSSC10, FSSC11] laid down the foundations for a meaningful theory of low-codimensional rectifiability.

Recently, intrinsic Lipschitz graphs have also been used to study quantitative rectifiability. This approach was introduced independently in [CFO19b, NY18], which give quantitative bounds on how close intrinsic Lipschitz graphs are to vertical planes and vertical sets. See also [NY20, FOR18, Rig19] for some related recent results. One motivation for this work is that, as in the Euclidean case, the quantitative rectifiability of intrinsic Lipschitz graphs is related to the behavior of a natural singular integral arising in the study of removability for Lipschitz harmonic functions in \mathbb{H}_n , see [CFO19a, FO18].

In this paper, we bound the quantitative rectifiability of intrinsic Lipschitz graphs by proving bounds on the β -numbers of intrinsic Lipschitz graphs analogous to the bounds in Theorem 1.1. Given $E \subset \mathbb{H}_n$ (typically an intrinsic Lipschitz graph) and $x \in \mathbb{H}_n$, we define the codimension-1 version of the β -numbers as

$$(2) \quad \beta_E(x, r) = \inf_{L \in \text{VP}} \left[r^{-2n-1} \int_{B(x,r) \cap E} \left(\frac{d(y, L)}{r} \right)^2 d\mathcal{H}^{2n+1}(y) \right]^{1/2}$$

where in the infimum, L ranges over the set of vertical planes (planes containing the center) in \mathbb{H}_n . This is scale-invariant in the sense that $\beta_{\delta_t(E)}(\delta_t(x), tr) = \beta_E(x, r)$ for any $t > 0$.

Codimension-1 β -numbers in the Heisenberg group were previously considered in [CFO19b], where it was shown that a weaker qualitative analogue of (1), known as the

weak geometric lemma, holds for intrinsic Lipschitz graphs in \mathbb{H}_n . This result was written only for \mathbb{H}_1 in [CFO19b], but the proof extends to all \mathbb{H}_n . In this paper we will show that intrinsic Lipschitz graphs in \mathbb{H}_n , for $n \geq 2$, satisfy a local version of (1) known as the (strong) geometric lemma. We note that the (strong) geometric lemma implies the weak geometric lemma, see [DS93, Part I, 1.1.5] for a relevant discussion in Euclidean spaces.

Our main result is the following theorem.

Theorem 1.2. *Let $n \geq 2$ and let Γ be an intrinsic λ -Lipschitz graph in \mathbb{H}_n . Then, for any $R > 0$ and any ball $B = B(y, R) \subset \mathbb{H}_n$,*

$$(3) \quad \int_0^R \int_{B \cap \Gamma} \beta_{\Gamma}(x, r)^2 d\mathcal{H}^{2n+1}(x) \frac{dr}{r} \lesssim_{\lambda} R^{2n+1}.$$

This generalizes Dorronsoro's Theorem, which proves a similar inequality for Lipschitz graphs in \mathbb{R}^n . More generally, an Ahlfors regular subset of \mathbb{R}^n satisfies a Euclidean analogue of (3) if and only if it is uniformly rectifiable.

We prove Theorem 1.2 using a slicing technique inspired by [NY18]. In that paper, the authors decomposed intrinsic Lipschitz graphs into slices using cosets of a codimension-1 subgroup of \mathbb{H}_n whose intersection with the vertical plane $\{y_n = 0\}$ is a subgroup P isomorphic to \mathbb{H}_{n-1} . Each slice of the graph can then be viewed as the graph of a Lipschitz function over cosets of P . This allowed the authors to use results about functions from \mathbb{H}_{n-1} to \mathbb{R} to study intrinsic Lipschitz graphs of \mathbb{H}_n . Because the authors in [NY18] were concerned with bounding vertical fluctuations of intrinsic Lipschitz graphs, they only needed to use one subgroup in the slicing as any codimension-1 subgroup of \mathbb{H}_n contains the z -axis.

Unlike [NY18], we want to control the behavior of the intrinsic Lipschitz graph on the entirety of a ball, not just the vertical fluctuations. Thus, we need to slice the graph along cosets of multiple codimension-1 subgroups of \mathbb{H}_n . As these slices interact nonlinearly, we can extract information about the intrinsic Lipschitz graph in all directions via harmonic analytic techniques. A similar slicing argument in the Euclidean setting was used recently in [Orp18] although the methods employed there were more geometric.

Having reduced the problem to slices of the intrinsic Lipschitz graph, we are then able to use a result of Fässler and Orponen [FO19] which proved a version of Dorronsoro's Theorem for Lipschitz functions on Heisenberg groups. This gives a bound analogous to (3) for the slices.

Results of [NY20] strongly suggest that Theorem 1.2 does not generalize to the three-dimensional Heisenberg group. Indeed, Section 3 of [NY20] constructs examples of surfaces $\Gamma(\alpha, \rho)$, where α and ρ are large real numbers, such that

$$\int_0^R \int_{B \cap \Gamma(\alpha, \rho)} \beta_{\Gamma(\alpha, \rho)}(x, r)^p d\mathcal{H}^3(x) \frac{dr}{r} \gtrsim \alpha^{4-p}$$

for any $1 \leq p < 4$. These surfaces are not intrinsic Lipschitz, but they satisfy an intrinsic Sobolev condition with constant independent of α and ρ , and we believe the construction can be modified to produce intrinsic Lipschitz graphs with Lipschitz constant independent of α and ρ . We will investigate the optimal exponent for (3) in \mathbb{H}_1 in future work. More generally, the question of whether the strong geometric lemma holds for surfaces in a general Carnot group G remains open.

Roadmap. In Section 2, we introduce some basic notation and definition for intrinsic Lipschitz graphs. To prove Theorem 1.2, we need some results on balls in intrinsic graphs and their projections and a description of codimension-1 slices of intrinsic

graphs, which we present in Section 3. We will also need to pass between the non-parametric β -numbers defined in (2) (which are defined in terms of a graph Γ) and parametric β -numbers (defined in terms of a function f). We prove the necessary lemmas in Section 4. Finally, in Section 5, we prove Theorem 1.2, modulo one last lemma that bounds the β -numbers of a graph in terms of the β -numbers of its slices, and in Section 6, we prove this last lemma.

2. PRELIMINARIES

We define the Heisenberg group \mathbb{H}_n as the Lie group $(\mathbb{R}^{2n+1}, \cdot)$ where elements of \mathbb{R}^{2n+1} are written as (x, y, z) for $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}$. The group product is defined as

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \Omega((x, y), (x', y'))/2)$$

where $\Omega((x, y), (x', y')) = \sum_{i=1}^n (x_i y'_i - x'_i y_i)$ is the symplectic form on \mathbb{R}^{2n} . Note that two elements $(x, y, z), (x', y', z') \in \mathbb{H}_n$ do not commute unless $\Omega((x, y), (x', y')) = 0$. We will also use the standard commutator notation $[u, v] = uvu^{-1}v^{-1}$ for $u, v \in \mathbb{H}_n$. The identity element in \mathbb{H}_n is $\mathbf{0} = (0, 0, 0)$.

Let $X_1, \dots, X_n, Y_1, \dots, Y_n, Z$ be the coordinate vectors of \mathbb{H}_n and let $x_1, \dots, x_n, y_1, \dots, y_n, z: \mathbb{H}_n \rightarrow \mathbb{R}$ be the coordinate functions. The center of the group is $\langle Z \rangle = \{(0, 0, z) : z \in \mathbb{R}\}$. A subgroup $W \subset \mathbb{H}_n$ is *vertical* if it contains the center $\langle Z \rangle$. An element $w \in \mathbb{H}_n$ is said to be a *horizontal vector* if $z(w) = 0$, and we let A be the set of horizontal vectors. Let d be the Carnot–Carathéodory metric on \mathbb{H}_n . For any $h \in \mathbb{H}_n$ we let $\|h\| = d(\mathbf{0}, h)$.

We define a family of automorphisms

$$\begin{aligned} \delta_t: \mathbb{H}_n &\rightarrow \mathbb{H}_n \\ (x, y, z) &\mapsto (tx, ty, t^2z) \end{aligned}$$

for $t \in \mathbb{R}$. The δ_t dilate the metric in that for any $g, h \in \mathbb{H}_n$, we have

$$d(\delta_t(g), \delta_t(h)) = |t|d(g, h).$$

For any horizontal vector $w \in \mathbb{H}_n$ and $\alpha \in \mathbb{R}$, we define $w^\alpha = \delta_\alpha(w)$; when $\alpha \in \mathbb{Z}$, this agrees with the usual notion of exponentiation.

The projection

$$\begin{aligned} \pi: \mathbb{H}_n &\rightarrow \mathbb{R}^{2n} \\ (x, y, z) &\mapsto (x, y) \end{aligned}$$

is a Lipschitz homomorphism. A function $f: \mathbb{H}_n \rightarrow \mathbb{R}^k$ is *affine* if there is an affine function $\tau: \mathbb{R}^{2n} \rightarrow \mathbb{R}^k$ so that $f = \tau \circ \pi$.

A *vertical plane* V is a subset $V = \pi^{-1}(P) \subset \mathbb{H}_n$ where $P \subset \mathbb{R}^{2n}$ is a $(2n-1)$ -dimensional affine plane. One particular vertical plane, which we'll use very often is $V_0 = \{y_n = 0\}$. A function $f: V_0 \rightarrow \mathbb{R}$ is *vertical* if it is constant on cosets of $\langle Z \rangle$. We will denote by VF the set of vertical functions on V_0 , and by Aff the set of vertical affine functions on V_0 ; i.e., restrictions to V_0 of affine function $\mathbb{H}_n \rightarrow \mathbb{R}$. Note that these functions are of the form

$$f(v) = \sum_{i=1}^n \alpha_i x_i(v) + \sum_{i=1}^{n-1} \beta_i y_i(v) + \gamma.$$

Let $w \in A$ be a horizontal vector such that $y_n(w) = 1$, we let $\Pi_w: \mathbb{H}_n \rightarrow V_0$ be the map such that $\Pi_w(h)$ is the unique point of intersection of the coset $h\langle w \rangle$ and the vertical plane V_0 . This map projects \mathbb{H}_n to V_0 along cosets of $\langle w \rangle$, and we have $\Pi_w(h) =$

$hw^{-y_n(h)}$. For any $g, h \in \mathbb{H}_n$,

$$(4) \quad \Pi_w(gh) = gh \cdot w^{-y_n(gh)-y_n(h)} = g \left(h \cdot w^{-y_n(h)} \right) \cdot w^{-y_n(g)} = \Pi_w(g \Pi_w(h)).$$

For any function $f: V_0 \rightarrow \mathbb{R}$ and any horizontal vector w with $y_n(w) = 1$, we define the w -intrinsic graph of f as

$$\Gamma_{f,w} = \{v w^{f(v)} : v \in V_0\}.$$

For $0 < \lambda < 1$, let Cone_λ be the open double cone

$$\text{Cone}_\lambda = \{p \in \mathbb{H}_n : \lambda d(\mathbf{0}, p) < |y_n(p)|\}.$$

We say that a subset $\Gamma \subset \mathbb{H}_n$ is an *intrinsic λ -Lipschitz graph* if for every $x \in \Gamma$, $(x \cdot \text{Cone}_\lambda) \cap \Gamma = \emptyset$. For any such Γ and any $w \in \text{Cone}_\lambda$ with $y_n(w) = 1$, the restriction $\Pi_w|_\Gamma$ is injective, so we can define a function $f_w: \Pi_w(\Gamma) \rightarrow \mathbb{R}$, $f_w(\Pi_w(p)) = y_n(p)$ such that $\Gamma = \Gamma_{f_w,w}$. Conversely, if $\Gamma_{f,w}$ is an intrinsic λ -Lipschitz graph, we say that f is an *intrinsic (w, λ) -Lipschitz function*.

We finally record that the Hausdorff $(2n+1)$ -measure \mathcal{H}^{2n+1} is an Ahlfors $(2n+1)$ -regular measure when restricted to an intrinsic λ -Lipschitz graph Γ , see e.g. [FS16, Theorem 3.9]. That is, there exists a constant $C > 1$ depending only on λ so that

$$C^{-1} r^{2n+1} \leq \mathcal{H}^{2n+1}(B(x, r) \cap \Gamma) \leq C r^{2n+1}, \quad \forall x \in \Gamma, r > 0.$$

3. PROJECTIONS AND SLICES OF INTRINSIC LIPSCHITZ GRAPHS

We will sometimes need to pass between balls in \mathbb{H}_n and their projections to V_0 , which can be highly distorted. In this section, we define some quasiballs that will make this more convenient. Moreover, we will introduce a slicing method in order to obtain a family of decompositions of an intrinsic Lipschitz graph Γ into graphs of real valued Lipschitz functions with domain \mathbb{H}_{n-1} .

We start with some notation. For any subspace $S \subset A$, let S^Ω be the symplectic complement of S , i.e.,

$$S^\Omega = \{v \in A : \Omega(v, s) = 0 \text{ for all } s \in S\}.$$

The subspace $S \subset A$ is called *symplectic* if $S \cap S^\Omega = \emptyset$. In a similar manner, for any subspace $S \subset A$ we define

$$S^{\bar{\Omega}} = \{h \in \mathbb{H}_n : \bar{\Omega}(h, s) = 0 \text{ for all } s \in S\},$$

where $\bar{\Omega}: \mathbb{H}_n \times \mathbb{H}_n \rightarrow \mathbb{R}$ is the alternating form $\bar{\Omega}(g, h) = \Omega(\pi(g), \pi(h))$. Note that $[g, h] = \bar{\Omega}(g, h)Z$ for all $g, h \in \mathbb{H}_n$.

Slightly abusing notation, for a vector $w \in A$ we let

$$v^\Omega = \langle w \rangle^\Omega \text{ and } v^{\bar{\Omega}} = \langle w \rangle^{\bar{\Omega}} = \pi^{-1}(w^\Omega).$$

Note that since $\Omega(w, w) = 0$, we have $w \in w^\Omega$. It is well known that for any subspace $S \subset A$ it holds that $\dim S + \dim S^\Omega = 2n$, see e.g. [Ber01, Chapter 1]. Hence, for every horizontal vector w such that $y_n(w) = 1$, the complement w^Ω is a $(2n-1)$ -dimensional horizontal subspace that contains w . For such w , we let $C_w = V_0 \cap w^\Omega$; this is a $(2n-2)$ -dimensional horizontal subspace of V_0 . Note that C_w does not uniquely determine w , since we have $C_w = C_{w+tX_n}$ for any $t \in \mathbb{R}$. Let also

$$P_w = V_0 \cap w^{\bar{\Omega}} = C_w + \langle Z \rangle,$$

and observe that this is a $(2n-1)$ -dimensional vertical subspace of V_0 .

Let w and C_w be as above and let $v \in A \cap V_0$ be a horizontal unit vector orthogonal to C_w ; this is unique up to sign. Let

$$R_w = \{sv + p + tZ \in V_0 : p \in B_{C_w}(\mathbf{0}, 1), |s| \leq 1, |t| \leq 1\},$$

where $B_{C_w}(\mathbf{0}, 1)$ is the unit ball in C_w . For any $g \in \mathbb{H}_n$ and any $r > 0$, we define

$$Q_w(g, r) = \Pi_w(g\delta_r(R_w)).$$

Since $R_w \subset V_0$, when $g = \mathbf{0}$, we simply have $Q_w(\mathbf{0}, r) = \delta_r(R_w)$. Note that the map $x \mapsto \Pi_w(gx)$ preserves Lebesgue measure on V_0 , so if μ is Lebesgue measure, then

$$(5) \quad \mu(Q_w(g, r)) = \mu(\delta_r(R_w)) \approx r^{2n+1}.$$

We now recall the definition of quasiballs. Let (X, ρ) be a metric space and let $\lambda \geq 1$. Recall that a λ -quasiball (or simply a quasiball if λ is understood) is any set $E \subset X$ for which there exist $x \in X$ and $R > 0$ for which

$$B_\rho(x, R) \subseteq E \subseteq B_\rho(x, \lambda R).$$

In the next lemma we'll show that when we slice $Q_w(g, r)$ with cosets of P_w we obtain quasiballs.

Lemma 3.1. *Let $w \in A$ with $y_n(w) = 1$. For any $g \in \mathbb{H}_n$, any $r > 0$, and any $u \in V_0$, the intersection $Q_w(g, r) \cap uP_w$ is either a quasiball of radius $\approx r$ or empty.*

Proof. Let R_w and v be as above. We first consider the case $g = \mathbf{0}$. For $a \in \mathbb{R}$, let $D_a = R_w \cap v^a P_w$. When $|a| \leq 1$,

$$D_a = \{av + p + tZ : p \in B_{C_w}(\mathbf{0}, 1), |t| \leq 1\}$$

is a quasiball of radius ≈ 1 in $v^a P_w$. Otherwise, when $|a| > 1$, $D_a = \emptyset$.

Every coset uP_w can be written $v^s P_w$ for some $s \in \mathbb{R}$, and

$$Q_w(\mathbf{0}, r) \cap v^s P_w = \delta_r(R_w \cap v^{\frac{s}{r}} P_w) = \delta_r(D_{\frac{s}{r}}).$$

This is either a quasiball of radius $\approx r$ or the empty set.

Suppose $g \in \mathbb{H}_n$ and $s \in \mathbb{R}$. Then

$$\begin{aligned} Q_w(g, r) \cap gv^s P_w &= \Pi_w(gQ_w(\mathbf{0}, r) \cap gv^s \text{span}(P_w, w)) \\ &= \Pi_w(g\delta_r(D_{\frac{s}{r}})) = \delta_r(\Pi_w(\delta_{r^{-1}}(g)D_{\frac{s}{r}})). \end{aligned}$$

We claim that $\Pi_w(\delta_{r^{-1}}(g)D_{\frac{s}{r}})$ is isometric to $D_{\frac{s}{r}}$. It suffices to consider the case $r = 1$.

Let $B = D_s$ and let $\alpha = -y_n(g)$. Since $B \subset V_0$, we have $y_n(gb) = y_n(g)$ for all $b \in B$, so $\Pi_w(gb) = gbw^\alpha$. Since $B \subset v^s P_w = sv + P_w$ and $\bar{\Omega}(w, P_w) = 0$, we have $\bar{\Omega}(b, w) = \bar{\Omega}(sv, w)$. That is, $[b, w]$ is independent of the choice of b . Therefore,

$$\Pi_w(gb) = gbw^\alpha = g[b, w^\alpha]w^\alpha b = gZ^{\alpha\Omega(b, w)}w^\alpha b = gZ^{s\alpha\Omega(v, w)}w^\alpha b.$$

Therefore, $\Pi_w(gB) = gZ^{s\alpha\Omega(v, w)}w^\alpha B$ is a left-translate of B , so $\Pi_w(gB)$ is a quasiball of radius $\approx r$ in $gv^s P_w$. \square

Let Γ be an intrinsic Lipschitz graph. When $g \in \Gamma$ is large, $Q_w(g, r)$ can be highly distorted, but the following lemma shows that $Q_w(g, r)$ is the projection of a quasiball with respect to the induced metric on Γ . Using this property, we'll show that the pushforward measure of $\mathcal{H}^{2n+1}|_\Gamma$ by Π_w is globally equivalent to the Lebesgue measure on V_0 .

Lemma 3.2. *Let $0 < \lambda < \lambda' < 1$. There is a $c > 0$ depending on λ and λ' such that for any intrinsic λ -Lipschitz graph Γ such that $\Pi_{Y_n}(\Gamma) = V_0$, any $g \in \Gamma$, any $r > 0$, and any horizontal vector $w \in A \cap \text{Cone}_{\lambda'}$ such that $y_n(w) = 1$:*

- (1) There is a function $f_w: V_0 \rightarrow \mathbb{R}$ such that $\Gamma = \Gamma_{f_w, w}$.
- (2) $Q_w(g, c^{-1}r) \subset \Pi_w(B(g, r) \cap \Gamma) \subset Q_w(g, cr)$.
- (3) Let μ be Lebesgue measure on V_0 . Then $\mu \approx (\Pi_w)_*(\mathcal{H}^{2n+1}|_\Gamma)$.

Proof. We start with the proof of (1). Let $L = (\frac{1}{\lambda} - \frac{1}{\lambda'})^{-1} > 0$. Since $\Pi_{Y_n}(\Gamma) = V_0$, the complement $\mathbb{H} \setminus \Gamma$ has two connected components, Γ^+ and Γ^- . Since $\mathbf{0} \in \Gamma$, the double cone Cone_λ is disjoint from Γ , and we can label its two connected components Cone_λ^+ and Cone_λ^- so that $\text{Cone}_\lambda^\pm \subset \Gamma^\pm$.

Let $v \in V_0$. We claim that there is a unique $t_0(v) = t_0 \in \mathbb{R}$ such that $vw^{t_0} \in \Gamma$ and that $|t_0| \leq Ld(\mathbf{0}, v)$. Suppose that $t > Ld(\mathbf{0}, v)$ and let $h = vw^t$. Note that $h \in \text{Cone}_\lambda^+$. Indeed, since $w \in \text{Cone}_{\lambda'}$, we have $|y_n(w)| > \lambda' d(\mathbf{0}, w) = \lambda' \|w\|$, i.e., $\|w\| < (\lambda')^{-1}$. By the triangle inequality,

$$d(\mathbf{0}, h) - \frac{y_n(h)}{\lambda} \leq d(\mathbf{0}, v) + t \cdot \|w\| - \frac{t}{\lambda} \leq d(\mathbf{0}, v) + \frac{t}{\lambda'} - \frac{t}{\lambda} = d(\mathbf{0}, v) - Lt < 0,$$

so $d(\mathbf{0}, h) < \frac{y_n(h)}{\lambda}$ and thus $vw^t \in \text{Cone}_\lambda^+$. Likewise, $vw^{-t} \in \text{Cone}_\lambda^-$. These points are on different sides of Γ , so there is some t_0 such that $vw^{t_0} \in \Gamma$; in fact, we can take $|t_0| \leq Ld(\mathbf{0}, v)$. Since $w^\alpha \in \text{Cone}_\lambda$ for any $\alpha \neq 0$, this t_0 is unique. Therefore, Γ is a w -intrinsic graph with $\Pi_w(\Gamma) = V_0$, and we define $f_w: V_0 \rightarrow \mathbb{R}$ to be the function such that $vw^{f_w(v)} \in \Gamma$ for all $v \in V_0$. In particular, $f_w(v) = t_0(v)$.

For the proof of (2), note that by scaling, we may assume $r = 1$. We first prove (2) for $g = \mathbf{0}$ (so we are assuming $\mathbf{0} \in \Gamma$). If $p \in B(\mathbf{0}, 1)$, then $|y_n(p)| \leq 1$, so

$$d(\mathbf{0}, \Pi_w(p)) \leq d(\mathbf{0}, p) + d(p, \Pi_w(p)) \leq 1 + |y_n(p)| \cdot \|w\| \lesssim_{\lambda'} 1.$$

Therefore, there is a $C > 0$ depending on λ' such that $\Pi_w(B(\mathbf{0}, 1)) \subset V_0 \cap B(\mathbf{0}, C)$, and if c is sufficiently large (depending only on λ'), then $\Pi_w(B(\mathbf{0}, 1)) \subset Q_w(\mathbf{0}, c)$. This proves one inclusion.

We now prove the other inclusion. Note first, that

$$d(\mathbf{0}, vw^{f_w(v)}) \leq d(\mathbf{0}, v) + \|w\| |f_w(v)| \leq \left(1 + \frac{L}{\lambda'}\right) d(\mathbf{0}, v)$$

and consequently $vw^{f_w(v)} \in \Gamma \cap B(\mathbf{0}, Dd(\mathbf{0}, v))$ for all $v \in V_0$, where $D = 1 + \frac{L}{\lambda'}$. Thus, if $d(\mathbf{0}, v) < D^{-1}$, then $v \in \Pi_w(B(\mathbf{0}, 1) \cap \Gamma)$. If c is sufficiently large (depending only on λ, λ'), then

$$Q_w(\mathbf{0}, c^{-1}) \subset V_0 \cap B(\mathbf{0}, D^{-1}) \subset \Pi_w(B(\mathbf{0}, 1) \cap \Gamma),$$

as desired.

Now assume $g \neq \mathbf{0}$. Then $g^{-1}\Gamma$ is still an intrinsic Lipschitz graph which now contains $\mathbf{0}$, so

$$(6) \quad Q_w(\mathbf{0}, c^{-1}) \subset \Pi_w(B(\mathbf{0}, 1) \cap g^{-1}\Gamma) \subset Q_w(\mathbf{0}, c).$$

Let $h' \in B(g, 1) \cap \Gamma$. Then $h' = gh$ for some $h \in B(\mathbf{0}, 1) \cap g^{-1}\Gamma$. Let $y = \Pi_w(h)$; by the upper bound of (6), $y \in Q_w(\mathbf{0}, c) = \delta_c(R_w)$ and $y = hw^\alpha$ for some $\alpha \in \mathbb{R}$. Then $gy\langle w \rangle = gh\langle w \rangle = h'\langle w \rangle$, so $\Pi_w(h') = \Pi_w(gy)$ and thus $\Pi_w(h') \in \Pi_w(g\delta_c(R_w)) = Q_w(g, c)$. This proves the upper bound of part 2, and the lower bound follows similarly.

To prove the last part, let $\Psi: V_0 \rightarrow \Gamma$ be the map $\Psi(v) = vw^{f_w(v)}$, so that Ψ is inverse to $\Pi_w|_\Gamma$. By (5) and part 2, $\mu(\Pi_w(B(x, r) \cap \Gamma)) \approx r^{2n+1}$ for any $x \in \Gamma$. Thus, by Theorem 2.4.3 of [AT04], $\Psi_*(\mu)|_\Gamma \approx \mathcal{H}^{2n+1}|_\Gamma$ and thus $\mu \approx (\Pi_w)_*(\mathcal{H}^{2n+1}|_\Gamma)$. \square

For every $k \leq n$, we identify \mathbb{H}_k with $\text{span}(X_1, \dots, X_k, Y_1, \dots, Y_k, Z) \subset \mathbb{H}_n$. The vertical subspace $\{y_n = 0\}$ is then the internal direct product of \mathbb{H}_{n-1} and $\langle X_n \rangle$. In fact, every codimension-1 vertical subspace $P \subset \mathbb{H}_n$ is isomorphic to $\mathbb{H}_{n-1} \times \mathbb{R}$, so the cosets of P decompose \mathbb{H}_n into copies of $\mathbb{H}_{n-1} \times \mathbb{R}$. In the following, we will use these decompositions to construct a family of decompositions of an intrinsic Lipschitz graph Γ into a union of graphs of Lipschitz functions $\mathbb{H}_{n-1} \rightarrow \mathbb{R}$.

Recall that for every $w \in A$, we defined $P_w = V_0 \cap w^{\overline{\Omega}}$; this is a codimension-1 subspace of V_0 . Then $P_w \cap \langle w \rangle = \{\mathbf{0}\}$, $P_w \cdot \langle w \rangle = w^{\overline{\Omega}}$, and w and P_w commute, so $w^{\overline{\Omega}} \cong \langle w \rangle \times P_w$. Furthermore, since $\pi(P_w) = \text{span}(X_n, w)^{\overline{\Omega}}$ and $\text{span}(X_n, w)$ is a symplectic subspace of A , $\pi(P_w)$ is a symplectic subspace and thus P_w is isomorphic to \mathbb{H}_{n-1} .

By Lemma 3.2, if Γ is an intrinsic Lipschitz graph, then for every horizontal vector $w \in A$ that is sufficiently close to Y_n , there is a function f_w such that $\Gamma = \Gamma_{f_w, w}$. These functions satisfy a Lipschitz condition on cosets of P_w .

Lemma 3.3. *Let $0 < \lambda < \lambda' < 1$. There is an $L > 0$ such that if Γ is an intrinsic λ -Lipschitz graph, $w \in A \cap \text{Cone}_{\lambda'}$, $y_n(w) = 1$, and $f_w: V_0 \rightarrow \mathbb{R}$ is the parametrizing function of Γ , i.e., $\Gamma = \Gamma_{f_w, w}$, then for any $g \in V_0$, the restriction $f_w|_{gP_w}$ is L -Lipschitz with respect to $d_{\mathbb{H}_n}$.*

Proof. Let $f_w: V_0 \rightarrow \mathbb{R}$ be the function such that $\Gamma = \Gamma_{f_w, w}$, as in Lemma 3.2 and let $g \in \mathbb{H}_n$. We claim that $f_w|_{gP_w}$ is Lipschitz. We may assume $g = \mathbf{0}$ as d is left-invariant. Define $\Psi: \Pi_w(\Gamma) \rightarrow \Gamma$, $\Psi(u) = uw^{f_w(u)}$.

Let $u, v \in P_w \cap \Pi_w(\Gamma)$. Since $\Psi(u), \Psi(v) \in \Gamma$, the intrinsic Lipschitz condition implies that $\Psi(u) \notin \Psi(v) \cdot \text{Cone}_{\lambda}$ or

$$h := \Psi(v)^{-1}\Psi(u) \notin \text{Cone}_{\lambda}.$$

Since $v^{-1}u \in P_w$ is in the $\overline{\Omega}$ -complement of w , it commutes with w , so we can decompose h as

$$h = (vw^{f_w(v)})^{-1}(uw^{f_w(u)}) = w^{-f_w(v)}v^{-1}uw^{f_w(u)} = v^{-1}u \cdot w^{f_w(u)-f_w(v)}.$$

Then, since $v^{-1}u \in V_0$,

$$(7) \quad |y_n(h)| = |y_n(w^{f_w(u)-f_w(v)})| = |f_w(u) - f_w(v)|.$$

Moreover, since $w \in \text{Cone}_{\lambda'}$ and $y_n(w) = 1$,

$$(8) \quad \begin{aligned} d(h, \mathbf{0}) &\leq d(u, v) + d(w^{f_w(u)}, w^{f_w(v)}) \\ &= d(u, v) + d(\mathbf{0}, w)|f_w(v) - f_w(u)| \\ &\leq d(u, v) + \frac{1}{\lambda'}|f_w(v) - f_w(u)|. \end{aligned}$$

Since $h \notin \text{Cone}_{\lambda}$, we have $|y_n(h)| \leq \lambda d(\mathbf{0}, h)$. Then,

$$|f_w(u) - f_w(v)| \stackrel{(7)}{=} |y_n(h)| \leq \lambda d(\mathbf{0}, h) \stackrel{(8)}{\leq} \lambda \left(d(u, v) + \frac{1}{\lambda'}|f_w(v) - f_w(u)| \right)$$

and by the fact that $\lambda' > \lambda$,

$$|f_w(u) - f_w(v)| \leq \frac{\lambda}{1 - \frac{\lambda}{\lambda'}} d(u, v) = \frac{\lambda\lambda'}{\lambda' - \lambda} d(u, v).$$

□

4. COMPARING β -NUMBERS

Since Theorem 1.2 deals with intrinsic graphs, its proof will use parametric versions of β -numbers, i.e., quantities that measure how close a function f is to an affine function rather than how close the graph Γ_f is to a plane. In Lemma 3.3, we saw that there are many different ways of writing Γ as a graph, all of which lead to different parametric β -numbers. In this section, we prove some inequalities comparing parametric and non-parametric β -numbers.

We first prove the following simple lemma.

Lemma 4.1. *Let $0 < \lambda < \lambda' < 1$. There is a $c > 0$ with the following property. Let $w \in A \cap \text{Cone}_{\lambda}$ be such that $y_n(w) = 1$. Let $f: V_0 \rightarrow \mathbb{R}$ be an intrinsic (w, λ) -Lipschitz function and let $\Gamma := \Gamma_{f,w}$ be its intrinsic graph. For any $r > 0$ and any $x \in \Gamma$,*

$$(9) \quad \beta_{\Gamma}(x, r) \lesssim_{\lambda, \lambda'} r^{-\frac{2n+3}{2}} \inf_{h \in \text{Aff}} \|f - h\|_{L_2(Q_w(x, cr))}.$$

Proof. Let c be as in Lemma 3.2 so that $\Pi_w(B(x, r) \cap \Gamma) \subset Q_w(x, cr)$. Let $Q = Q_w(x, cr)$. Let $h \in \text{Aff}$ be the affine function that minimizes $\|h - f\|_{L_2(Q)}$. Let $S = \Gamma_{h,w}$ and let μ be Lebesgue measure on V_0 . Then S is a vertical plane, so

$$\begin{aligned} \beta_{\Gamma}(x, r)^2 &\leq r^{-2n-3} \int_{B(x, r) \cap \Gamma_f} d(y, S)^2 d\mathcal{H}^{2n+1}(y) \\ &\stackrel{\text{Lem. 3.2}}{\approx}_{\lambda, \lambda'} r^{-2n-3} \int_{\Pi_w(B(x, r) \cap \Gamma_f)} d(vw^{f(v)}, S)^2 d\mu(v) \\ &\leq r^{-2n-3} \int_Q d(vw^{f(v)}, vw^{h(v)})^2 d\mu(v) \\ &\lesssim_{\lambda, \lambda'} r^{-2n-3} \|f - h\|_{L_2(Q)}^2. \end{aligned}$$

Taking square roots of both sides gives the desired inequality. \square

Let $P \subset V_0$ be a vertical subgroup. We say that a measurable function f is P -slice affine if for every $v \in V_0$, the restriction of f to vP is a vertical affine function. Let SAff_P be the set of P -slice affine functions. Note that we have the following inclusion of function spaces:

$$(10) \quad \text{Aff} \subseteq \text{SAff}_P \subseteq \text{VF}.$$

The second result we will need deals with different parameterizing functions for the same graph. Suppose that Γ is an intrinsic Lipschitz graph, w and w' are horizontal vectors with $y_n(w) = y_n(w') = 1$ and $[w, w'] = 0$. Moreover, assume that f and f' are two parameterizing functions for Γ , so that $\Gamma = \Gamma_{f,w} = \Gamma_{f',w'}$. The relationship between f and f' is nonlinear. Indeed, by Lemma 3.3, f is Lipschitz on cosets of $P = P_w = V_0 \cap w\bar{\Omega}$, while f' need not be. Regardless, we will show that if f is close to P -slice affine, then f' is close to P -slice affine too.

Lemma 4.2. *Let $0 < \lambda < 1$. There exist $c > 0$ and $\lambda' \in (\frac{1+\lambda}{2}, 1)$ depending only on λ so that the following holds. Let $w, w' \in A \cap \text{Cone}_{\lambda}$ be horizontal vectors with $y_n(w) = y_n(w') = 1$ and $[w, w'] = 0$.*

Let Γ be a λ -intrinsic Lipschitz graph with parameterizing functions $f, f': V_0 \rightarrow \mathbb{R}$ such that $\Gamma = \Gamma_{f,w} = \Gamma_{f',w'}$. Then, for any $x \in \Gamma$ and $r > 0$,

$$(11) \quad \min_{\sigma \in \text{SAff}_{P_w}} \|f' - \sigma\|_{L_2(Q_{w'}(x, r))} \lesssim \min_{\sigma \in \text{SAff}_{P_w}} \|f - \sigma\|_{L_2(Q_w(x, cr))}.$$

The key to this bound is that since $[w, w'] = 0$, we have $w - w' \in P_w$ and thus $P_w + \langle w \rangle = P_w + \langle w' \rangle$. Thus, if $F \in \text{SAff}_{P_w}$ satisfies a Lipschitz bound on each slice, then there is an $F' \in \text{SAff}_{P_w}$ such that $\Gamma_{F,w} = \Gamma_{F',w'}$.

We will need two lemmas dealing with affine functions on slices.

Lemma 4.3. *Let $\mu > 0$. There is a constant $c > 1$ depending only on μ and n so that if $f: \mathbb{H}_n \rightarrow \mathbb{R}$ is a Lipschitz function and $U \subset \mathbb{H}_n$ is a μ -quasiball, then the affine function $F: \mathbb{H}_n \rightarrow \mathbb{R}$ such that*

$$\|f - F\|_{L_2(U)} = \inf_{T \in \text{Aff}} \|f - T\|_{L_2(U)}$$

satisfies $\|F\|_{\text{Lip}} \leq c \|f\|_{\text{Lip}}$, where the infimum is taken over affine functions on \mathbb{H}_n .

Proof. Since U is a μ -quasiball, there is a $u_0 \in U$ and an $r > 0$ such that

$$B(u_0, r) \subset U \subset B(u_0, \mu r).$$

By rescaling and translation, we may suppose that $r = 1$ and $u_0 = \mathbf{0}$. Let $\langle g, h \rangle_U = \int_U g h d\mathcal{H}^{2n+2}$ be the inner product on $L_2(U)$ and let $\|\cdot\|_2$ denote the norm in $L_2(U)$. Let $f_0 = f - f(\mathbf{0})$, and let $F_0 = F - f(\mathbf{0})$. Then

$$\|f_0 - F_0\|_2 = \inf_{T \in \text{Aff}} \|f_0 - T\|_2,$$

so it suffices to prove the lemma for f_0 and F_0 .

Let

$$C_1 = \max_{K \in \text{Aff}} \frac{\|K\|_{\text{Lip}}}{\|K\|_{L_2(B(\mathbf{0},1))}} = \max_{K \in \text{Aff}} \frac{\|K\|_{\text{Lip}}}{\|K - f_{B(\mathbf{0},1)}\|_{L_2(B(\mathbf{0},1))} d\mathcal{H}^{2n+2}}.$$

This is finite by compactness, and if $K \in \text{Aff}$, then

$$\|K\|_{\text{Lip}} \leq C_1 \|K\|_{L_2(B(\mathbf{0},1))} \leq C_1 \|K\|_2.$$

Let

$$C_2 = \max_{\substack{\|g\|_{\text{Lip}} < \infty \\ g(\mathbf{0})=0}} \frac{\|g\|_2}{\|g\|_{\text{Lip}}}.$$

If $g(\mathbf{0}) = 0$, then $\|g\|_{L_\infty(U)} \leq \mu \|g\|_{\text{Lip}}$, so

$$C_2 \leq \mu \sqrt{\mathcal{H}^{2n+2}(U)} \lesssim \mu^{n+2}.$$

Since $f_0 - F_0$ is orthogonal to F_0 , we have $\|F_0\|_2 \leq \|f_0\|_2$ and thus

$$\|F_0\|_{\text{Lip}} \leq C_1 \|F_0\|_2 \leq C_1 \|f_0\|_2 \leq C_1 C_2 \|f_0\|_{\text{Lip}} \lesssim \mu^{n+2} \|f_0\|_{\text{Lip}}. \quad \square$$

Lemma 4.4. *Let $L > 0$. There is a $\lambda \in (0, 1)$ such that if $w, w' \in A \cap \text{Cone}_\lambda$, $y_n(w) = y_n(w') = 1$, $[w, w'] = 0$, and $T: P_w \rightarrow \mathbb{R}$ is an affine function with $\text{Lip}(T) < L$, then there is an affine function $T': P_w \rightarrow \mathbb{R}$ such that $\Gamma_{T,w} = \Gamma_{T',w'}$ and $\text{Lip}(T') < 2L$.*

Proof. Let $C_w = \pi(P_w)$, which we think of as a vector space with norm $\|\cdot\|$. As affine functions do not depend on the z -coordinate, we see that $T = \tau \circ \pi$ where $\tau: C_w \rightarrow \mathbb{R}$ is affine and $\text{Lip}(T) = \text{Lip}(\tau)$. Let $\lambda \in (0, 1)$ be such that if $w, w' \in A \cap \text{Cone}_\lambda$ and $y_n(w) = y_n(w') = 1$, then $\|w - w'\| < (2L)^{-1}$. Let $s = w - w'$.

Since $[w, s] = -[w, w'] = 0$ and $y_n(s) = 0$, we have $s \in P_w$. Let $M: P_w \rightarrow P_w$ be the map

$$M(p) = \Pi_{w'}(p w^{T(p)}) = p w^{T(p)} (w')^{-T(p)} = p s^{T(p)} \quad \forall p \in P_w.$$

Since $M(v + tZ) = M(v) + tZ$ for all $v \in P_w$ and $t \in \mathbb{R}$, M descends to a map $m: C_w \rightarrow C_w$. Let $m = \pi \circ M|_{C_w}$. By our choice of λ , for all $p, q \in C_w$,

$$\|m(p) - m(q)\| = \|p - q + (\tau(p) - \tau(q))s\| \geq \|p - q\| - \text{Lip}(\tau)\|p - q\| \cdot \|s\| \geq \frac{\|p - q\|}{2}.$$

Therefore m and M are invertible and $\text{Lip}(m^{-1}) \leq 2$.

Let $T'(M(p)) = T(p)$ for all $p \in P_w$. This is affine and by construction,

$$pw^{T(p)} = M(p)(w')^{T(p)} = M(p)(w')^{T'(M(p))},$$

so $\Gamma_{T,w} = \Gamma_{T',w'}$. Finally, $T' = \tau \circ m^{-1} \circ \pi$, so $\text{Lip}(T') = \text{Lip}(\tau \circ m^{-1}) < 2L$. \square

Now, we prove Lemma 4.2.

Proof of Lemma 4.2. By Lemma 3.2, there is a $c_0 > 0$ such that for every $x \in \Gamma$ and every $r > 0$, $Q_{w'}(x, r) \subset \Pi_{w'}(B(x, c_0 r) \cap \Gamma)$ and $\Pi_w(B(x, r) \cap \Gamma) \subset Q_w(x, c_0 r)$. Let $c = c_0^2$.

Let $P = P_w$. We first consider a single slice $Q_w(x, cr) \cap vP$; without loss of generality, we may take $v = \mathbf{0}$. Let $D := Q_w(x, cr) \cap P$ and $D' := Q_{w'}(x, r) \cap P$. Let $T: P \rightarrow \mathbb{R}$ be the affine function minimizing $\|f - T\|_{L_2(D)}$. We know from Lemma 3.3 that f is Lipschitz on P , and if $\frac{1+\lambda}{2} < \lambda' < 1$, then $\text{Lip}(f|_P)$ is bounded by a function of λ . Therefore, by Lemma 4.3, $\text{Lip}(T)$ is also bounded by a function of λ , say $\text{Lip}(T) < L = L(\lambda)$.

The intrinsic graph $\Gamma_{T,w}$ is a plane which we call R . We suppose that $\lambda' \in (0, 1)$ is close enough to 1 that we may apply Lemma 4.4 to find an affine function $T': P \rightarrow \mathbb{R}$ such that $R = \Gamma_{T',w'}$ and $\text{Lip}(T') \leq 2L$.

Let $g \in \Gamma \cap (P + \langle w \rangle)$ and $u = \Pi_w(g)$. Then $g = uw^{f(u)}$ and thus $f(u) = y_n(g)$. Let $h = gw^{T(u)-f(u)}$; then $h = uw^{T(u)} \in R$. Let $u' = \Pi_{w'}(g)$ and $v' = \Pi_{w'}(h)$, so that $f(u) = f'(u')$ and $T(u) = T'(v')$. Since $[w, w'] = 0$, we let $s = w - w'$ and write

$$u' = uw^{f(u)}(w')^{-f(u)} = us^{f(u)}$$

and $v' = us^{T(u)}$.

We have

$$\begin{aligned} |f'(u') - T'(u')| &\leq |f(u) - T(u)| + |T(u) - T'(u')| \\ &= |f(u) - T(u)| + |T'(v') - T'(u')| \leq |f(u) - T(u)| + \text{Lip}(T')d(u', v'). \end{aligned}$$

Since $d(u', v') = \|s\| \cdot |f(u) - T(u)|$,

$$(12) \quad |f'(\Pi_{w'}(g)) - T'(\Pi_{w'}(g))| \lesssim |f(\Pi_w(g)) - T(\Pi_w(g))|$$

for every $g \in \Gamma \cap (P + \langle w \rangle)$.

Applying the argument above to every coset of P shows that if $\sigma \in \text{SAff}_P$ minimizes $\|f - \sigma\|_{L_2(Q_w(x, cr))}$, then there is a slice-affine function $\sigma' \in \text{SAff}_P$ such that

$$(13) \quad |f'(\Pi_{w'}(g)) - \sigma'(\Pi_{w'}(g))| \lesssim |f(\Pi_w(g)) - \sigma(\Pi_w(g))|$$

for every $g \in \Gamma$.

By Lemma 3.2, if μ is Lebesgue measure on V_0 , then

$$(14) \quad \mu \approx (\Pi_w)_*(\mathcal{H}^{2n+1}|_\Gamma) \approx (\Pi_{Y_n})_*(\mathcal{H}^{2n+1}|_\Gamma).$$

Therefore,

$$\begin{aligned}
\|f' - \sigma'\|_{L_2(Q_{w'}(x,r))}^2 &\stackrel{(13)}{\lesssim} \int_{\Gamma \cap B(x, \sqrt{cr})} (f'(\Pi_{w'}(g)) - \sigma'(\Pi_{w'}(g)))^2 d\mathcal{H}^{2n+1}(g) \\
&\stackrel{(14)}{\lesssim} \int_{\Gamma \cap B(x, \sqrt{cr})} (f(\Pi_w(g)) - \sigma(\Pi_w(g)))^2 d\mathcal{H}^{2n+1}(g) \\
&\stackrel{(13)}{\lesssim} \|f - \sigma\|_{L_2(Q_w(x,cr))}^2.
\end{aligned}$$

Thus

$$\|f' - \sigma'\|_{L_2(Q_{w'}(x,r))} \lesssim \|f - \sigma\|_{L_2(Q_w(x,cr))} = \min_{\sigma \in \text{SAff}_p} \|f - \sigma\|_{L_2(Q_w(x,cr))},$$

as desired. \square

5. STRONG GEOMETRIC LEMMA FOR INTRINSIC LIPSCHITZ GRAPHS

In this section, we will prove Theorem 1.2, modulo a bound that we will prove in Section 6. Our strategy is to reduce from the intrinsic Lipschitz graph Γ to Lipschitz graphs on cosets of \mathbb{H}_{n-1} by slicing. By the results of Section 3, there are codimension-1 vertical subgroups $W \subset \mathbb{H}_n$ such that the intersections $\nu W \cap \Gamma$ are graphs of Lipschitz functions defined on subgroups isomorphic to \mathbb{H}_{n-1} .

We start with some standard notation. Let $f: \mathbb{H}_n \rightarrow \mathbb{R}$ be a smooth function. The *horizontal derivatives* of f in the directions $X_i, Y_i, i = 1, \dots, n$, are the left invariant vector fields:

$$X_i f(h) := \frac{\partial f}{\partial x_i}(h) - \frac{1}{2} y_i(h) \frac{\partial f}{\partial z}(h) \text{ and } Y_i f(h) := \frac{\partial f}{\partial y_i}(h) + \frac{1}{2} x_i(h) \frac{\partial f}{\partial z}(h), \quad h \in \mathbb{H}_n.$$

The *horizontal gradient* of f is defined as

$$\nabla_H f(h) = (X_1 f(h), \dots, X_n f(h), Y_1 f(h), \dots, Y_n f(h)), \quad h \in \mathbb{H}_n.$$

Fässler and Orponen proved that Lipschitz functions on Heisenberg groups satisfy the following version of Dorrnsoro's Theorem. For any function $f: \mathbb{H}_n \rightarrow \mathbb{R}$ and any $x \in \mathbb{H}_n, r > 0$, let

$$(15) \quad \theta_f(B(x, r)) = r^{-2n-4} \inf_{g \in \text{Aff}} \|f - g\|_{L_2(B(x,r))}^2 \approx \inf_{g \in \text{Aff}} \int_{B(x,r)} \frac{(f(x) - g(x))^2}{r^2} dx.$$

Theorem 5.1. [FO19, Theorem 6.1] *Let $f \in L_2(\mathbb{H}_n)$ with $\nabla_H f \in L_2(\mathbb{H}_n)$. Then*

$$\int_{\mathbb{H}_n} \int_0^\infty \theta_f(B(x, r)) \frac{dr}{r} dx \lesssim \|\nabla_H f\|_{L_2(\mathbb{H}_n)}^2.$$

This will let us bound how well each slice $\nu W \cap \Gamma$ can be approximated by planes, but even if $\nu W \cap \Gamma$ is close to a plane for every ν , the whole graph Γ may not be. We thus prove Theorem 1.2 by considering slices of Γ parallel to several different planes W_i . Using a bound that will be proved in Section 6, we will show that if $\nu W_i \cap \Gamma$ is close to a plane for many different choices of W_i , then the whole graph Γ is close to a plane. We then deduce the strong geometric lemma for Γ by applying Theorem 5.1 to each slice.

We will need a weak local version of Theorem 5.1.

Corollary 5.2. *Let $f: \mathbb{H}_n \rightarrow \mathbb{R}$ be a Lipschitz function and $B(y, R) \subset \mathbb{H}_n$ a ball. Then*

$$\int_{B(y,R)} \int_0^R \theta_f(B(x, r)) \frac{dr}{r} dx \lesssim \|f\|_{Lip}^2 R^{2n+2}.$$

Proof. We translate so that $y = \mathbf{0}$ and $f(\mathbf{0}) = 0$. We first construct a Lipschitz function $F: \mathbb{H}_n \rightarrow \mathbb{R}$ such that $F = f$ on $B(\mathbf{0}, 2R)$ and $\|\nabla_H F\|_{L_2(\mathbb{H}_n)}^2 \lesssim \|f\|_{\text{Lip}}^2 R^{2n+2}$.

Let $S = B(\mathbf{0}, 2R) \cup (\mathbb{H} \setminus B(\mathbf{0}, 4R))$ and define $F: S \rightarrow \mathbb{R}$ by

$$F(x) = \begin{cases} f(x), & d(\mathbf{0}, x) \leq 2R, \\ 0 & d(\mathbf{0}, x) \geq 4R. \end{cases}$$

If $p \in B(\mathbf{0}, 2R)$ and $d(\mathbf{0}, q) \geq 4R$, then

$$\frac{|f(p) - f(q)|}{d(p, q)} \leq \frac{\|f\|_{L_\infty(B(\mathbf{0}, 2R))}}{2R} \leq \text{Lip}(f),$$

so $\text{Lip}(F) = \text{Lip}(f)$. Since \mathbb{R} is an absolute 1-Lipschitz retract [BL00, Ch. 1], we can extend F to a Lipschitz function from \mathbb{H} to \mathbb{R} with $\text{Lip}(F) = \text{Lip}(f)$. This function is supported in $B(\mathbf{0}, 4R)$, so

$$\|\nabla_H F\|_{L_2(\mathbb{H}_n)}^2 \leq \text{Lip}(F)^2 \mathcal{H}^{2n+2}(B(\mathbf{0}, 4R)) \lesssim \text{Lip}(f)^2 R^{2n+2}$$

as desired.

If $x \in B(\mathbf{0}, R)$ and $r < R$, then $f|_{B(x, r)} = F|_{B(x, r)}$ and thus $\theta_F(B(x, r)) = \theta_f(B(x, r))$.

Then

$$\begin{aligned} \int_{B(y, R)} \int_0^R \theta_f(B(x, r)) \frac{dr}{r} dx &= \int_{B(y, R)} \int_0^R \theta_F(B(x, r)) \frac{dr}{r} dx \\ &\leq \int_{\mathbb{H}_n} \int_0^\infty \theta_F(B(x, r)) \frac{dr}{r} dx \\ &\lesssim \|\nabla_H F\|_{L_2(\mathbb{H}_n)}^2 \\ &\leq \|f\|_{\text{Lip}}^2 R^{2n+2}. \end{aligned}$$

□

We now prove the main theorem.

Proof of Theorem 1.2. Let $Q(x, r) = Q_{Y_n}(x, r)$. Let $\lambda' \in (\frac{1+\lambda}{2}, 1)$ be as in Lemma 4.2.

We first construct some $(2n-1)$ -dimensional vertical subspaces P, P_1, \dots, P_{2n-1} . Recall that $P_w = V_0 \cap w^\Omega$ and let

$$P := P_{Y_n} = \text{span}(X_1, \dots, X_{n-1}, Y_1, \dots, Y_{n-1}, Z).$$

Let $W_0 = P + \langle Y_n \rangle = \{x_n = 0\}$ and let $w_1, \dots, w_{2n-1} \in A \cap W_0 \cap \text{Cone}_{\lambda'}$ be linearly independent vectors such that $y_n(w_i) = 1$. Let $P_i := P_{w_i} = V_0 \cap w_i^\Omega$. Each vector w_i is close to Y_n , so each plane P_i is close to P . Since the w_i 's span $A \cap W_0$, we have

$$\bigcap_i w_i^\Omega = (A \cap W_0)^\Omega = \text{span}(Z, Y_n),$$

and $\bigcap_i P_i = V_0 \cap \bigcap_i w_i^\Omega = \langle Z \rangle$. For all i and all $x \in \Gamma$, let $Q_i(x, r) := Q_{w_i}(x, r)$.

In Lemma 6.2 in Section 6, we will show that there is a $c > 1$ depending on λ and the P_i 's such that if $f^\nu: V_0 \rightarrow \mathbb{R}$ is a vertical function, then

$$\min_{h \in \text{Aff}} \|f^\nu - h\|_{L_2(Q(x, r))} \lesssim \sum_i \min_{\sigma \in \text{SAff}_{P_i}} \|f^\nu - \sigma\|_{L_2(Q(x, cr))}$$

for any $x \in \Gamma$ and $r > 0$. We also suppose that c is large enough to satisfy Lemma 4.2, Lemma 3.2, and Lemma 4.1. In particular,

$$r^{n+3/2} \beta_\Gamma(x, r) \lesssim \min_{h \in \text{Aff}} \|f - h\|_{L_2(Q(x, cr))} = (*)$$

for any $x \in \Gamma$.

Now, fix some $x \in \Gamma$ and let $f^\nu \in \text{VF}$ be the vertical function such that

$$(16) \quad \|f - f^\nu\|_{L_2(Q(x, cr))} = \min_{h \in \text{VF}} \|f - h\|_{L_2(Q(x, cr))}.$$

Let $f_{w_i}: V_0 \rightarrow \mathbb{R}$ be the parameterizing functions such that $\Gamma = \Gamma_{f_{w_i}, w_i}$. Then, by the triangle inequality,

$$\begin{aligned} (*) &\leq \|f - f^\nu\|_{L_2(Q(x, cr))} + \min_{h \in \text{Aff}} \|f^\nu - h\|_{L_2(Q(x, cr))} \\ &\stackrel{\text{Lem. 6.2}}{\lesssim} \|f - f^\nu\|_{L_2(Q(x, cr))} + \sum_i \min_{\sigma \in \text{SAff}_{p_i}} \|f^\nu - \sigma\|_{L_2(Q(x, c^2 r))} \\ &\lesssim \|f - f^\nu\|_{L_2(Q(x, cr))} + \sum_i \min_{\sigma \in \text{SAff}_{p_i}} (\|f^\nu - f\|_{L_2(Q(x, c^2 r))} + \|f - \sigma\|_{L_2(Q(x, c^2 r))}) \\ &\stackrel{\text{Lem. 4.2}}{\lesssim} \|f - f^\nu\|_{L_2(Q(x, c^2 r))} + \sum_i \min_{\sigma \in \text{SAff}_{p_i}} \|f_{w_i} - \sigma\|_{L_2(Q_i(x, c^2 r))}. \end{aligned}$$

By (16) and the inclusion $\text{SAff}_p \subseteq \text{VF}$,

$$(17) \quad \begin{aligned} r^{n+3/2} \beta_\Gamma(x, r) &\leq \|f - f^\nu\|_{L_2(Q(x, c^2 r))} + \sum_i \min_{\sigma \in \text{SAff}_{p_i}} \|f_{w_i} - \sigma\|_{L_2(Q_i(x, c^2 r))} \\ &\leq \min_{\sigma \in \text{SAff}_p} \|f - \sigma\|_{L_2(Q(x, c^2 r))} + \sum_i \min_{\sigma \in \text{SAff}_{p_i}} \|f_{w_i} - \sigma\|_{L_2(Q_i(x, c^2 r))} \end{aligned}$$

Thus, in order to prove Theorem 1.2, it suffices to bound how well f and the f_{w_i} 's can be approximated by slice-affine functions.

Let $p_0 \in \Gamma$, $R > 0$, and $B = B(p_0, R)$. After a translation, we may suppose $p_0 = \mathbf{0}$. Let

$$\gamma(x, r) = r^{-2n-1} \min_{\sigma \in \text{SAff}_p} \frac{\|f - \sigma\|_{L_2(Q(x, r))}^2}{r^2}.$$

We consider

$$I = \int_0^R r^{-2n-1} \int_{B \cap \Gamma} \min_{\sigma \in \text{SAff}_p} \frac{\|f - \sigma\|_{L_2(Q(x, c^2 r))}^2}{r^2} dx \frac{dr}{r} \approx \int_0^R \int_{B \cap \Gamma} \gamma(x, c^2 r) dx \frac{dr}{r}.$$

We first pass from the integral above to a sum over the balls in a family of bounded-multiplicity covers \mathcal{C}_k of Γ .

By Lemma 3.2 and our choice of c , we have

$$(18) \quad Q(x, c^{-1}r) \subset \Pi(B(x, r) \cap \Gamma) \subset Q(x, cr)$$

for any $x \in \Gamma$ and $r > 0$. For $k \geq 0$, let \mathcal{N}_k be a maximal $R2^{-k}$ -net in $\Gamma \cap B$. Let $r_k = 2c^3 R2^{-k}$ and let $\mathcal{C}_k = \{Q(p, r_k) : p \in \mathcal{N}_k\}$. By the Ahlfors regularity of Γ , \mathcal{C}_k has bounded multiplicity, i.e., for any $x \in V_0$, $|\{Q \in \mathcal{C}_k : x \in Q\}| \lesssim 1$. Furthermore, there is a $D > 0$ such that if $Q \in \mathcal{C}_k$, then $Q \subset Q(\mathbf{0}, DR)$.

For any $x \in \Gamma \cap B$, there is an $p \in \mathcal{N}_k$ such that $d(x, p) \leq R2^{-k}$, so by (18),

$$Q(x, R2^{-k}) \subset \Pi(B(x, c^2 R2^{-k}) \cap \Gamma) \subset \Pi(B(p, 2c^2 R2^{-k}) \cap \Gamma) \subset Q(p, 2c^3 R2^{-k}),$$

and thus $\gamma(x, R2^{-k}) \lesssim \gamma(p, r_k)$. It follows that

$$(19) \quad \begin{aligned} I &\lesssim \sum_{k=0}^{\infty} \sum_{p \in \mathcal{N}_k} \mathcal{H}^{2n+1}(B(p, R2^{-k}) \cap \Gamma) \gamma(p, r_k) \\ &\lesssim \sum_{k=0}^{\infty} \sum_{p \in \mathcal{N}_k} R^{2n+1} 2^{-k(2n+1)} \gamma(p, r_k), \end{aligned}$$

that is, I is bounded by a sum over the quasiballs in the \mathcal{C}_k 's.

Next, we write the sum in (19) as an integral over slices. For notational convenience, we shorten x_n^u to just u . For nonempty sets $S \subset uP$, let

$$\theta^u(S) = \text{diam}(S)^{-2n-2} \inf_{g \in \text{Aff}} \|f - g\|_{L_2(S)}^2,$$

and let $\theta^u(\emptyset) = 0$. By Section 3, uP is isomorphic to \mathbb{H}_{n-1} .

Since V_0 is a product,

$$\begin{aligned} \gamma(x, r) &= r^{-2n-1} \int_{\mathbb{R}} \min_{g \in \text{Aff}} \frac{\|f - g\|_{L_2(Q(x, r) \cap uP)}^2}{r^2} du \\ &= r^{-1} \int_{\mathbb{R}} \frac{\text{diam}(Q(x, r) \cap uP)^{2n+2}}{r^{2n+2}} \theta^u(Q(x, r) \cap uP) du \\ &\approx r^{-1} \int_{\mathbb{R}} \theta^u(Q(x, r) \cap uP) du, \end{aligned}$$

where the last line follows from the fact that $Q(x, r) \cap uP$ is either empty or an r -quasiball (Lemma 3.1). Therefore,

$$(20) \quad I \lesssim \int_{\mathbb{R}} \sum_{k=0}^{\infty} \sum_{Q \in \mathcal{C}_k} R^{2n} 2^{-2nk} \theta^u(Q \cap uP) du.$$

Finally, we apply Corollary 5.2 in each slice and integrate. We can slice each cover \mathcal{C}_k into a family of bounded-multiplicity covers of slices. Let $\mathcal{C}_k^u = \{Q \cap uP : Q \in \mathcal{C}_k\} \setminus \{\emptyset\}$. This is a set of quasiballs in uP with bounded multiplicity, and by Fubini's Theorem, we can rewrite (20) as

$$I \lesssim \int_{\mathbb{R}} \sum_{k=0}^{\infty} \sum_{S \in \mathcal{C}_k^u} R^{2n} 2^{-2nk} \theta^u(S) du.$$

Let $a > 0$ be such that $\text{diam } S \leq ar_k$ for any $S \in \mathcal{C}_k^u$ and let $F_u = f|_{uP}$. Then for any $S \in \mathcal{C}_k^u$ and any $s \in S$, we have $\theta^u(S) \lesssim \theta_{F_u}(B_{uP}(s, ar_k))$, where θ_{F_u} is as in (15). Since $\bigcup_{S \in \mathcal{C}_k^u} S \subset Q(\mathbf{0}, DR)$, the bounded multiplicity of \mathcal{C}_k^u implies

$$\begin{aligned} \sum_{S \in \mathcal{C}_k^u} R^{2n} 2^{-2nk} \theta^u(S) &\lesssim \sum_{S \in \mathcal{C}_k^u} R^{2n} 2^{-2nk} \mathcal{H}^{2n}(S)^{-1} \int_S \theta_{F_u}(B_{uP}(s, ar_k)) ds \\ &\lesssim \int_{Q(\mathbf{0}, DR) \cap uP} \theta_{F_u}(B_{uP}(s, ar_k)) ds. \end{aligned}$$

This is zero when $|u| > DR$, so

$$\begin{aligned} I &\lesssim \int_{-DR}^{DR} \sum_{k=0}^{\infty} \int_{Q(\mathbf{0}, DR) \cap uP} \theta_{F_u}(B_{uP}(s, ar_k)) ds du \\ &\lesssim \int_{-DR}^{DR} \int_0^{2ar_0} \int_{Q(\mathbf{0}, DR) \cap uP} \theta_{F_u}(B_{uP}(s, r)) ds \frac{dr}{r} du. \end{aligned}$$

By Lemma 3.1, there is a $D' > 0$ such that for each $u \in [-DR, DR]$, there is a ball $B_u \subset uP$ of radius $D'R$ such that $Q(p_0, DR) \cap uP \subset B_u$. Thus, by Corollary 5.2,

$$\begin{aligned} I &\lesssim \int_{-DR}^{DR} \int_0^{\infty} \int_{B_u} \theta^u(B_{uP}(s, r)) ds \frac{dr}{r} du \\ &\lesssim \int_{-DR}^{DR} \|F_u\|_{\text{Lip}}^2 (D'R)^{2n} du \\ &\lesssim_{\lambda} 2DR \cdot (D'R)^{2n} \approx R^{2n+1}. \end{aligned}$$

That is,

$$\int_0^R r^{-2n-1} \int_{B \cap \Gamma} \inf_{g \in \text{SAff}_p} \frac{\|f - g\|_{L_2(Q(x, c^2 r))}^2}{r^2} dx \frac{dr}{r} \lesssim_\lambda R^{2n+1}.$$

Similarly, we can prove

$$\int_0^R r^{-2n-1} \int_{B \cap \Gamma} \inf_{g \in \text{SAff}_{P_i}} \frac{\|f w_i - g\|_{L_2(Q_i(x, c^2 r))}^2}{r^2} dx \frac{dr}{r} \lesssim_\lambda R^{2n+1}.$$

This, together with (17) allows us to conclude

$$\int_{B \cap \Gamma} \int_0^R \beta_\Gamma(x, r)^2 \frac{dr}{r} dx \lesssim_\lambda R^{2n+1},$$

as desired. \square

6. SLICING VERTICAL FUNCTIONS

In this section, we prove bounds on functions that are close to affine on many families of parallel hyperplanes. The simplest version of this bound deals with functions on the cube that are close to affine on any axis-parallel hyperplane. For V a vector space and $U \subset V$ a subspace, we define $\text{SAff}_U(V)$ to be the set of measurable functions that are affine on every coset of U .

Proposition 6.1. *Let $d \geq 3$ and let $R_1, \dots, R_d \subset \mathbb{R}^d$ be the coordinate $(d-1)$ -planes, so that R_i consists of points whose i th coordinate is zero. Let $I = [-1, 1]$ and let $\text{SAff}_{R_i}(\mathbb{R}^d) \subset L_2(I^d)$ be the set of functions that are affine on each plane parallel to R_i . Then, for any $g \in L_2(I^d)$,*

$$(21) \quad \min_{\lambda \in \text{Aff}} \|g - \lambda\|_2 \approx_d \sum_i \min_{\lambda \in \text{SAff}_{R_i}(\mathbb{R}^d)} \|g - \lambda\|_2.$$

This proposition implies the bounds on slice-affine functions on \mathbb{H}_n used in the proof of Theorem 1.2.

Lemma 6.2. *Let $n \geq 2$ and let $P_1, \dots, P_{2n-1} \subset V_0$ be vertical hyperplanes in general position, i.e., vertical $(2n-1)$ -planes such that $\bigcap_i P_i = \langle Z \rangle$. There is a $c > 1$ such that for any vertical function $f \in L_2(V_0)$, any $x \in \Gamma_f$, and any $r > 0$,*

$$(22) \quad \min_{g \in \text{Aff}} \|f - g\|_{L_2(Q(x, r))} \lesssim_{P_1, \dots, P_{2n-1}} \sum_i \min_{g \in \text{SAff}_{P_i}(V_0)} \|f - g\|_{L_2(Q(x, cr))},$$

where for convenience we define $Q(x, r) = Q_{Y_n}(x, r)$.

Proof. After applying a translation, we may suppose that $x = \mathbf{0}$.

Let $A_0 = A \cap V_0$ and let $P = P_{Y_n} = Y_n^{\overline{\Omega}} \cap V_0$. Then $Q(x, r) = \Pi_{Y_n}(\delta_r(K))$ where

$$K = \{sX_n + a + tZ : a \in B_A(x, 1) \cap P, |s| \leq 1, |t| \leq 1\}.$$

Let $S_r = \pi(Q(x, r))$; this is isometric to the product of an r -ball and an interval of length $2r$, so it is a quasiball of radius r in A_0 . Furthermore, for any $p \in S_r$, the intersection $\pi^{-1}(p) \cap Q(x, r)$ is an interval of length $2r^2$.

We identify A_0 with \mathbb{R}^{2n-1} by an isomorphism. The projections $\pi(P_i)$ are subspaces of A_0 in general position, so there is a linear transformation $M: A_0 \rightarrow A_0$ such that $M(\pi(P_i)) = R_i$ for all i . Let $D_t = [-t, t]^{2n-1} \subset A_0$. This is a quasiball in A_0 , and MS_r and $M^{-1}S_r$ are quasiballs with constants depending on M , so there is a $b > 0$ depending only on M such that $MS_t \subset D_{bt}$ and $D_t \subset M^{-1}S_{bt}$ for all $t > 0$.

Since any $g \in \text{Aff}$ is a vertical function,

$$\|f - g\|_{L_2(Q(x,r))}^2 = 2r^2 \|f - g\|_{L_2(S_r)}^2.$$

Let $h: A_0 \rightarrow \mathbb{R}$ be the function $h(v) = f(M^{-1}v)$; this is also a vertical function, and by a change of variables,

$$\begin{aligned} \min_{\lambda \in \text{Aff}} \|f - \lambda\|_{L_2(Q(x,r))}^2 &= \min_{\lambda \in \text{Aff}} 2r^2 \|f - \lambda\|_{L_2(S_r)}^2 \\ &= \min_{\lambda \in \text{Aff}} 2r^2 |\det(M)|^{-1} \cdot \|h - \lambda \circ M^{-1}\|_{L_2(MS_r)}^2 \\ &\leq \min_{\lambda \in \text{Aff}} 2r^2 |\det(M)|^{-1} \|h - \lambda\|_{L_2(D_{br})}^2. \end{aligned}$$

By Proposition 6.1,

$$\min_{\lambda \in \text{Aff}} \|h - \lambda\|_{L_2(D_{br})}^2 \lesssim \sum_i \min_{\lambda \in \text{SAff}_{R_i}(\mathbb{R}^{2n-1})} \|h - \lambda\|_{L_2(D_{br})}^2.$$

Let J_i be the determinant of the restriction $M^{-1}|_{R_i}: R_i \rightarrow \pi(P_i)$. If $\lambda \in \text{SAff}_{R_i}(\mathbb{R}^{2n-1})$, then $\lambda \circ M \in \text{SAff}_{\pi(P_i)}(A_0)$ and $\lambda \circ M \circ \pi \in \text{SAff}_{P_i}(V_0)$, so by another change of variables,

$$\begin{aligned} \min_{\lambda \in \text{SAff}_{R_i}(\mathbb{R}^{2n-1})} \|h - \lambda\|_{L_2(D_{br})}^2 &= \min_{\lambda \in \text{SAff}_{R_i}(\mathbb{R}^{2n-1})} |J_i|^{-1} \|f - \lambda \circ M\|_{L_2(M^{-1}D_{br})}^2 \\ &\leq \min_{\lambda \in \text{SAff}_{\pi(P_i)}(A_0)} |J_i|^{-1} \|f - \lambda\|_{L_2(S_{b^2r})}^2 \\ &= 2b^{-4} r^{-2} \min_{\lambda \in \text{SAff}_{P_i}(V_0)} |J_i|^{-1} \|f - \lambda\|_{L_2(Q(x,b^2r))}^2. \end{aligned}$$

Combining these calculations, we find

$$\begin{aligned} \min_{\lambda \in \text{Aff}} \|f - \lambda\|_{L_2(Q(x,r))}^2 &\lesssim_M \min_{\lambda \in \text{Aff}} r^2 \|h - \lambda\|_{L_2(D_{br})}^2 \\ &\lesssim \sum_i r^2 \min_{\lambda \in \text{SAff}_{R_i}(\mathbb{R}^{2n-1})} \|h - \lambda\|_{L_2(D_{br})}^2 \\ &\lesssim_M b^{-4} r^{-2} \sum_i \min_{\lambda \in \text{SAff}_{P_i}(V_0)} r^2 \|f - \lambda\|_{L_2(Q(x,b^2r))}^2. \end{aligned}$$

Since b and M depend on P_1, \dots, P_n , we conclude

$$\min_{\lambda \in \text{Aff}} \|f - \lambda\|_{L_2(Q(x,r))}^2 \lesssim_{P_1, \dots, P_n} \sum_i \min_{\lambda \in \text{SAff}_{P_i}(V_0)} \|f - \lambda\|_{L_2(Q(x,b^2r))}^2.$$

as desired. \square

In the rest of this section, we prove Proposition 6.1 using a wavelet decomposition. Let $\psi_{j,k} \in L_2([-1, 1])$ be the Haar wavelet basis for $L_2([-1, 1])$. That is, let $\psi_{0,0}(t) = 1$, and for $j > 0$ and $0 \leq k < 2^{j-1}$, let

$$\psi_{j,k}(t) = \begin{cases} -1 & 2k \cdot 2^{-j+1} - 1 \leq t < (2k+1) \cdot 2^{-j+1} - 1 \\ 1 & (2k+1) \cdot 2^{-j+1} - 1 \leq t < (2k+2) \cdot 2^{-j+1} - 1 \\ 0 & \text{otherwise.} \end{cases}$$

When $j > 0$, the support of $\psi_{j,k}(t)$ is an interval of width 2^{-j+2} .

For multi-indices $\mathbf{j} = (j_1, \dots, j_d)$, $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d$ with $0 \leq k_i < \max\{1, 2^{j_i-1}\}$, let

$$\Psi_{\mathbf{j}, \mathbf{k}}(t_1, \dots, t_d) = \prod_{i=1}^d \psi_{j_i, k_i}(t_i).$$

Let $\text{supp}(\mathbf{j}) = \{i \mid j_i \neq 0\}$. It is straightforward to check that the $\Psi_{\mathbf{j}, \mathbf{k}}$'s are orthogonal.

Let

$$c_{\mathbf{j},\mathbf{k}} = \frac{\langle g, \Psi_{\mathbf{j},\mathbf{k}} \rangle}{\langle \Psi_{\mathbf{j},\mathbf{k}}, \Psi_{\mathbf{j},\mathbf{k}} \rangle}$$

so that

$$g = \sum_{\mathbf{j},\mathbf{k}} c_{\mathbf{j},\mathbf{k}} \Psi_{\mathbf{j},\mathbf{k}}.$$

We partition this sum in two ways. For every subset $S \subset \{1, \dots, d\}$, define

$$(23) \quad f_S = \sum_{\substack{\mathbf{j},\mathbf{k} \\ \text{supp}(\mathbf{j})=S}} c_{\mathbf{j},\mathbf{k}} \Psi_{\mathbf{j},\mathbf{k}}.$$

For each $0 \leq i \leq d$, let

$$g_i = \sum_{\substack{\mathbf{j},\mathbf{k} \\ |\text{supp}(\mathbf{j})|=i}} c_{\mathbf{j},\mathbf{k}} \Psi_{\mathbf{j},\mathbf{k}} = \sum_{|S|=i} f_S.$$

Note that $f_S(t_1, \dots, t_d)$ depends only on the t_i such that $i \in S$; in particular, $f_\emptyset = g_0$ is constant. The f_S 's are pairwise orthogonal, as are the g_i 's.

Let $\lambda: L_2([0, 1]^d) \rightarrow \text{Aff}$ be the orthogonal projection to Aff . Let $\text{SAff}_i = \text{SAff}_{R_i}(\mathbb{R}^d)$ and let λ_i be the orthogonal projection to SAff_i . For every \mathbf{j}, \mathbf{k} with $|\text{supp} \mathbf{j}| \geq 2$, we have $\Psi_{\mathbf{j},\mathbf{k}} \perp \text{Aff}$. Likewise, for every \mathbf{j}, \mathbf{k} with $|\text{supp} \mathbf{j} \setminus \{i\}| \geq 2$, we have $\Psi_{\mathbf{j},\mathbf{k}} \perp \text{SAff}_i$. Therefore,

$$(24) \quad \lambda(g) = \lambda(g_0 + g_1) = g_0 + \lambda(g_1),$$

and

$$(25) \quad \lambda_i(g) = \lambda_i(g_0 + g_1 + g_2) = g_0 + \lambda_i(g_1 + g_2)$$

for all i .

We will need two lemmas.

Lemma 6.3. *Let g_i and f_S be as above. For any $1 \leq l, m \leq d$, $l \neq m$, we have $\lambda_l(f_{l_l}) = f_{l_l}$, $\lambda_m(f_{l_l}) = \lambda(f_{l_l})$, and*

$$\lambda_l(g_1) = f_{l_l} + \sum_{m \neq l} \lambda(f_{l_m}).$$

Proof. Since $f_{l_l}(t_1, \dots, t_l)$ depends only on t_l , it is constant on each plane parallel to R_l . Therefore, $\lambda_l(f_{l_l}) = f_{l_l}$.

When $m \neq l$, then f_{l_l} is independent of t_m , so $\lambda_m(f_{l_l})$ is independent of t_m and affine on every plane parallel to R_m . Therefore, $\lambda_m(f_{l_l})$ is affine and $\lambda_m(f_{l_l}) = \lambda(f_{l_l})$. This implies

$$\lambda_l(g_1) = \sum_m \lambda_l(f_{l_m}) = \lambda_l(f_{l_l}) + \sum_{m \neq l} \lambda(f_{l_m}).$$

□

Lemma 6.4.

$$(26) \quad \|g - \lambda(g)\|_2^2 = \|g_1 - \lambda(g_1)\|_2^2 + \sum_{i=2}^{\infty} \|g_i\|_2^2,$$

and for any l ,

$$(27) \quad \begin{aligned} \|g - \lambda_l(g)\|_2^2 &= \left\| g_1 - \lambda_l(g_1) + g_2 - \lambda_l(g_2) + \sum_{i=3}^{\infty} g_i \right\|_2^2 \\ &= \|g_1 - \lambda_l(g_1)\|_2^2 + \|g_2 - \lambda_l(g_2)\|_2^2 + \sum_{i=3}^{\infty} \|g_i\|_2^2. \end{aligned}$$

Proof. By (24) and the orthogonality of the g_i 's,

$$\|g - \lambda(g)\|_2^2 = \left\| g_1 - \lambda(g_1) + \sum_{i=2}^{\infty} g_i \right\|_2^2 = \|g_1 - \lambda(g_1)\|_2^2 + \sum_{i=2}^{\infty} \|g_i\|_2^2 + 2 \sum_{i=2}^{\infty} \langle g_1 - \lambda(g_1), g_i \rangle.$$

For $i \geq 2$, g_i is orthogonal to g_1 and to Aff , so $\langle g_1 - \lambda(g_1), g_i \rangle = 0$. This implies (26).

By (25),

$$g - \lambda_l(g) = (g_1 - \lambda_l(g_1)) + (g_2 - \lambda_l(g_2)) + \sum_{i=3}^{\infty} g_i.$$

We claim that the terms in this sum are orthogonal. When $i \geq 3$, g_i is orthogonal to g_1, g_2 , and SAff_l , so it suffices to check that $\langle g_1 - \lambda_l(g_1), g_2 - \lambda_l(g_2) \rangle = 0$.

Since $(g_1 - \lambda_l(g_1)) \perp \text{SAff}_l$ and $\langle g_1, g_2 \rangle = 0$,

$$\langle g_1 - \lambda_l(g_1), g_2 - \lambda_l(g_2) \rangle = \langle g_1 - \lambda_l(g_1), g_2 \rangle = -\langle \lambda_l(g_1), g_2 \rangle.$$

By Lemma 6.3,

$$\langle \lambda_l(g_1), g_2 \rangle = \langle \alpha, g_2 \rangle + \langle f_{[l]}, g_2 \rangle,$$

where $\alpha = \sum_{m \neq l} \lambda(f_{[m]})$ is affine. Since g_2 is a sum of $\Psi_{\mathbf{j}, \mathbf{k}}$'s with $|\text{supp}(\mathbf{j})| = 2$, it is orthogonal to any affine function. Likewise, $f_{[l]}$ is a sum of $\Psi_{\mathbf{j}, \mathbf{k}}$'s with $|\text{supp}(\mathbf{j})| = 1$, so $f_{[l]}$ is orthogonal to g_2 . Therefore,

$$\langle g_1 - \lambda_l(g_1), g_2 - \lambda_l(g_2) \rangle = -\langle \lambda_l(g_1), g_2 \rangle = 0,$$

as desired. \square

Next, we use these lemmas to show that

$$(28) \quad \|g_1 - \lambda(g_1)\|^2 \approx_d \sum_{l=1}^d \|g_1 - \lambda_l(g_1)\|^2$$

and

$$(29) \quad \|g_2\|^2 \approx_d \sum_{l=1}^d \|g_2 - \lambda_l(g_2)\|^2.$$

To prove (28), we decompose the left and right sides. For the left side, note that $g_1 = \sum_i f_{[i]}$, so

$$g_1 - \lambda(g_1) = \sum_i f_{[i]} - \lambda(f_{[i]}).$$

Each function $f_{[i]}(t_1, \dots, t_d) - \lambda(f_{[i]})(t_1, \dots, t_d)$ depends only on t_i and satisfies

$$\int_{[-1,1]^d} f_{[i]} - \lambda(f_{[i]}) d\mu = 0,$$

so when $i \neq j$,

$$\langle f_{[i]} - \lambda(f_{[i]}), f_{[j]} - \lambda(f_{[j]}) \rangle = 0.$$

Therefore,

$$\|g_1 - \lambda(g_1)\|^2 = \sum_m \|f_{[m]} - \lambda(f_{[m]})\|^2.$$

For the right side, by Lemma 6.3,

$$g_1 - \lambda_l(g_1) = \sum_{m \neq l} (f_{[m]} - \lambda(f_{[m]})).$$

As noted above, the terms in the sum are pairwise orthogonal, so

$$\|g_1 - \lambda_l(g_1)\|^2 = \sum_{m \neq l} \|f_{[m]} - \lambda(f_{[m]})\|^2.$$

Summing over l , we get

$$\sum_l \|g_1 - \lambda_l(g_1)\|^2 = \sum_m (d-1) \|f_{\{m\}} - \lambda(f_{\{m\}})\|^2 = (d-1) \|g_1 - \lambda(g_1)\|^2,$$

which proves (28).

To prove (29), note that if $|S \setminus \{l\}| \geq 2$, then f_S is orthogonal to SAff_l . Let

$$h_l = \sum_{\substack{S \subset \{1, \dots, d\} \setminus \{l\} \\ |S|=2}} f_S$$

and $k_l = g_2 - h_l$, so that h_l is orthogonal to k_l and to SAff_l .

Then

$$\|g_2 - \lambda_l(g_2)\|^2 = \|h_l + k_l - \lambda_l(g_2)\|^2 = \|h_l\|^2 + \|k_l - \lambda_l(g_2)\|^2 \geq \|h_l\|^2$$

and

$$\sum_l \|g_2 - \lambda_l(g_2)\|^2 \geq \sum_l \|h_l\|^2 = \sum_l \sum_{\substack{S \subset \{1, \dots, d\} \setminus \{l\} \\ |S|=2}} \|f_S\|^2.$$

Each subset S of order 2 appears $d-2$ times in the sum on the right, so

$$\sum_l \|g_2 - \lambda_l(g_2)\|^2 \geq (d-2) \sum_{|S|=2} \|f_S\|^2 = (d-2) \|g_2\|^2.$$

Conversely, $\|g_2 - \lambda_l(g_2)\| \leq \|g_2 - \lambda(g_2)\|$, so $\sum_l \|g_2 - \lambda_l(g_2)\|^2 \leq d \|g_2 - \lambda(g_2)\|^2$. This proves (29).

Finally, we prove Proposition 6.1. By (26),

$$\min_{\lambda \in \text{Aff}} \|g - \lambda\|^2 = \|g - \lambda(g)\|^2 = \|g_1 - \lambda(g_1)\|^2 + \sum_{i=2}^{\infty} \|g_i\|^2.$$

By (27)–(29),

$$\begin{aligned} \sum_i \min_{\lambda \in \text{SAff}_i} \|g - \lambda\|_2 &= \sum_l \|g - \lambda_l(g)\|^2 \\ &= \sum_l \left(\|g_1 - \lambda_l(g_1)\|^2 + \|g_2 - \lambda_l(g_2)\|^2 + \sum_{i=3}^{\infty} \|g_i\|^2 \right) \\ &\approx_d \|g_1 - \lambda(g_1)\|^2 + \|g_2\|^2 + \sum_{i=3}^{\infty} \|g_i\|^2 \\ &= \min_{\lambda \in \text{Aff}} \|g - \lambda\|^2, \end{aligned}$$

as desired.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT

Email address: `vasileios.chousionis@uconn.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT

Email address: `sean.li@uconn.edu`

COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY

Email address: `ryoung@cims.nyu.edu`