(11) Let $k^2 = \frac{g}{L}$, which is a positive constant.

\[ \therefore \ x'' = -k^2 \sin x \quad \text{(1)} \]

Thus \[ F(x) = \int_0^x f(t) \, dt = -\int_0^x k^2 \sin t \, dt = -k^2 (1 - \cos x) \]

Hence potential energy \[ U(x) = -F(x) = k^2 (1 - \cos x) \]

Write (1) as a 1st order system:

\[
\begin{align*}
  x' &= y \\
  y' &= -k^2 \sin x
\end{align*}
\quad \text{(2)}
\]

It is known that this is a Hamiltonian system with

\[ H(x, y) = \frac{1}{2} y^2 + U(x) \]

\[ \therefore H(x, y) = \text{constant along a solution trajectory} \]

i.e. \[ \frac{y^2}{2} + k^2 (1 - \cos x) = \text{constant along trajectories} \]

Equilibrium solutions of (1) are \[ y = 0 \quad \text{and} \quad \sin x = 0 \]

Thus \( (x, y) = (0, 0), (\pm \pi, 0), (\pm 2\pi, 0), \ldots \) are the equilibrium points.

We plot the graph of \( U \):
It is clear that $U(x) > \min U = U(0) = 0$.

Thus $H(x, y) = \frac{y^2}{2} + U(x) > U(x) > 0$.

Energy $H$ associated with equilibrium points are special.

$H(0, 0) = 0$, $H(\pm \pi, 0) = H(-\pi, 0) = 2k^2$

In fact $H(2m\pi, 0) = 0$, $m = \pm 1, \pm 2, \ldots$

$H((2m+1)\pi, 0) = 2k^2$, $m = 0, \pm 1, \pm 2, \ldots$

(i) For $H(x, y) = c = 0$, soln are $(x, y) = (2m\pi, 0)$, $m = \pm 1, \pm 2, \ldots$

(ii) For $0 < c < 2k^2$, soln trajectories of $H(x, y) = c$ are periodic orbits.

(iii) For $H(x, y) = c = 2k^2$, soln trajectories are heteroclinic orbits.

\[ \lim_{t \to +\infty} x(t) = -\pi \quad \lim_{t \to -\infty} x(t) = +\pi \]

There is another orbit such that

\[ \lim_{t \to +\infty} x(t) = +\pi \quad \lim_{t \to -\infty} x(t) = -\pi \]

(iv) For $c > 2k^2$ & $H(x, y) = c$, soln $x(t)$ can keep on increasing to $+\infty$.

or keep on decreasing to $-\infty$. Physically it corresponds to the pendulum keeps on swinging.
(21) This is the same as Example 8.4.2. The phase plane has already been represented in Figure 8.9.

Know: \[ H(x, y) = \frac{1}{2} y^2 + \frac{x^2}{2} - \frac{x^4}{4} = \text{constant along colu trajectory.} \]

In the heteroclinic orbit that goes through the equilibrium pt. \((x, y) = (1, 0)\) is:

\[ H(x, y) = \text{constant} = H(1, 0) = \frac{1}{4} \]

\[
\begin{align*}
\text{For } (x(0) = 2) \\
\text{y } (0) = 0, & \quad H(x, y) = H(2, 0) = \frac{2^2}{2} - \frac{2^4}{4} = -2.
\end{align*}
\]

The graph \( H(x, y) = -2 \):

\[ \Leftrightarrow \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^4}{4} = -2 \]

Along this graph, \( x \rightarrow \infty \) \& \( y \rightarrow \infty \). More precisely, \( y \approx \pm \frac{1}{2} x^2 \) for large \( x \).

Thus as \( t \) gets larger, then \( x(t) \rightarrow \infty \) \& \( y(t) \rightarrow \infty \).

(In fact one can show there is a finite \( t \) s.t. \( t \rightarrow t_0 \),
then \( x(t) \rightarrow \infty \) \& \( y(t) \rightarrow \infty \).)
(30) \[ x'' = \frac{x - 3x^2}{f(x)} \]  

i. \[ F(x) = \int_0^x f(t) \, dt = \int_0^x (t - 3t^2) \, dt = \frac{x^2}{2} - x^3 \]

Hence potential energy \( U(x) = -F(x) = -\frac{x^2}{2} + x^3 \).

Write (i) as a system,

\[
\begin{cases}
    x' = y \\
    y' = x - 3x^2
\end{cases}
\]

It is a Hamiltonian system with

\[ H(x, y) = \frac{1}{2} y^2 + U(x) = \frac{1}{2} y^2 - \frac{x^2}{2} + x^3 \]

\[ \therefore H(x, y) = \text{constant along a soln trajectory} \]

i.e. \[ \frac{1}{2} y^2 - \frac{x^2}{2} + x^3 = C \] along a soln trajectory.

Like to plot these curves for various \( C \) in the phase plane.

Equilibrium solns are \((x, y) = (0, 0) \) and \((x, y) = (\frac{1}{3}, 0)\).

Note that \( x = 0 \) and \( x = \frac{1}{3} \) are the roots of \( x - 3x^2 = 0 \).

We plot the graph of \( U \): This is a cubic polynomial with critical points at \( x = 0 \) and \( x = \frac{1}{3} \).

\[ z = \frac{1}{54} \]

\[ z = U(x) = -\frac{x^2}{2} + x^3 \]

The roots of \( U \) are \( x = 0 \) and \( x = \frac{1}{2} \).

At \( x = \frac{1}{3} \),

\[ U(\frac{1}{3}) = -\frac{1}{2} \left( \frac{1}{3} \right)^2 + \left( \frac{1}{3} \right)^3 = -\frac{1}{54} \]
Energy $H$ associated with equilibrium points are special. We compute such values:

\[
\begin{align*}
H(0,0) &= 0, \\
H\left(\frac{1}{3},0\right) &= -\frac{1}{54}.
\end{align*}
\]

The phase plane is

- $H = 0$ (homoclinic orbit)
- $H > 0$
- $H < 0$
- $H = -\frac{1}{54}$

(i) There is a homoclinic orbit with $H(x,y) = 0$. It forms a loop around the equilibrium point $(x,y) = \left(\frac{1}{3},0\right)$.

(ii) With initial condition $(x,y) = (0,1)$, it is outside of the homoclinic orbit loop. It is clear from the phase plane that as $t$ progresses, the solution trajectory goes to $x \to -\infty$ and $y \to -\infty$.

(In fact there is a finite time $t_0 > 0$ s.t. $x(t) \to -\infty$ and $y(t) \to -\infty$ as $t \to t_0$. )