\[ \int x'' = 2x'(x-1) \quad (1) \]

\[ x(1) = 0, \quad x'(1) = 1 \quad (2) \]

Let \( z = x' \). Then \( x'' = z' = \frac{d^2 z}{dx^2} = z \frac{dz}{dx} \quad (3) \)

**Remark**: Strictly speaking, \( z(t) = \tilde{z}(x(t)) \) with \( z \neq \tilde{z} \) being different functions. Then

\[
\frac{dz}{dt} = \frac{d}\frac{dz}{dx} \frac{dx}{dt} = z(t) \frac{d\tilde{z}}{dx} = \tilde{z} \frac{d\tilde{z}}{dx}
\]

But we follow the book's calculation as in (3).

Put (3) into (1),

\[
z \frac{d\tilde{z}}{dx} = 2 \tilde{z} (x-1)
\]

\[
z = 0 \quad \text{or} \quad \frac{d\tilde{z}}{dx} = 2 (x-1)
\]

**Case 1** : \( z = 0 \).

\[
\frac{dx}{dt} = 0 \quad \text{so that} \quad x(t) = \text{constant} \quad \text{for all} \quad t.
\]

With (2), we see that \( x(t) = x(1) = 0 \) for all \( t \).

But then \( x'(t) = 0 \) for all \( t \). This contradicts \( x'(1) = 1 \).

This case cannot exist.

**Case 2** : \( \frac{d\tilde{z}}{dx} = 2 (x-1) \).

\[
z = \int 2(x-1) dx = x^2 - 2x + C
\]
Put into (2),

\[ z(0) = 1 = 0^2 - 0 + C \]
\[ \therefore C = 1. \]

Hence

\[ z = \frac{dx}{dt} = x^2 - 2x + 1 \]
\[ = (x - 1)^2 \]
\[ \int \frac{dx}{(x-1)^2} = \int dt \]
\[ \frac{-1}{x-1} = t + C_1 \tag{4} \]

Put into (2),

\[ \frac{-1}{0-1} = 0 + C_1 \]
\[ \therefore C_1 = 1 \]

Thus (4) leads to:

\[ \frac{-1}{x-1} = t + 1 \]
\[ x - 1 = \frac{-1}{t+1} \]
\[ x = 1 - \frac{1}{t+1} \]
\[ = \frac{t}{t+1} \]